Theory of Composite Shells

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Week - 07

Lecture - 02

Buckling of shell

Dear learners, welcome to Week-7, Lecture-2. In the week-7 in lecture-1, I have discussed about the Buckling of shells and various theoretical background that why we want to do study the buckling of shells, and why only the cylindrical shell is considered most, and definitely the other shells are also studied but the cylindrical shell is analysis is mostly published in the literature.

So, today, I will just do some special cases and brief formulations for the buckling of cylindrical shells.

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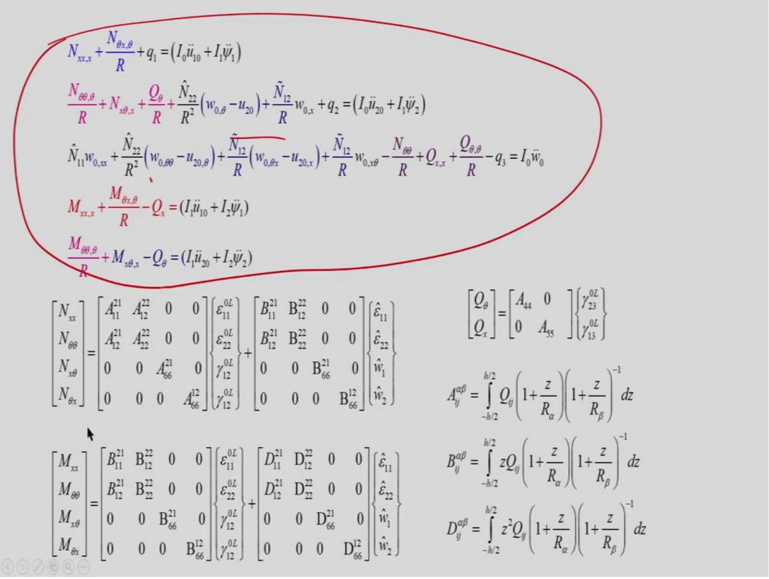

Today, I will do some special cases and brief formulations for the buckling of cylindrical shells. Initially, we were having these 5 partial differential equations having non-linear terms:

$$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1 \quad equation(1)$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + \frac{Q_\theta}{R} + \frac{N_{22}}{R^2}\left(w_0{,}_\theta - u_{20}\right) + \frac{N_{12}}{R}\left(w_0{,}_x\right) + q_2 = \left(I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2\right) \quad equation(2)$$

$$N_{11} w_0{,}_{xx} + \frac{N_{22}}{R^2}\left(w_0{,}_{\theta\theta} - u_{20,\theta}\right) + \frac{N_{12}}{R}\left(w_0{,}_{\theta x} - u_{20,x}\right) + \frac{N_{12}}{R} w_0{,}_{x\theta} - \frac{N_{\theta\theta}}{R}$$

$$+Q_{x,x} + \frac{Q_{\theta,\theta}}{R} - q_3 = I_0 \ddot{w}_0 \quad equation(3)$$

$$M_{xx,x} + \frac{M_{\theta x,\theta}}{R} - Q_x = I_1 \ddot{u}_{10} + I_2 \ddot{\psi}_1 \quad equation(4)$$

$$\frac{M_{\theta\theta,\theta}}{R} + M_{x\theta,x} - Q_\theta = I_1 \ddot{u}_{20} + I_2 \ddot{\psi}_2 \quad equation(5)$$

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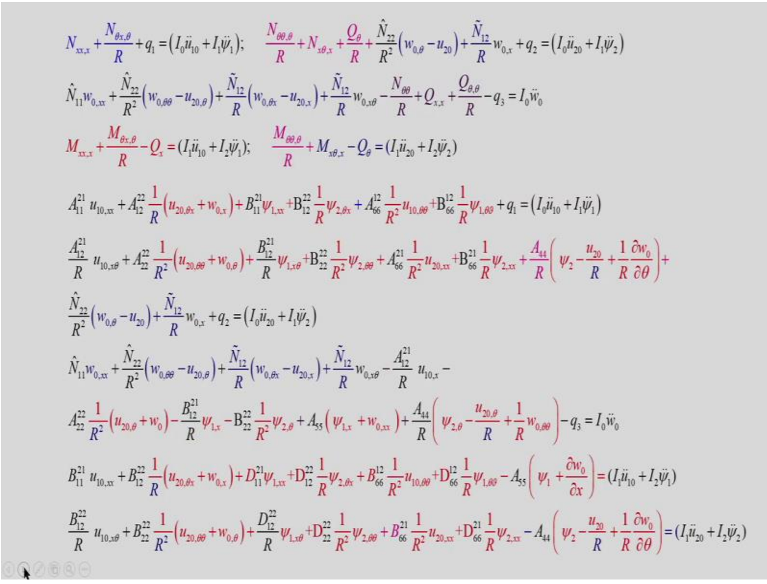

In lecture-01 of week-07, I explained that for the cylindrical shell case, this can be reduced to these equations:

$$A_{11}^{21}u_{10,xx}+A_{12}^{22}\frac{1}{R}\left(u_{20,\theta x}+w_{0,x}\right)+B_{11}^{21}\psi_{1,xx}+B_{12}^{22}\frac{1}{R}\psi_{2,\theta x}+A_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta}+B_{66}^{12}\frac{1}{R}\psi_{1,\theta\theta}+q_1$$
$$=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)\ equation(1)$$

$$\frac{A_{12}^{21}}{R}u_{10,x\theta}+A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{B_{12}^{21}}{R}\psi_{1,x\theta}+B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta}+A_{66}^{21}\frac{1}{R^2}u_{20,xx}+B_{66}^{21}\frac{1}{R}\psi_{2,xx}$$
$$+\frac{A_{44}}{R}\left(\psi_2-\frac{u_{20}}{R}+\frac{1}{R}\frac{\partial w_0}{\partial\theta}\right)+\frac{\widehat{N_{22}}}{R^2}\left(w_{0,\theta}-u_{20}\right)+\frac{\widehat{N_{12}}}{R}w_{0,x}+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)\ \ equation(2)$$

$$\widehat{N_{11}}w_{0,xx}+\frac{\widehat{N_{22}}}{R^2}\left(w_{0,\theta\theta}-u_{20,\theta}\right)+\frac{\widehat{N_{12}}}{R}\left(w_{0,\theta x}-u_{20,x}\right)+\frac{\widehat{N_{12}}}{R}w_{0,x\theta}-\frac{A_{12}^{21}}{R}u_{10,x}$$
$$-A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{B_{12}^{21}}{R}\psi_{1,x}-B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta}+A_{55}\left(\psi_{1,x}+w_{0,xx}\right)+$$
$$\frac{A_{44}}{R}\left(\psi_{2,\theta}-\frac{u_{20,\theta}}{R}+\frac{1}{R}w_{0,\theta\theta}\right)-q_3=I_0\ddot{w}_0\quad equation(3)$$

$$B_{11}^{21}u_{10,xx}+B_{12}^{22}\frac{1}{R}\left(u_{20,\theta x}+w_{0,x}\right)+D_{11}^{21}\psi_{1,xx}+D_{12}^{22}\frac{1}{R}\psi_{2,\theta x}+B_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta}+D_{66}^{12}\frac{1}{R}\psi_{1,\theta\theta}$$
$$-A_{55}\left(\psi_1+\frac{\partial w_0}{\partial x}\right)=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)\ \ equation(4)$$

$$\frac{B_{12}^{22}}{R}u_{10,x\theta}+B_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{D_{12}^{22}}{R}\psi_{1,x\theta}+D_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta}+B_{66}^{21}\frac{1}{R^2}u_{20,xx}+D_{66}^{21}\frac{1}{R}\psi_{2,xx}$$
$$-\frac{A_{44}}{R}\left(\psi_2-\frac{u_{20}}{R}+\frac{1}{R}\frac{\partial w_0}{\partial\theta}\right)=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)\quad equation(5)$$

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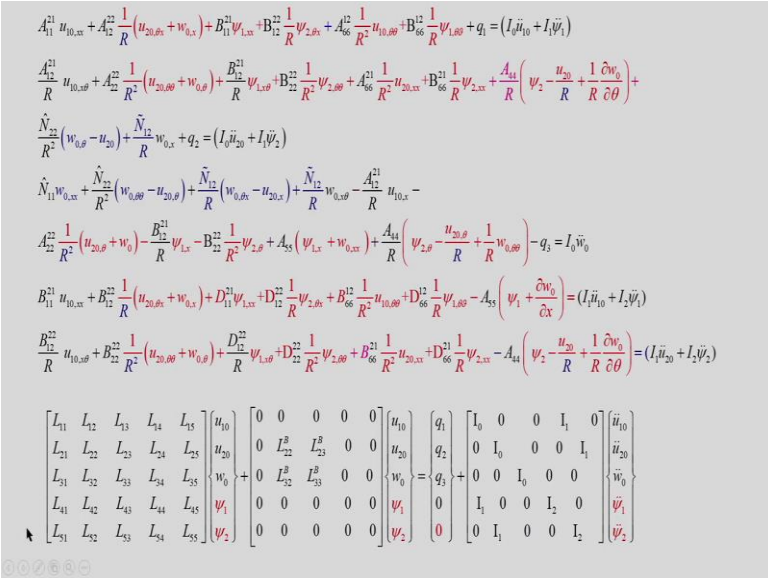

And substituting the shell constitutive relations, ultimately, this can be expressed as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & L_{22}^B & L_{23}^B & 0 & 0 \\ 0 & L_{32}^B & L_{33}^B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I_0 & 0 & 0 & I_1 & 0 \\ 0 & I_0 & 0 & 0 & I_1 \\ 0 & 0 & I_0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 \\ 0 & I_1 & 0 & 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \\ \ddot{w}_0 \\ \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{bmatrix}$$

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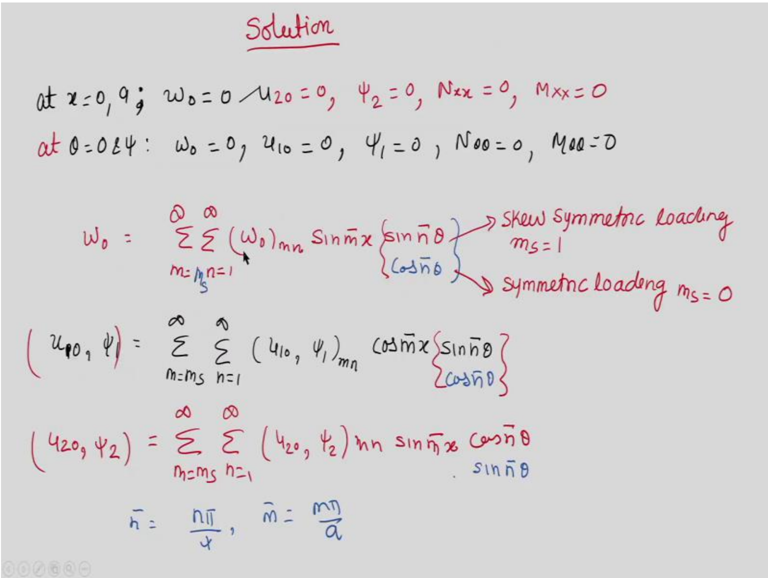

Then using the simply supported boundary condition that both edges are simply supported i.e.,

$$
\begin{aligned}
&at\, x = 0, a \quad w_0 = 0;\, u_{20} = 0;\, \psi_2 = 0;\, N_{xx} = 0;\; M_{xx} = 0 \\
&at\, \theta = 0, \alpha \quad w_0 = 0;\, u_{10} = 0;\, \psi_1 = 0;\, N_{\theta\theta} = 0;\; M_{\theta\theta} = 0
\end{aligned}
$$

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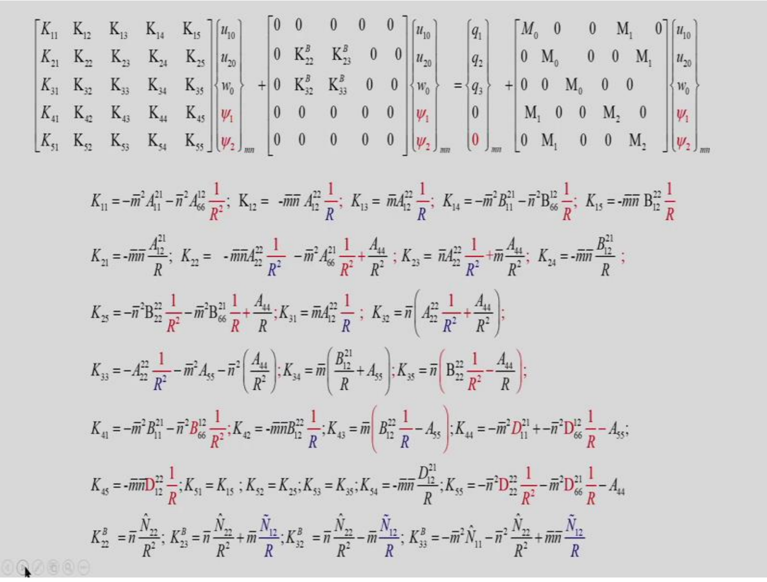

If we do so it reduces to matrix K:

$$
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn}
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & K_{22}^B & K_{23}^B & 0 & 0 \\
0 & K_{32}^B & K_{33}^B & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn}
=
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix}_{mn}
+
$$

$$
\begin{bmatrix}
M_0 & 0 & 0 & M_1 & 0 \\
0 & M_0 & 0 & 0 & M_1 \\
0 & 0 & M_0 & 0 & 0 \\
M_1 & 0 & 0 & M_2 & 0 \\
0 & M_1 & 0 & 0 & M_2
\end{bmatrix}
\begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn}
$$

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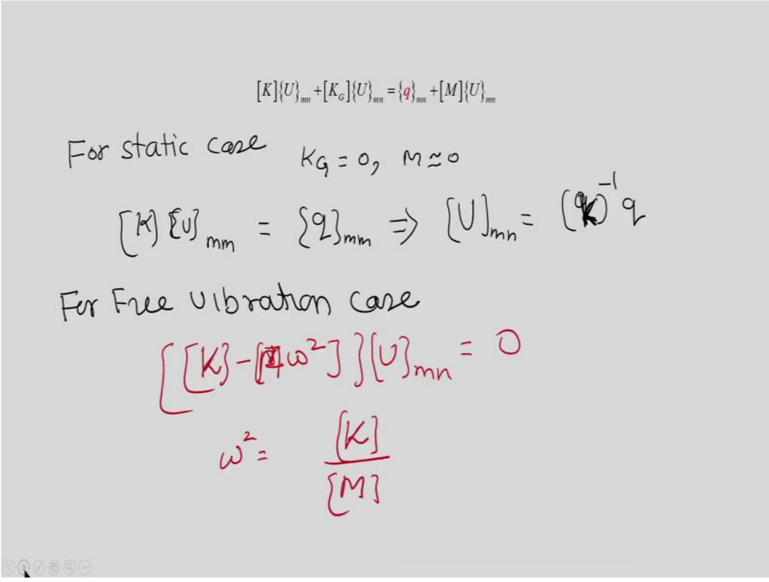

And depending upon the situation if it is the static case, then K_G = 0, and the inertia matrix = 0, and we can solve the deflection of the shell. And for the case of free vibration K_G = 0, q = 0, then it will be an eigenvalue problem.

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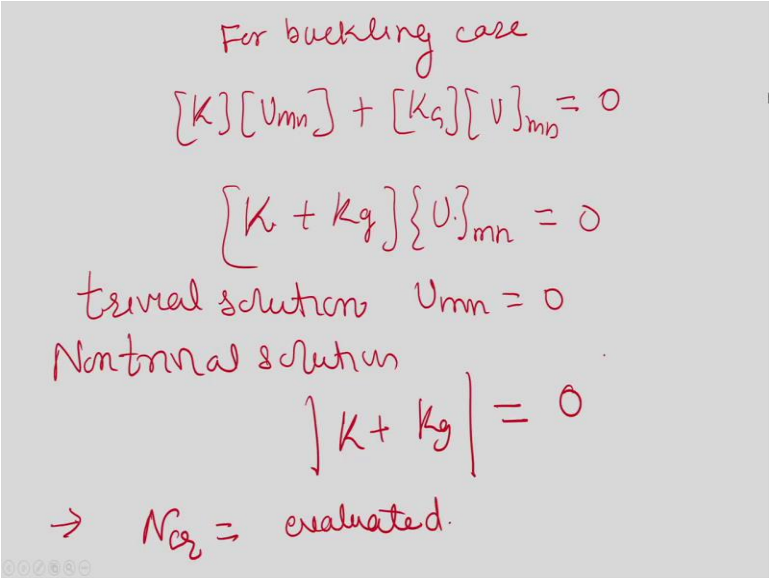

We can find the critical buckling using this.

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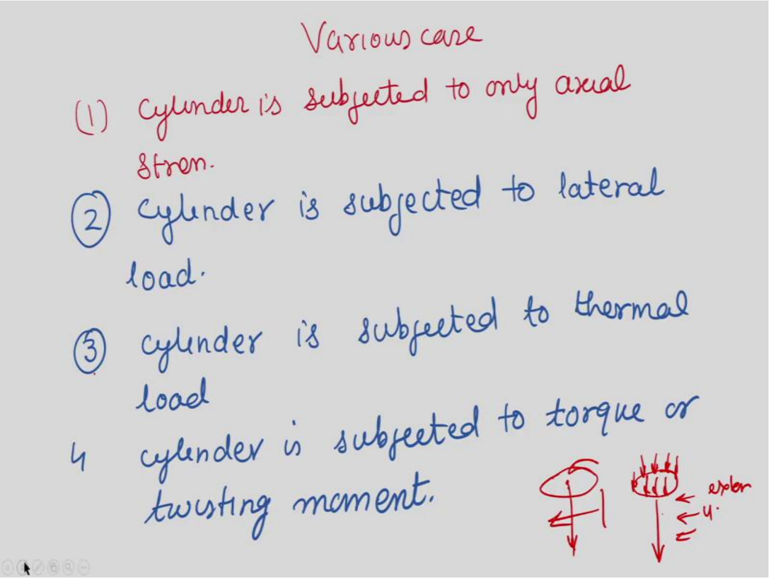

Now, I would like to say that there are various causes for a cylinder:

First: A cylinder is subjected to only axial stress let us say a cylinder is subjected to only compressive axial stress along the longitudinal direction.

Second: A cylinder is subjected to the lateral load under external pressure.

Third: A cylinder is subjected to buckling due to the thermal load. The thermal load may be the temperature difference along axial, temperature difference along the radial, and temperature difference between top and bottom, in this way, we can do a different kind of analysis of buckling.

Fourth: A cylinder is subjected to torque or the twisting moment.

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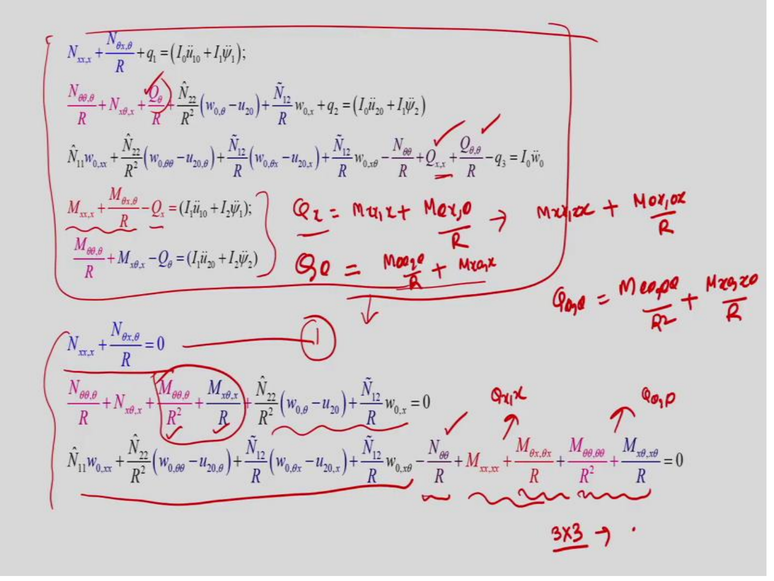

The thermal load may be the temperature difference along axial, temperature difference along the radial, and temperature difference between top and bottom,

in this way, we can do a different kind of analysis of buckling. Fourth: A cylinder is subjected to torque or the twisting moment. Before going into the solution zone, I would like to discuss one more issue that though we have 5 differential equations in most of the literature if you go to general articles, initially they will express these 5 differential equations, and ultimately, they convert those into 3 differential equations.

Because you can see that from here $Q_x = M_{xx,x} + \frac{M_{\theta x,\theta}}{R}$, for a static case because for buckling the inertia terms = 0 and loading terms = 0.

Similarly, $Q_\theta = \frac{M_{\theta\theta,\theta}}{R} + M_{x\theta,x}$

And then it substituted here. If you do so, then the first equation will be:

$$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} + q_1 = 0 \ .$$

In the second equation, instead of Q_θ, substituting this value and dividing by R,

$$Q_{\theta,\theta} = \frac{M_{\theta\theta,\theta\theta}}{R^2} + \frac{M_{x\theta,x\theta}}{R}.$$

The second equation will be:

$$\frac{N_{\theta\theta,\theta}}{R}+N_{x\theta,x}+\frac{M_{\theta\theta,\theta}}{R^2}+\frac{M_{x\theta,\theta}}{R}+\frac{N_{22}}{R^2}\left(w_0,_{\theta}-u_{20}\right)+\frac{N_{12}}{R}\left(w_0,_{x}\right)=0$$

In the third equation here $Q_{\theta,\theta}$ and $Q_{x,x}$, if we do the derivative of $Q_{x,x}$, it becomes $M_{xx,xx}+\frac{M_{\theta x,\theta x}}{R}$ and $Q_{\theta,\theta}$ becomes $\frac{M_{\theta\theta,\theta\theta}}{R^2}+\frac{M_{x\theta,x\theta}}{R}$.

The third equation will be:

$$N_{11}w_0,_{xx}+\frac{N_{22}}{R^2}\left(w_0,_{\theta\theta}-u_{20,\theta}\right)+\frac{N_{12}}{R}\left(w_0,_{\theta x}-u_{20,x}\right)+\frac{N_{12}}{R}w_0,_{x\theta}-\frac{N_{\theta\theta}}{R}+M_{xx,xx}$$
$$+\frac{M_{\theta x,\theta x}}{R}+\frac{M_{\theta\theta,\theta\theta}}{R^2}+\frac{M_{x\theta,x\theta}}{R}=0$$

In this way, we can solve these three equations together. The reason to solve instead of 3 equations instead of 5 equations is if we are interested to develop a closed-form solution, then the solution of a 3 by 3 matrix done by hand is comparatively easy.

Ultimately, we have to find the determinant of that matrix. If we have a 3 by 3 matrix, we can easily handle that. If we have a 5 by 5 matrix, finding the determinant by hand, is difficult. We can do programming in MATLAB or in any programming language i.e., Mathematica, Fortran, or C ++. If you do programming, there will be no problem either you solve 3 equations or 5 equations.

But when we try to find a closed-form solution using the calculator, then working with a 3 by 3 equation is easy. In most of the literature, these 3 equations are solved for the case of buckling.

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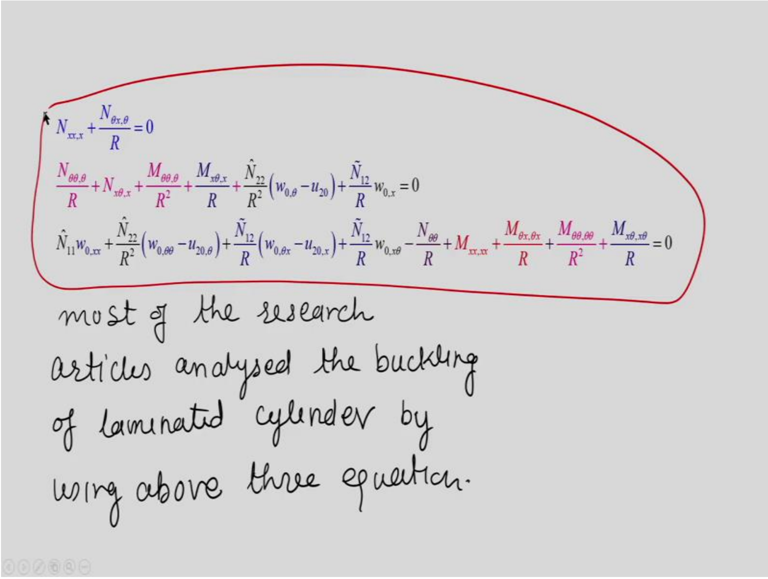

First, convert these equations into a primary variable form.

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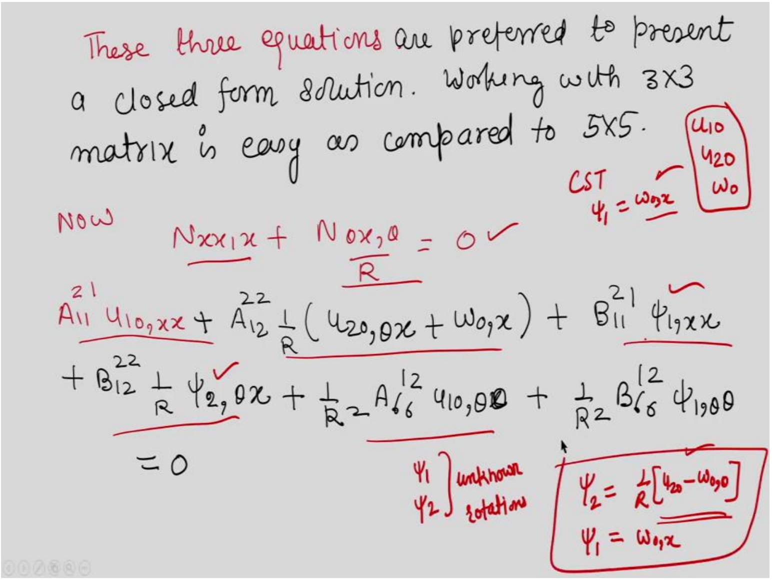

This is the first equation, using the self-constitutive relations, which we have given in the lecture-01 of week-07:

$$A_{11}^{21}u_{10,xx} + A_{12}^{22}\frac{1}{R}\left(u_{20,\theta x} + w_{0,x}\right) + B_{11}^{21}\psi_{1,xx} + B_{12}^{22}\frac{1}{R}\psi_{2,\theta x} + A_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta} + B_{66}^{12}\frac{1}{R^2}\psi_{1,\theta\theta} = 0$$

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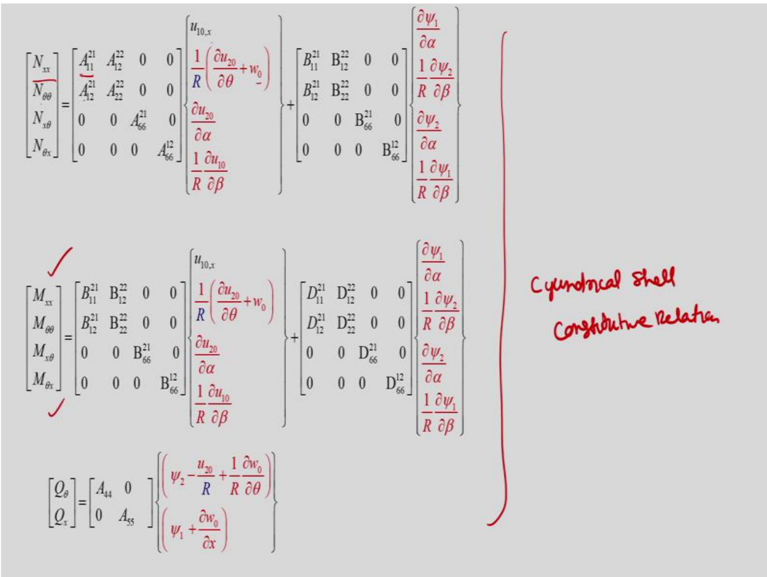

For the sake of completeness, I am also having this:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \begin{bmatrix} A_{11}^{21} & A_{12}^{22} & o & o \\ A_{12}^{21} & A_{22}^{22} & o & o \\ o & o & A_{66}^{21} & o \\ o & o & o & A_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(\frac{\partial u_{20}}{\partial \theta} + w_0\right) \\ \frac{\partial u_{20}}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial u_{10}}{\partial \beta}\right) \end{bmatrix} + \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_1}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial \psi_2}{\partial \beta}\right) \\ \frac{\partial \psi_2}{\partial \alpha} \\ \frac{1}{R}\left\{\frac{\partial \psi_1}{\partial \beta}\right\} \end{bmatrix}$$

$$\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(\frac{\partial u_{20}}{\partial \theta} + w_0\right) \\ \frac{\partial u_{20}}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial u_{10}}{\partial \beta}\right) \end{bmatrix} + \begin{bmatrix} D_{11}^{21} & D_{12}^{22} & o & o \\ D_{12}^{21} & D_{22}^{22} & o & o \\ o & o & D_{66}^{21} & o \\ o & o & o & D_{66}^{12} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_1}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial \psi_2}{\partial \beta}\right) \\ \frac{\partial \psi_2}{\partial \alpha} \\ \frac{1}{R}\left\{\frac{\partial \psi_1}{\partial \beta}\right\} \end{bmatrix}$$

$$\begin{bmatrix} Q_\theta \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & o \\ o & A_{55} \end{bmatrix} \begin{bmatrix} \left(\psi_2 - \dfrac{u_{20}}{R_2} + \dfrac{1}{R} \dfrac{\partial w_0}{\partial \theta} \right) \\ \left(\psi_1 + \dfrac{\partial w_0}{\partial x} \right) \end{bmatrix}$$

So, you can see that $N_{xx} = A_{11}^{21} u_{10,x} + A_{12}^{22} \frac{1}{R}\left(u_{20,\theta x} + w_{0,x}\right)$.

Similarly, $N_{\theta\theta}$, $N_{x\theta}$, *and* $N_{\theta x}$ These shell constitutive relations are specifically for the cylindrical shell.

Similarly, the concept of the moment if we substitute it here then the equation becomes like this. But here you can note that we have three differential equations, the first equation corresponding to u_{10}, second equation corresponding to u_{20}, and third equation w_0. But we have expressed the constitutive relations using the FSDT in terms of ψ_1 and ψ_2. Ultimately, we have to express these ψ_1 and ψ_2 for the case of classical shell theory that ψ_1 reduces to $w_{0,x}$.

And $\psi_2 = \frac{1}{R}\left(u_{20} - w_{0,x}\right)$. If you remember, in week-02, I explained that if it is a classical shell theory, then ψ_1 and ψ_2 will be represented like this. For the case of FSDT or higher-order theories ψ_1 and ψ_2 are unknown rotations. When we finally do that, we have to replace this ψ_1 and ψ_2 with this.

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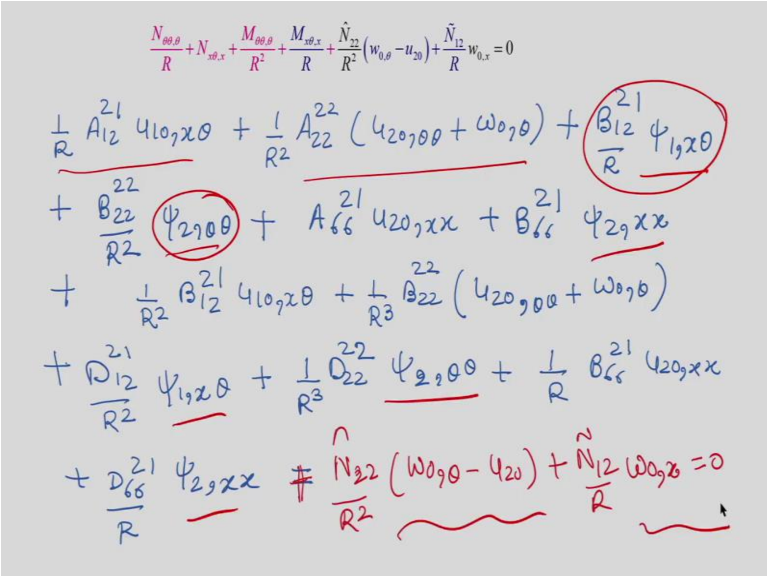

Now, the second equation, by using the shell constitutive relations becomes:

$$\frac{1}{R}\frac{A_{12}^{21}}{R}u_{10,x\theta}+A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{B_{12}^{21}}{R}\psi_{1,x\theta}+B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta}+A_{66}^{21}u_{20,xx}+B_{66}^{21}\psi_{2,xx}$$
$$+\frac{1}{R^2}B_{12}^{21}u_{10,x\theta}+\frac{1}{R^3}B_{22}^{22}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{D_{12}^{21}}{R^2}\psi_{1,x\theta}+\frac{1}{R^3}D_{22}^{22}\psi_{2,\theta\theta}+\frac{1}{R}B_{66}^{21}u_{20,xx}$$
$$+\frac{D_{66}^{21}}{R}\psi_{2,xx}+\frac{N_{22}}{R^2}\left(w_{0,\theta}-u_{20}\right)+\frac{N_{12}}{R}w_{0,x}=0$$

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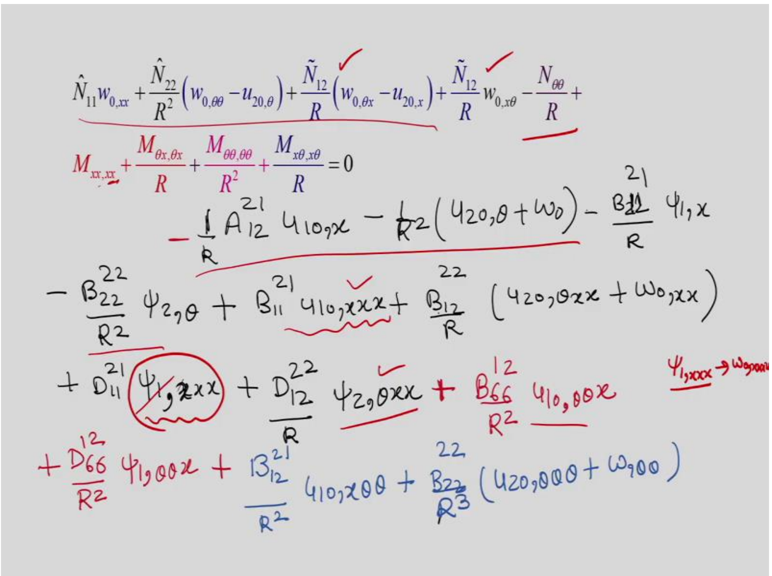

This can be modified later on, I have tried to modify these governing equations and substituting all the values of $M_{x\theta}$ and $M_{\theta x}$.

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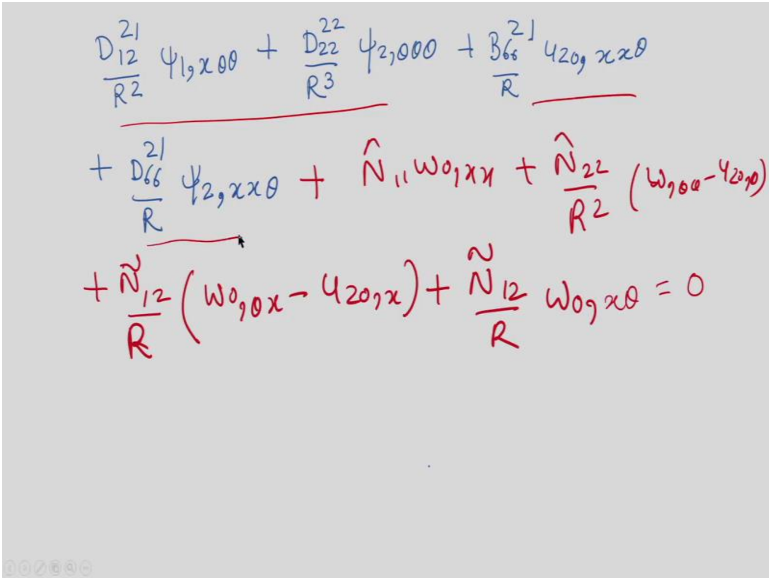

One thing is that we have to do it very carefully. If we make a mistake of a minus sign or plus sign, then there will be a problem. Maybe in the formulation, which I am deriving here, you may find that I missed some R_1 term or sometimes, instead of a R^3, R^2 may come up. We have to be very much careful that there should be no error. A good programmer can only do good programming, we can get the results only when our theoretical formulation is error-free.

Whenever you are going to develop a program, your theoretical formulation should be error-free. If it contains an error, then you will be in a VCs circle i.e., you may have other errors in programming and theoretical formulation.

Therefore, your theoretical formulation should be very clear and error-free, there should be no issue with minus and plus sign or the constitutive relations and other things. If you make a program at least one step should be clear that your formulation is right, then you just have to check the code for any error.

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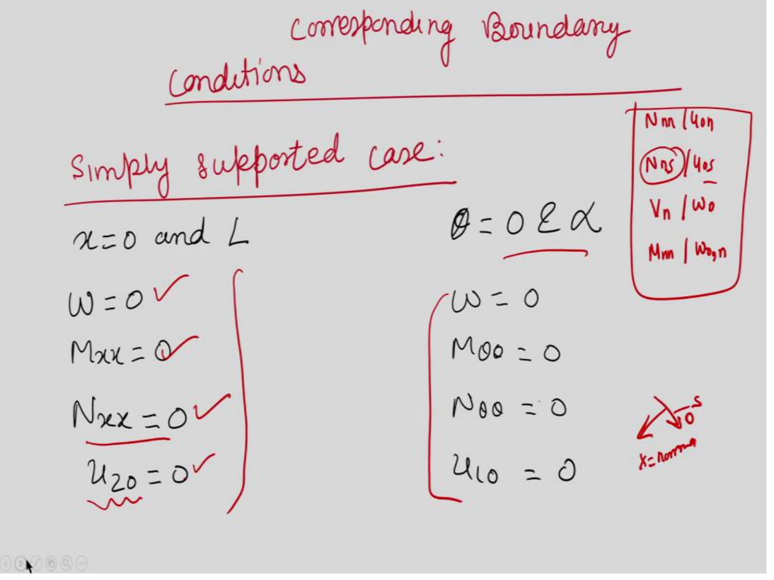

Now we have these three equations and the corresponding boundary conditions. This is our classical shell theory.

The variables we have:

N_{nn} or u_{0n}

N_{ns} or u_{ns}

V_n or w_0

M_{nn} or $w_{0,n}$

Out of these four variables, we need to specify. For the case of a simply supported where x is the normal direction and θ is the shear direction, in that case

$$w = 0;\ M_{xx} = 0;\ N_{xx} = 0;\ u_{20} = 0$$

If we choose this type of support condition, then we can get the exact solution. In the same way along the θ direction the following variables need to be specified:

$w = 0;\ M_{\theta\theta} = 0;\ N_{\theta\theta} = 0;\ u_{10} = 0$.

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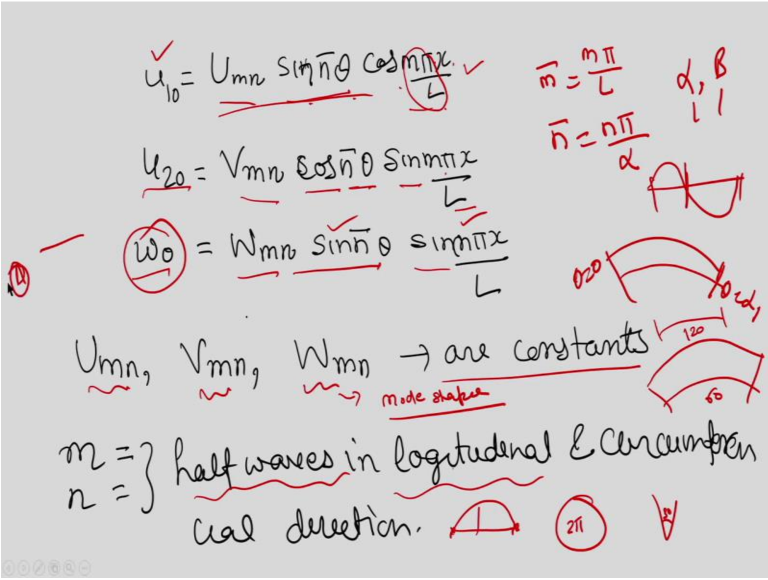

If these are the conditions along x-direction and θ direction, then if we assume the solution like that $u_{10} = U_{mn} \sin \bar{n}\theta \cos \frac{m\pi x}{L}$.

$\frac{m\pi x}{L}$ = $\bar{m}$ and $\bar{n}$ = $\frac{n\pi}{\alpha}$ or $\frac{n\pi}{\theta}$ depending upon the complete angle. Here, we have chosen α as our curvilinear coordinate.

Let us say that your span angle is 60˚ sometimes you want a big panel 120˚ or 30˚. Depending upon the span of a cylindrical shell or maybe a closed shell. If you want to study a closed shell then it will be 360˚, 2 π will be the angle.

Similarly, $u_{20} = V_{mn} \cos \bar{n}\theta \sin \frac{m\pi x}{L}$ and $w_0 = W_{mn} \sin \bar{n}\theta \sin \frac{m\pi x}{L}$.

Transverse deflection = 0 at both the edges along the x-axis and θ axis, it is assumed in both sine series. But in the case of u_{10} and u_{20} one is cos and another is sin. Here U_{mn}, V_{mn}, and W_{mn} are the constants that we need to find.

Generally, we can use to say that this help to find the mode shapes of a buckling, amplitude of that, then m and n will be the half-waves in the longitudinal and circumferential direction, for example, if we talk about 2 π, then it will be like this complete wave, we have chosen up to here.

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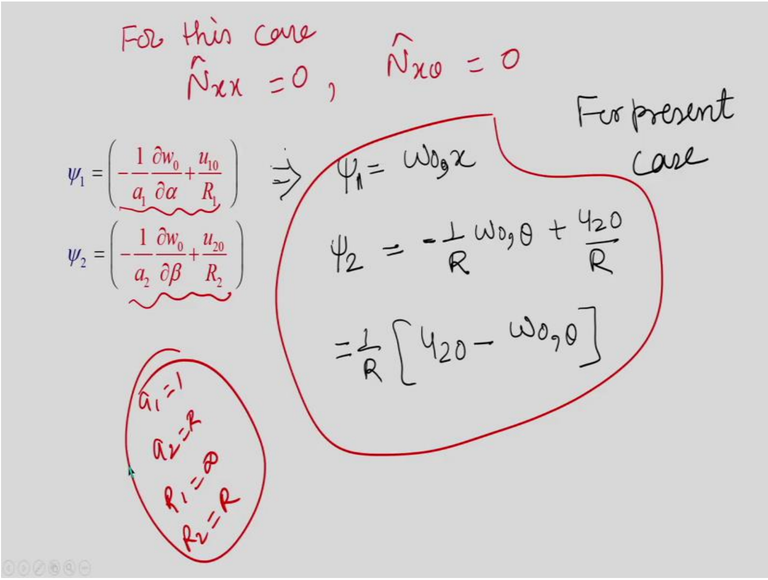

Already, I told you that for the case of a general shell:

$$\psi_1 = \left(-\frac{1}{a_1}\frac{\partial w_0}{\partial \alpha} + \frac{u_{10}}{R_1}\right)$$

ψ_2 is $\left(-\frac{1}{a_2}\frac{\partial w_0}{\partial \beta} + \frac{u_{20}}{R_2}\right)$.

For the present case of cylindrical shell:

$a_1 = 1$, $a_2 = \text{R}$, $R_1 = \infty$, and $R_2 = \text{R}$.

If we substitute all this parameter that it reduces to this:

$\psi_1 = w_{0,x}$ and $\psi_2 = \frac{1}{R}\left(u_{20} - w_{0,x}\right)$.

We have to consider this. Sometimes students may face the problem, they do all the shell constitutive relation, and then realize where is ψ_1 and ψ_2, we have to carefully do that.

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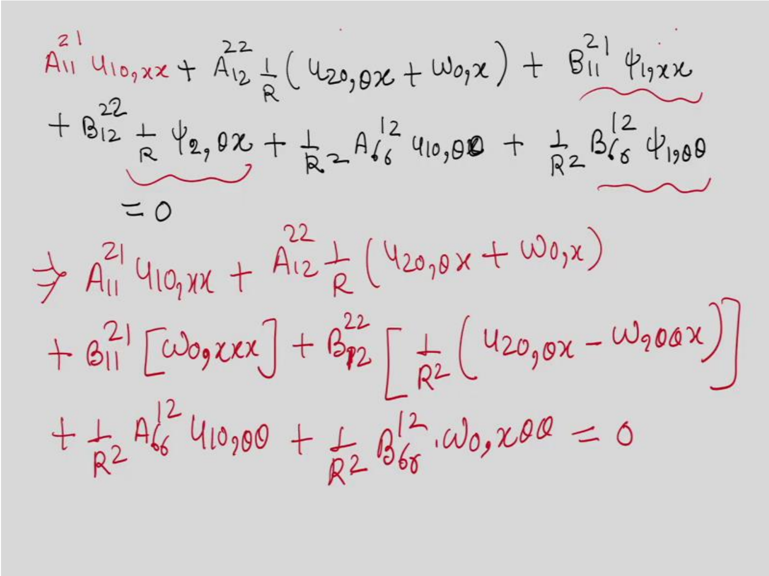

Already, I discussed that these terms need to be further explicitly written like this:

$$A_{11}^{21}u_{10,xx}+A_{12}^{22}\frac{1}{R}\left(u_{20,\theta x}+w_{0,x}\right)+B_{11}^{21}\left(w_{0,xxx}\right)+B_{12}^{22}\frac{1}{R^2}\left(u_{20,\theta x}+w_{0,\theta\theta x}\right)$$
$$+A_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta}+B_{66}^{12}\frac{1}{R^2}w_{0,x\theta\theta}=0$$

We assumed that the cylinder is simply supported in both the directions.

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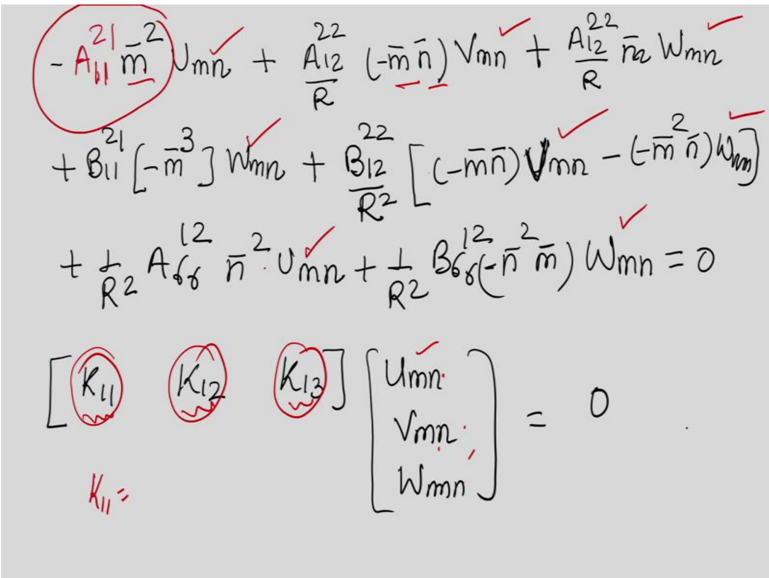

If we substitute this expression: $w_0 = W_{mn} \sin \bar{n}\theta \sin \frac{m\pi x}{L}$ into this equation, it reduces to:

$$-A_{11}^{21}\bar{m}^2 U_{mn} + \frac{A_{12}^{22}}{R}\left(-\bar{m}\bar{n}\right)V_{mn} + \frac{A_{12}^{22}}{R}\bar{m}W_{mn} + B_{11}^{21}\left(-\bar{m}^3\right)W_{mn} + \frac{B_{12}^{22}}{R^2}\left[\left(-\bar{m}\bar{n}\right)V_{mn} - \left(-\bar{m}^2\bar{n}\right)W_{mn}\right]$$
$$+\frac{1}{R^2}A_{66}^{12}\bar{n}^2 U_{mn} + \frac{1}{R^2}B_{66}^{12}\left(-\bar{n}^2\bar{m}\right)W_{mn} = 0$$

Ultimately, if you see this equation carefully, we can write in the form of some coefficients K_{11}, K_{12}, and K_{13}.

We are going to arrange coefficients of U_{mn} at one place, coefficient of V_{mn} at one place, coefficients of W_{mn} at one place.

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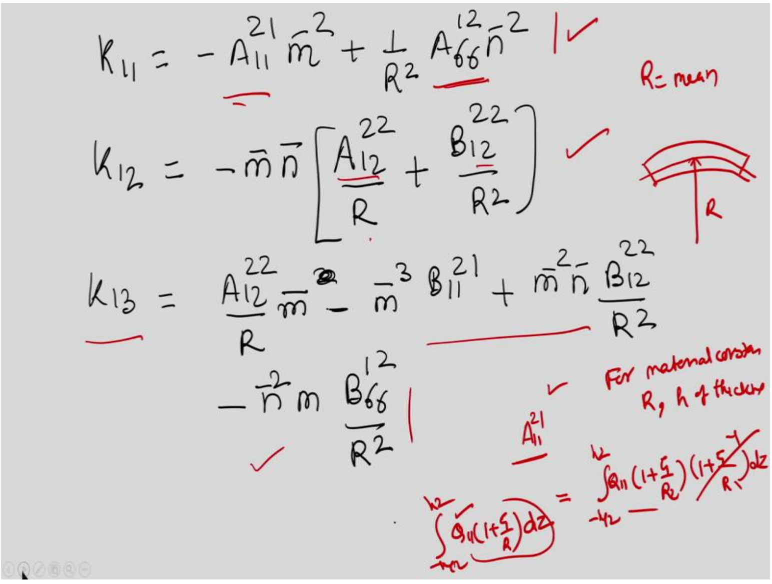

Here, I can say:

$$K_{11} = -A_{11}^{21}\bar{m}^2 + \frac{1}{R^2}A_{66}^{12}\bar{n}^2$$

$$K_{12} = -\bar{m}\bar{n}\left(\frac{A_{12}^{22}}{R} + \frac{B_{12}^{22}}{R^2}\right)$$

$$K_{13} = \frac{A_{12}^{22}}{R}\bar{m} - \bar{m}^3 B_{11}^{21} + \bar{m}^2\bar{n}\frac{B_{12}^{22}}{R^2} - \bar{n}^2 m\frac{B_{66}^{12}}{R^2}.$$

The important point here is that K_{11}, K_{12}, K_{13} all you know, A_{11}^{21} for a given material constant radius and thickness of the shell, you can evaluate.

It is $\int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)d\varsigma$, here $R_1 = 0$,

Therefore, $A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R}\right)d\varsigma$.

If we study a single layer orthotropic cylindrical shell, we substitute the value of Q_{11} and then integrate, it will give the value of A_{11}^{21}.

In the same way, we can evaluate A_{66}^{12}, A_{12}^{22}, B_{12}^{22}, etc., and the mean radius R , if this is the single layer cylinder, the mean radius will be this much because we are assuming that cylinder is very thin, Otherwise, if you talking about a thick cylinder, R plus z will come into that picture.

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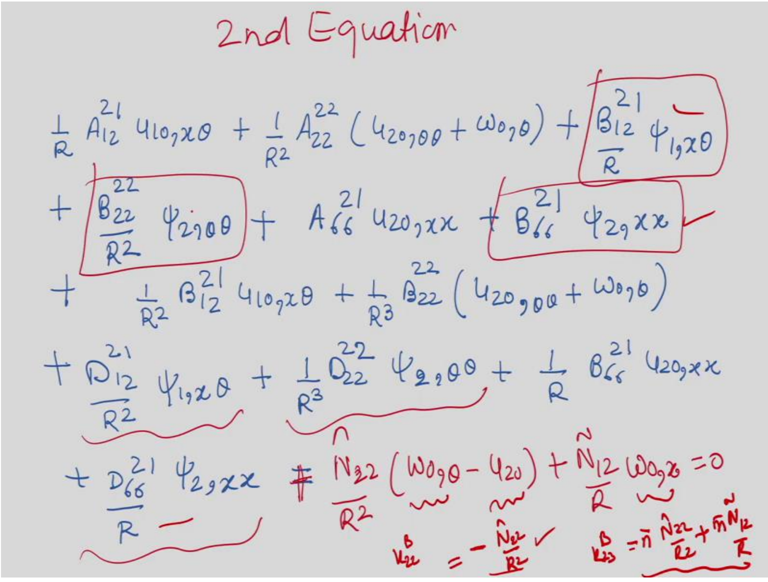

Similarly, we convert the second equation here:

$$\frac{A_{12}^{21}}{R}u_{10,x\theta}+A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{B_{12}^{21}}{R}\psi_{1,x\theta}+B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta}+A_{66}^{21}u_{20,xx}+B_{66}^{21}\psi_{2,xx}$$
$$+\frac{1}{R^2}B_{12}^{21}u_{10,x\theta}+\frac{1}{R^3}B_{22}^{22}\left(u_{20,\theta\theta}+w_{0,\theta}\right)+\frac{D_{12}^{21}}{R^2}\psi_{1,x\theta}+\frac{1}{R^3}D_{22}^{22}\psi_{2,\theta\theta}+\frac{1}{R}B_{66}^{21}u_{20,xx}$$
$$+\frac{D_{66}^{21}}{R}\psi_{2,xx}+\frac{N_{22}}{R^2}\left(w_{0,\theta}-u_{20}\right)+\frac{N_{12}}{R}w_{0,x}=0$$

The terms ψ_1 and ψ_2 need to be written explicitly. And using the expression given in A_{12}^{21}; $u_{10,x\theta}$ and $u_{20,\theta\theta}$ expressed in Fourier series.

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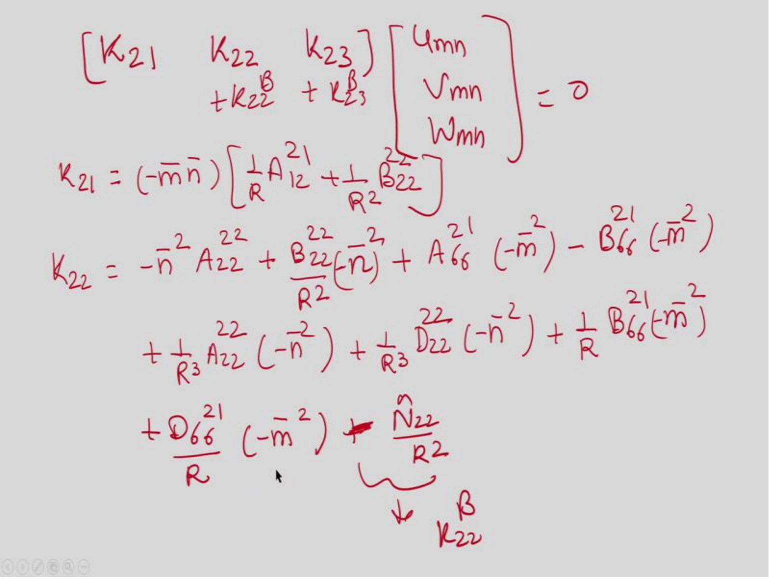

If you substitute all these things, it leads to an equation K_{21} which is the coefficient of U_{mn}, K_{22} coefficient of V_{mn}, and K_{23} is a coefficient of W_{mn}.

$$K_{21}=-\bar{m}\bar{n}\left(\frac{A_{12}^{21}}{R}+\frac{B_{22}^{22}}{R^2}\right)$$

$$K_{22}=-\bar{n}^2A_{22}^{22}+\frac{B_{22}^{22}}{R^2}\left(-\bar{n}^2\right)+A_{66}^{21}\left(-\bar{m}^2\right)-B_{66}^{21}\left(-\bar{m}^2\right)+\frac{1}{R^3}D_{22}^{22}\left(-\bar{n}^2\right)+\frac{1}{R}B_{66}^{21}\left(-\bar{m}^2\right)$$
$$+\frac{D_{66}^{21}}{R}\left(-\bar{m}^2\right)-\frac{\hat{N}_{22}}{R^2}$$

Here, K_{22}^{B} will be corresponding to $-\dfrac{N_{22}}{R^2}$, B means buckling.

$$K_{23}^{B} = \bar{n}\frac{\hat{N}_{22}}{R^2} + \bar{m}\frac{N_{12}}{R}.$$

These two are also contributed in the matrix of K_{22} and K_{23}.

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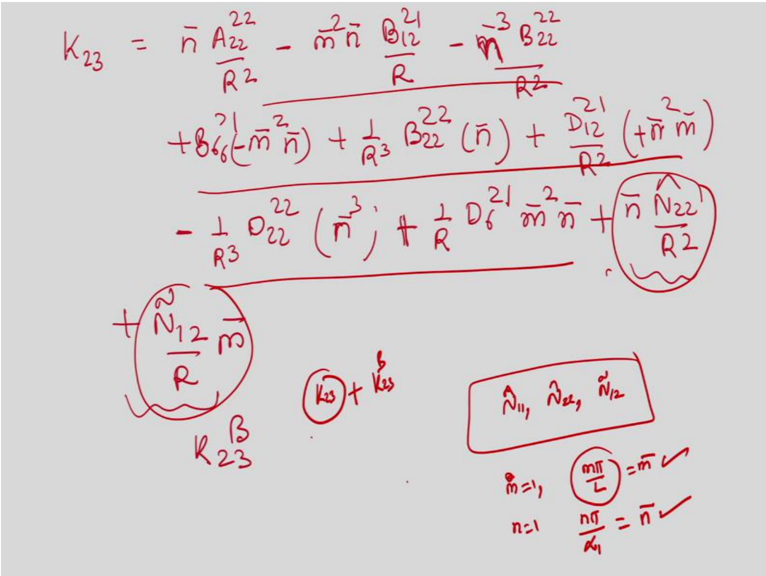

I have given a special name. K_{23} is:

$$\bar{n}\frac{A_{22}^{22}}{R^2} - \bar{m}^2\bar{n}\frac{B_{12}^{21}}{R} - \bar{n}^3\frac{B_{22}^{22}}{R^2} + B_{66}^{21}\left(-\bar{m}^2\bar{n}\right) + \frac{1}{R^3}B_{22}^{22}\left(\bar{n}\right) + \frac{D_{12}^{21}}{R^2}\left(+\bar{n}^2\bar{m}\right) - \frac{1}{R^3}D_{22}^{22}\left(\bar{n}^3\right)$$
$$+\frac{1}{R}D_{66}^{21}\bar{m}^2\bar{n} + \bar{n}\frac{\hat{N}_{22}}{R^2} + \frac{N_{12}}{R}\bar{m}$$

These we can evaluate, it will be $K_{23} + \bar{n}\dfrac{\hat{N}_{22}}{R^2} + \dfrac{N_{12}}{R}\bar{m}$, because we do not know the values of $\hat{N}_{22}$ or N_{12}, we are keeping these as it is.

Other things will give you one number,

if m=1, then $\bar{m} = \dfrac{m\pi}{L}$; if n=1 then $\bar{n} = \dfrac{n\pi}{\alpha_1}$.

$\bar{m}$ and $\bar{n}$ m will be evaluated; they will give you one number. And the value of A_{22}^{22},

B_{12}^{21}, B_{22}^{22}, if you substitute, K_{23} will be just a number plus K_{23}^{B}.

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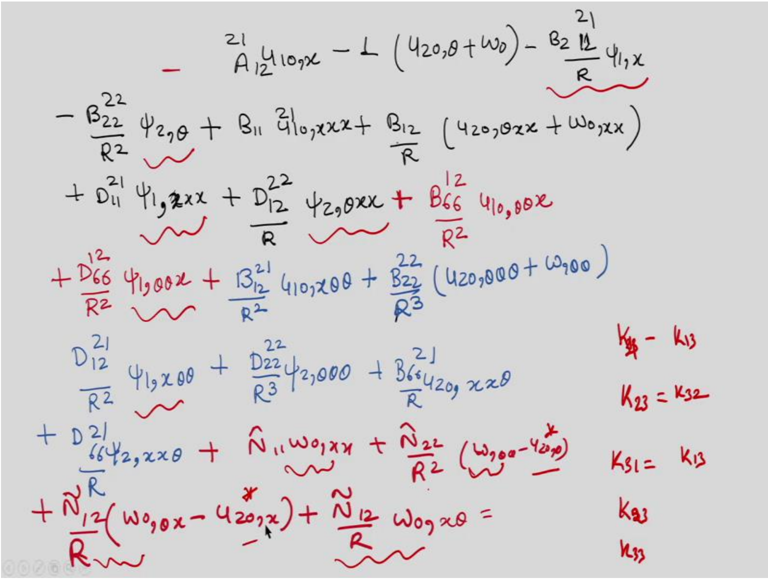

In the same way, if you substitute in this equation 3, we have to write explicitly.

$$-\frac{1}{R}A_{12}^{21}u_{10,x}-A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta}+w_{0,\theta}\right)-\frac{B_{12}^{21}}{R}\psi_{1,x}-B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta}+B_{11}^{21}u_{10,xxx}+\frac{B_{12}^{22}}{R}\left(u_{20,\theta xx}+w_{0,xx}\right)$$
$$+D_{11}^{21}\psi_{1,xxx}+\frac{D_{12}^{22}}{R}\psi_{2,\theta xx}+\frac{B_{66}^{12}}{R^2}u_{10,\theta\theta x}+\frac{D_{66}^{12}}{R^2}\psi_{1,\theta\theta x}+\frac{B_{12}^{21}}{R^2}u_{10,x\theta\theta}+\frac{B_{22}^{22}}{R^3}\left(u_{20,\theta\theta\theta}+w_{0,\theta\theta}\right)=0$$
$$\frac{D_{12}^{21}}{R^2}\psi_{1,x\theta\theta}+\frac{D_{22}^{22}}{R^3}\psi_{2,\theta\theta\theta}+\frac{B_{66}^{21}}{R^2}u_{20,xx\theta}+\frac{D_{66}^{21}}{R}\psi_{2,xx\theta}+\hat{N}_{11}w_{0,xx}+\frac{\hat{N}_{22}}{R^2}\left(w_{0,\theta\theta}+u_{20,\theta}\right)$$
$$+\frac{\hat{N}_{22}}{R}\left(w_{0,\theta x}+u_{20,x}\right)+\frac{\tilde{N}_{12}}{R}w_{0,x\theta}=0$$

If you, do it carefully, you will find that $K_{31} = K_{13}$, $K_{23} = K_{32}$. We can see that $\frac{\hat{N}_{22}}{R^2}$ and $\frac{\tilde{N}_{12}}{R}$, some terms may be same, we have to be very much careful. $K_{31} = K_{13}$, but some terms may not be there. K_{33} is now going to have

$$\hat{N}_{11}w_{0,xx}+\frac{\hat{N}_{22}}{R^2}\left(w_{0,\theta\theta}+u_{20,\theta}\right)+\frac{\hat{N}_{22}}{R}\left(w_{0,\theta x}+u_{20,x}\right)+\frac{\tilde{N}_{12}}{R}w_{0,x\theta}.$$

And K_{32} will have these two-term $w_{0,\theta x}+u_{20,x}$.

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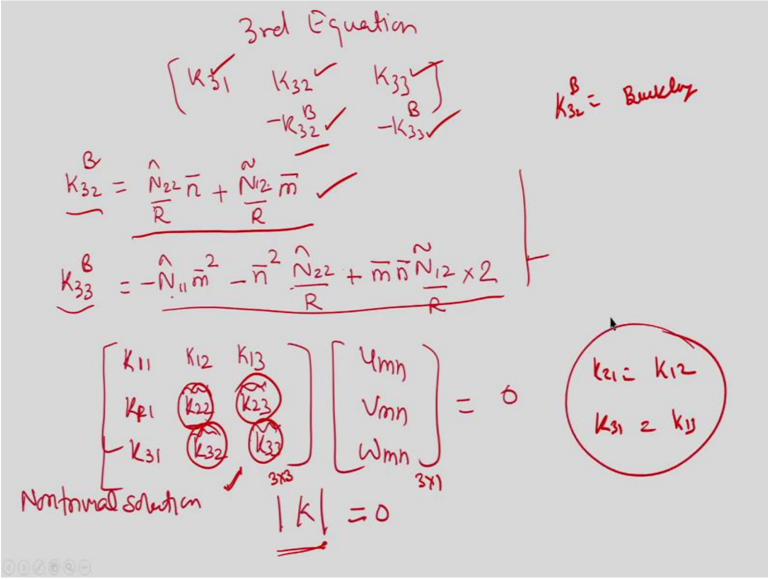

Ultimately, we can write the third equation $\left[K_{31} \;\; K_{32} \;\; K_{31} - K_{32}^{B} - K_{33}^{B}\right]$.

Here, K_{32}^{B} means stiffness corresponding due to buckling $\bar{n}\dfrac{\hat{N}_{22}}{R^2} + \dfrac{\tilde{N}_{12}}{R}\bar{m}$, and

$$K_{33}^{B} = -\hat{N}_{11}\bar{m}^2 - \bar{n}^2\frac{\hat{N}_{22}}{R} + \bar{m}\bar{n}\frac{\tilde{N}_{12}}{R}x\,2.$$

If we make a matrix and club all together:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = 0$$

It is a 3 by 3 matrix and U_{mn} = 3 by 1 matrix. For a solution of buckling, we say that for a non-trivial solution the determinant of the system $|K|$ = 0.

Herein K_{22}, K_{23}, K_{33} these $\hat{N}_{11}$, $\hat{N}_{22}$, $\tilde{N}_{12}$ are unknowns. The determinant of that matrix is written like this, $K_{21} = K_{12}$, and $K_{31} = K_{13}$.

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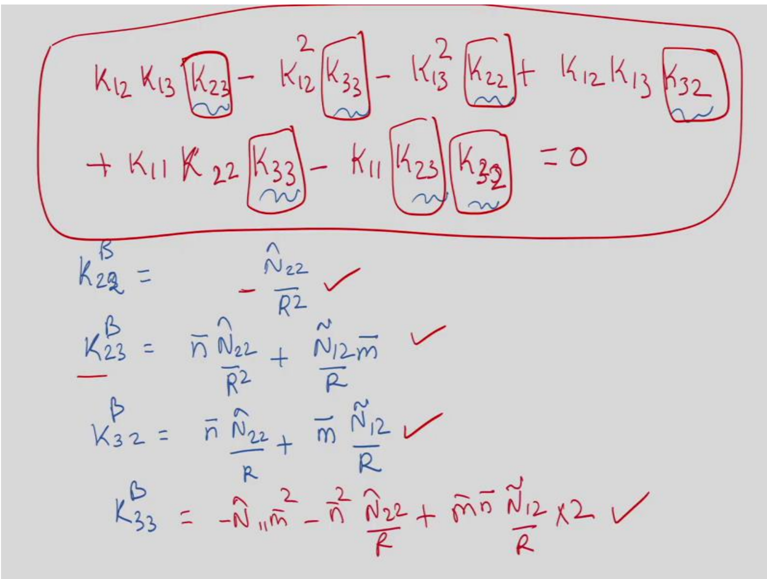

If we use those things, then the determinant can be written as:

$$K_{12}K_{13}K_{23} - K_{12}^2 K_{33} - K_{13}^2 K_{33} + K_{12}K_{13}K_{32} + K_{11}K_{22}K_{33} + K_{11}K_{23}K_{32} = 0$$

This is the closed-form solution six terms are there.

These terms will have some buckling parameters.

$$K_{22}^B \;=\; -\frac{\hat{N}_{22}}{R^2}$$

$$K_{23}^B \;=\; \bar{n}\,\frac{\hat{N}_{22}}{R^2} + \frac{\tilde{N}_{12}}{R}\,\bar{m}$$

$$K_{32}^B \;=\; \bar{n}\,\frac{\hat{N}_{22}}{R^2} + \frac{\tilde{N}_{12}}{R}\,\bar{m}$$

$$K_{33}^B \;=\; -\hat{N}_{11}\bar{m}^2 - \bar{n}^2\,\frac{\hat{N}_{22}}{R} + \bar{m}\,\bar{n}\,\frac{\tilde{N}_{12}}{R}\,x\,2\,.$$

If we have these parameters, then depending upon the situation, if the cylindrical shell is subjected to axial loading, twisting loading, or the external pressure, some of the terms will be 0 and some will contribute. Accordingly, some terms will be there and we can solve this matrix. I will explain for a particular case that if a cylinder is subjected to axial loading what will be the solution.

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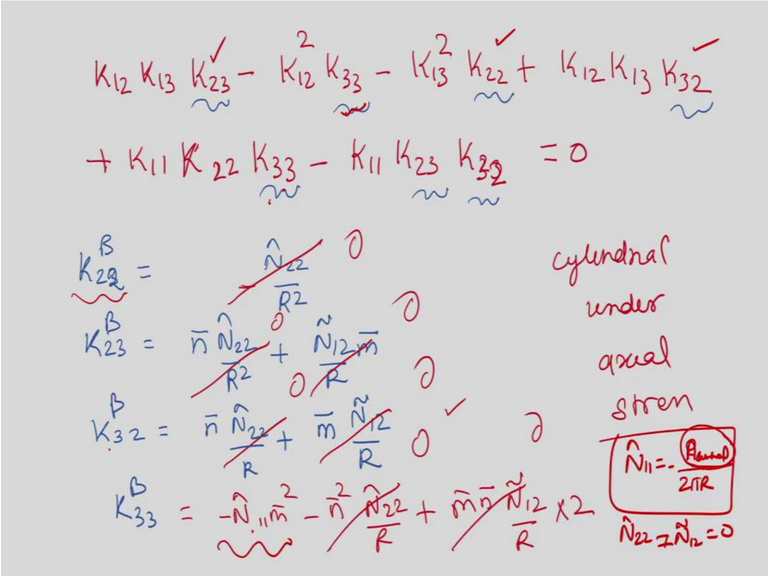

Here, one can see that for the case of a cylinder, there are several cases, I will discuss it later. But first, I will say that cylinder is subjected to axial stress, for that case

$$\hat{N}_{11} = \frac{-A}{2\pi R} .$$

If this is the load and $\hat{N}_{22} + \tilde{N}_{12} = 0$.

If you substitute that K_{22}^B, K_{23}^B, $K_{32}^B = 0$ and $K_{33}^B = -\hat{N}_{11}\bar{m}^2$.

Where, $\hat{N}_{11} = \frac{-A}{2\pi R}$, here, A is the axial load. We are interested to find the critical axial load under which this laminated cylinder is going to buckle. Here, you can see that in K_{33}^B there is no change, here no change only these two terms are going to change.

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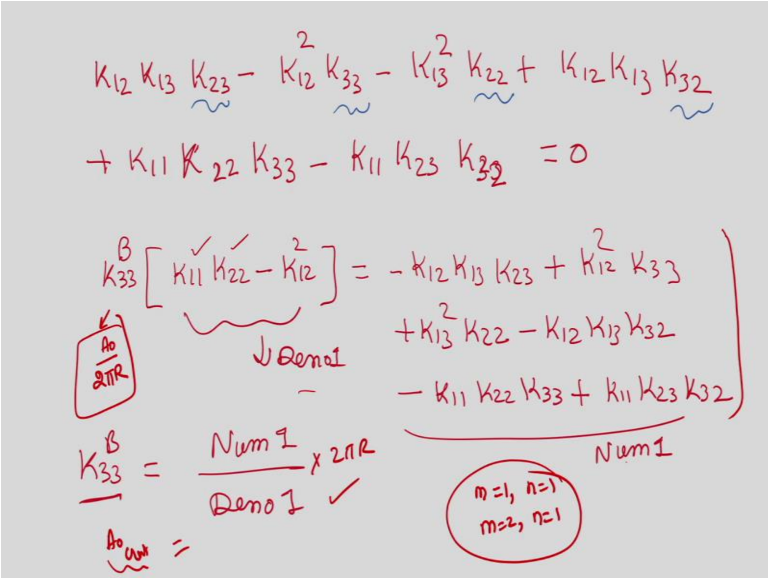

If we substitute the value, ultimately, K_{33}^{B} will have these coefficients $K_{11}\ \ K_{22}\ -K_{12}^{2}$. And other terms you can keep on the right-hand side:

$$\underbrace{K_{33}^{B}\left[K_{11}\ \ K_{22}\ -K_{12}^{2}\right]}_{Denominator}=\underbrace{K_{12}K_{13}K_{23}+K_{12}^{2}K_{33}+K_{13}^{2}K_{33}-K_{12}K_{13}K_{32}-K_{11}K_{22}K_{33}+K_{11}K_{23}K_{32}}_{Numerator}.$$

$K_{33}^{B} = \dfrac{A_{0critical}}{2\pi R}$, this is for a complete cylinder.

$K_{12}K_{13}K_{23}+K_{12}^{2}K_{33}+K_{13}^{2}K_{33}-K_{12}K_{13}K_{32}-K_{11}K_{22}K_{33}+K_{11}K_{23}K_{32}$ is numerator, and $K_{33}^{B}\left[K_{11}\ \ K_{22}\ -K_{12}^{2}\right]$ is denominator. If you multiply this with $2\pi R$.

It gives you the $A_{0critical}$. Based on this, we are interested to find m and n, axial buckling load can be found by choosing different values, let us say, you put m = 1 and n = 1 then, evaluate this term. Then put m = 2 and n = 1 and evaluate. Different combinations for which it is coming minimum will be the critical buckling load. We have to find for different m and n.

In this way, you can say that it is a 3 by 3 matrix, even a class-12 student can compute with the help of a calculator or just by pen and paper. This is the numerator term divided by the denominator. You can calculate all these things with the help of a simple calculator or you can do the programming. That is why a 3 by 3 matrix is preferred mostly in this closed-form solutions.

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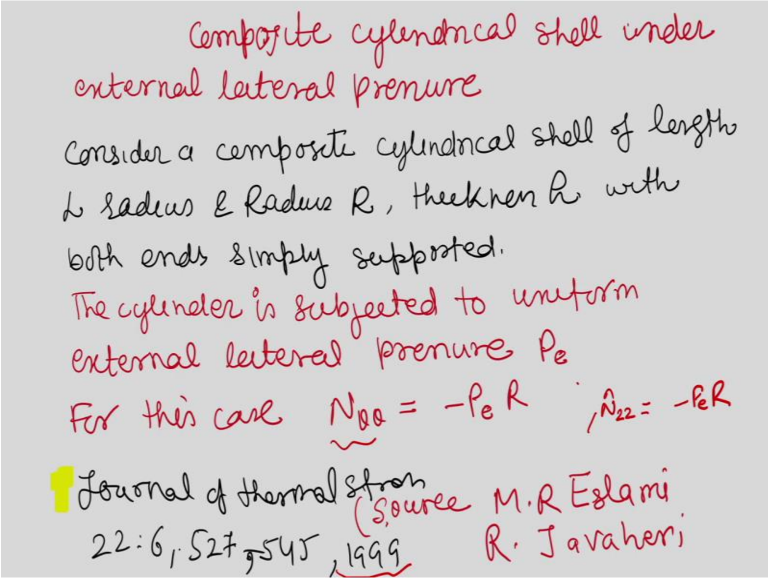

First, I would like to say that a cylinder may be subjected to a variety of load cases. The very first case is subjected to a uniform external lateral pressure which is p_e. For that case, $\hat{N}_{22} = -p_e R$. This is a journal article by Prof. M.R Eslami and R. Javaheri the work was published in the journal of Thermal Stress in 1999. They have done the buckling of a circular composite cylinder under mechanical and thermal loading.

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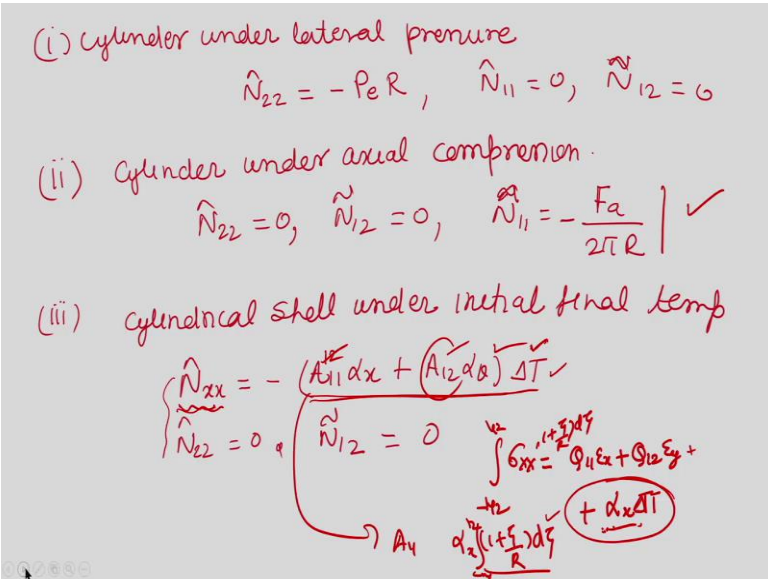

From that work, I am just presenting some of that. They have studied cylindrical shells under lateral pressure. For that case, $\hat{N}_{22} = 2\pi R$, and $\hat{N}_{11} = 0$ and $\tilde{N}_{12} = 0$. If you substitute all this thing into that equation, and then again you can find $\hat{N}_{22}$. In the same way, the cylinder is under axial compression which I have already discussed this term $\tilde{N}_{11} = -\dfrac{Fa}{2\pi R}$.

And the third case is the cylindrical shell under initial final temperature, then $\hat{N}_{xx} = -\left(A_{11}\alpha_x + A_{12}\alpha_\theta\right)\Delta T$ like this.

If we want to write, $\sigma_{xx} = Q_{11}\varepsilon_x + Q_{12}\varepsilon_y + \alpha_x \Delta T$. If you do integration, it will be

$$\int_{-h/2}^{h/2} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) d\varsigma .$$

Ultimately, the definition of A_{11} through thermal = $\displaystyle\int_{-h/2}^{h/2} \alpha_x \left(1+\frac{\varsigma}{R}\right) d\varsigma$.

This $\alpha_x \Delta T$ is the contribution of thermal load.

$\hat{N}_{xx} = -\left(A_{11}^{12}\alpha_x + A_{12}\alpha_\theta\right)\Delta T$, from here ΔT can find out. This will be the denominator.

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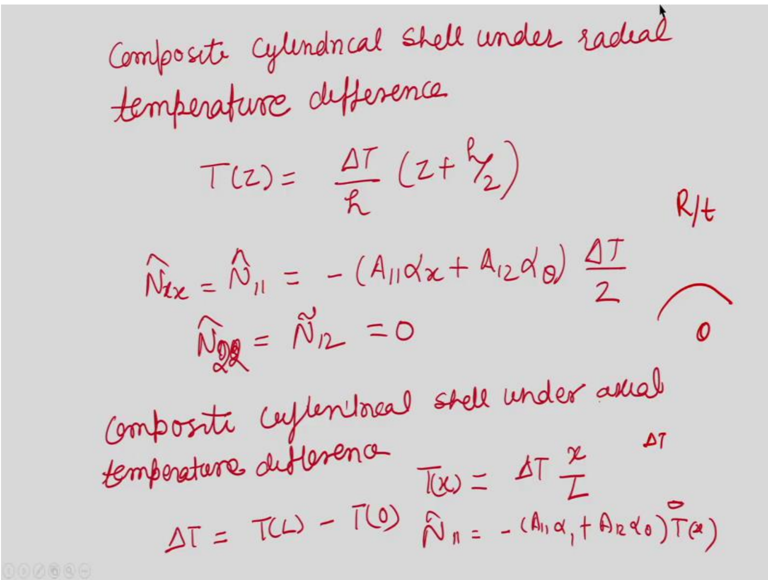

Similarly, some other works were done that the composite cylinder shell under radial temperature difference because in the thickness direction change in temperature.

$$T(z) = \frac{\Delta T}{h}\left(z + \frac{h}{2}\right) .$$

The temperature is varying along the thickness by this formula:

$\hat{N}_{xx} = \hat{N}_{11} = -\left(A_{11}\alpha_x + A_{12}\alpha_\theta\right)\frac{\Delta T}{2}$, and ΔT is the temperature difference.

From here, $\hat{N}_{11}$ can be found. In this case, $\hat{N}_{22}$ and $\tilde{N}_{12}$ = 0. The cylindrical shell under axial temperature difference along the x length $T(x) = \Delta T \frac{x}{L}$.

Substituting it into $\hat{N}_{11}$, we can find ΔT for that. In most cases, this equation:

$K_{12}K_{13}K_{23} + K_{12}^2 K_{33} + K_{13}^2 K_{33} - K_{12}K_{13}K_{32} - K_{11}K_{22}K_{33} + K_{11}K_{23}K_{32} = 0$ is the main characteristic equation, and depending upon the situation some terms are non-zero, and we can do it by our hand. But in actual practice, one can write a program and solve that. For different m and n, different radius to thickness, or for different ϕ angle we can find the critical buckling load for the cylinder under axial load or a cylinder under external pressure. In this way, we can study the buckling of a cylindrical shell.

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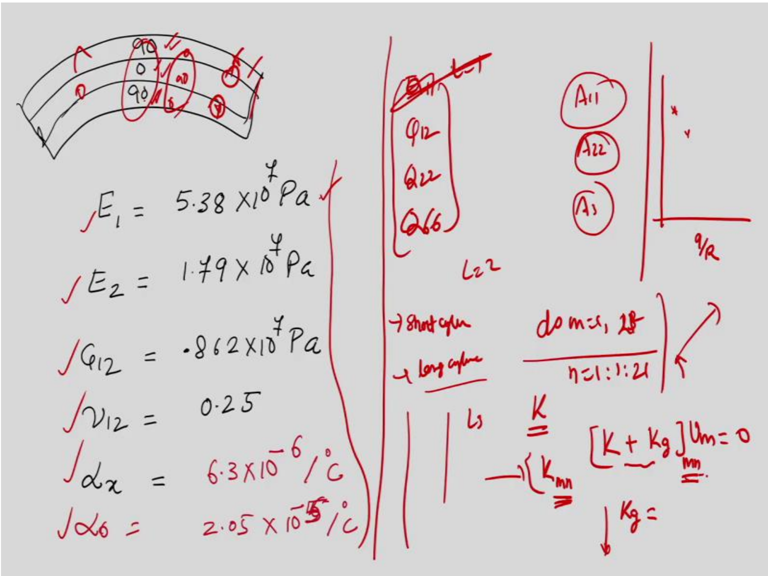

For the case of calculation, this problem is solved in most of the literature, a three-layer cylindrical shell in which a 90, 0, 90 thickness. Here,

$$E_1 = 5.38X10^7 Pa$$
$$E_2 = 5.38X10^7 Pa$$
$$G_{12} = 0.862X10^7 Pa$$
$$\mu_{12} = 0.25$$
$$\alpha_x = 6.3X10^{-6} / °C$$
$$\alpha_0 = 2.05X10^{-5} / °C$$

Represented in terms of pascal.

Let us say for the case of 0 E_1 is same, then you need to find Q_{11}, Q_{12}, Q_{22}, and Q_{66}, And then layer-wise, let us say

for layer 1: L = 1

for layer 2: L = 2

for layer 3: L = 3,

And then substitute the value of A_{11}, A_{12}, A_{23}, and so on. Already, I gave you a simple basic idea that how to write a program in MATLAB and evaluate these things.

Once you evaluate of $A_1, A_2, B_1, B_2, D_1, and\ D_2$ then we can evaluate a matrix K and we can write a loop that m = 1, 21, n = $1:3:21$, any number we can write a and we can evaluate K_{mn} for a particular combination of this. This matrix $\left[K + K_g\right]_{mn} U_{mn}$ will also change with respect to m and n.

For a particular m and n combination, we will get the value of buckling parameter, and sometimes, we may apply the optimization, but for simple cases, where m = 1 and n = 1; for the case of a plate, when it is subjected to a biaxial loading then m = 1, n = 1 may give you the lowest critical buckling load, but for the shell, it is not true.

Generally, it has been found that for the case of a higher m and n, there is a lower buckling load. In the next lecture, I will solve a problem by taking this, how do we evaluate A_{11}, A_{12}, A_{13}, and actual finding of buckling parameter.

It has been found that different $\frac{a}{R}$ ratio (length to radius ratio), we can say that there are two types of cylinders, one is a short cylinder, another is a long cylinder. When there is a short cylinder, the buckling load may be high.

As we increase the cylinder length, the buckling parameter may increase or may decrease depending upon the situation. Further, the influence of the stacking sequence sometimes you say 90, 0, 90, or 0, 90, 0. Even we can study the influence of the angle. For that case, I have given enough background that how to evaluate a buckling of a cylindrical shell subjected to an axial load, external pressure, or thermal loading. One can write a program and get the results.

Later on, what is the current state of research? These days the research is going on the buckling of a functionally graded, functionally graded cylindrical shell may have some imperfections like there is an effect of delamination, holes, or there may be some boundary conditions or there may be edges.

The analytical solution is possible for a cross-ply cylindrical shell with simply supported boundary conditions, but in a real application, this is not possible that always we have a simply supported boundary condition. If you are interested to solve a different kind of boundary condition, then we have to go for numerical solutions.

In the next lecture, I will also briefly explain that buckling of a Levy type cylindrical shell i.e., opposite edges are simply supported, and then cylindrical shell panel or a cylinder is subjected to the axial stress, then how do we get the solution. The solution procedure is almost similar to the case of bending, but there will be no bending for the case of a Levy-type vibration.

Similarly, we have to initially guess the buckling parameter and we can solve for that. In the literature, a lot of work is related to buckling of shells, buckling of different kinds of shells like turtle shells, conical shells, cylindrical shells, the shells having carbon nanotubes, the shells having graphene sheet, and the functionally graded shells having the piezoelectric patch.

Recently, I have found that one work is there that is the laminated cylindrical shell and a piezoelectric patch is over there. With the help of a piezoelectric patch, it is affecting the critical buckling load of that cylinder if you apply a piezoelectric patch.

Depending upon requirements. If it is high, it will be safe. We aim to find the first critical buckling load, but from a design point of view, we find at least three consecutive buckling loads and mode shapes. Mode shapes are very much important for the case of cylindrical shells because there may be a magnitude but the way it is deflected or its deflection comes up matters much.

With this, I end this lecture.

Thank you very much.