Theory of Composite Shells

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Week - 07

Lecture - 03

Buckling of shell

Dear learners welcome to the course of Theory of Composite Shells, Week- 07, Lecture-03. In this lecture, I will explain the remaining portion of the Buckling of Shells specifically developing a buckling solution of a Levy-supported finite shell panel.

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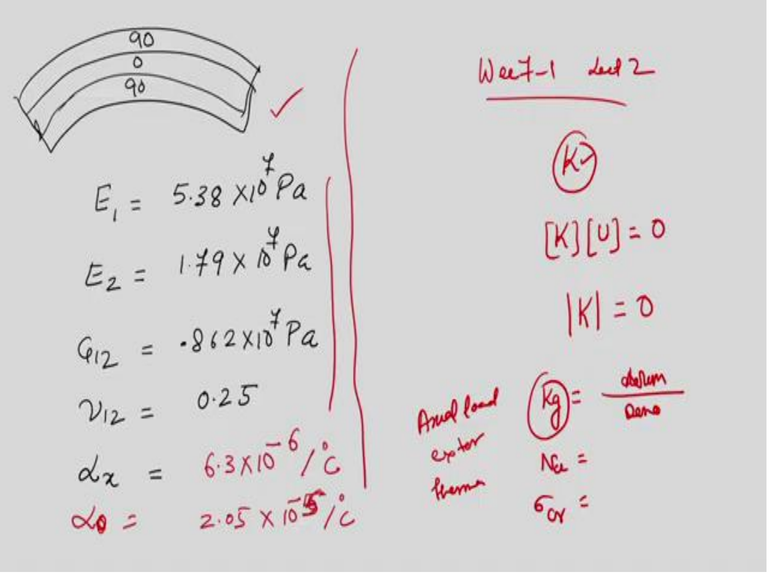

In week-07, lecture-02, I asked can we get the buckling parameters or can we get a small program for that buckling of a composite cylinder? I have taken a problem; it is given in an Islamic paper, in that a 3 - layer cylindrical composite shell with different cases is considered, i.e., it may be subjected to the axial load, it may be subjected to the external pressure or it may be subjected to the thermal temperature.

The property parameters are written here:

$$E_1 = 5.38X10^7 Pa$$
$$E_2 = 5.38X10^7 Pa$$
$$G_{12} = 0.862X10^7 Pa$$
$$\mu_{12} = 0.25$$
$$\alpha_x = 6.3X10^{-6} / °C$$
$$\alpha_0 = 2.05X10^{-5} / °C$$

If you are interested to develop a small program, you can write even in excel also but I have written this code in MATLAB just for your clarity. Most of the time students, post-graduate students, or Ph.D. students, when we talk about the theoretical formulation, they say that they can develop the theoretical formulation. But when we ask them regarding developing a program then they hesitate or do not have confidence.

With this example, I would like to say that with the very basic steps you can develop codes for the shell panels or cylindrical shells. First, you must make a theoretical formulation, either you type it or you make it in your copy and that should be error-free.

There should be no error, whether you talk about sign cases or loading. Our aim is to develop a $[K]$. If you remember that $[K][U]$ = 0 and then $|K|$= 0 and from determinant K_g comes or you can say $N_{critical}$ or $\sigma_{critical}$. Some factors will be on the numerator side and some will be on the denominator side as I explained in the lecture- 02 of the week-07.

Now, I shall explain step by step the development of a 3- layer program. Doing a single layer may be easy, but when we go for a 3- layer case, then we need to write a small program in any language. The way you develop a theoretical formulation in your copy is just a small tinkering you can develop in your MATLAB code.

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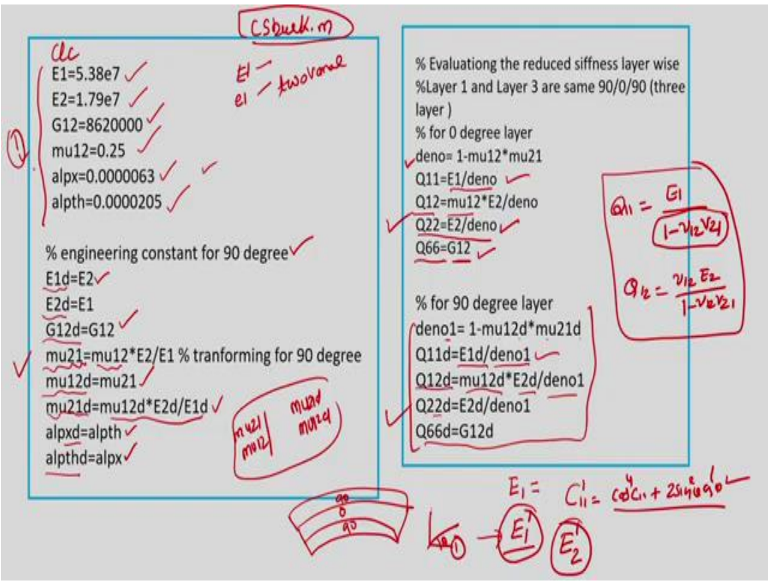

First, give the parameters in a MATLAB file; let us say, a cylindrical shell buck.m, we put the file name like this. And then we type clc to clear screen. Then assign:

E1=5.38e7

E2=1.79e7

G12=8620000

mu12=0.25

alpx=0.0000063

alpth=0.0000205.

We have to be very careful in MATLAB, it takes caps i.e., it considers e1 and E1 as two different variables.

Therefore, we have to be very much careful while naming the variables. Then we convert them because we require formulations for 0° as well as for 90° material properties. 0° material properties are given here, then we are interested to find for the 90° material property. We can say that if our one axis is making an angle θ, when we rotate it to 90°, it means 2 becomes 1.

E1 = E1d = $\bar{E}_1$, it transformed.

$\bar{E}_1$ = E2, for cross-ply, it is very easy but, difficult for an angle ply. In a week- 05, I gave the transformation matrix to develop the transformation of Young's modulus, stiffness matrix, or a compliance matrix.

If you remember, $C_{11}^{1} = \cos^4 C_{11} + 2\sin^2\theta\cos^2\theta$.

Similarly, we can find the engineering constants. But, for our case it is a 2- dimensional problem: we can easily find:

E1d = E2,

E2d = E1

Which means Young's modulus in the 2nd direction for 90° and I have written a comment % engineering constant for 90°.

When we write a program it is better to write a comment.

G12d = G12

mu21=mu12*E2/E1 % transforming for 90 degree

m12d is mu21

mu21d=mu12d*E2d/E1d

alpxd=alpth

alpthd=alpx.

In this way, we have evaluated that E1 is named for material properties along the 0° direction and d is used for the material properties along 90° direction. Then, we have to evaluate the Q11, Q22, Q12, and Q66.

$$Q_{11} = \frac{E_1}{1-\mu_{12}\mu_{21}}.$$

We call $1-\mu_{12}\mu_{21}$ is deno because it is same for all the cases. deno=1-mu12*mu21, therefore, we can say:

Q11=E1/deno

Q12=mu12*E2/deno.

These formulas were already explained in the week -01 lecture, or you can check any mechanics of composite book where the reduced stiffness is given or the book on theory of plate and shell.

Q22=E2/deno

Q66=G12.

Similarly, we have to create the reduced stiffness for the 90˚ layer, in that case, it is known as deno1.

deno1 = 1- mu12d*mu21d

Q11d= E1d/deno1

Q12 = mu12d* E2d /deno1.

This is a very simple thing; one can write either in MATLAB or in excel or one can type also. For a given property, you can evaluate this. The very first step is to transform the property and calculate the reduced stiffness.

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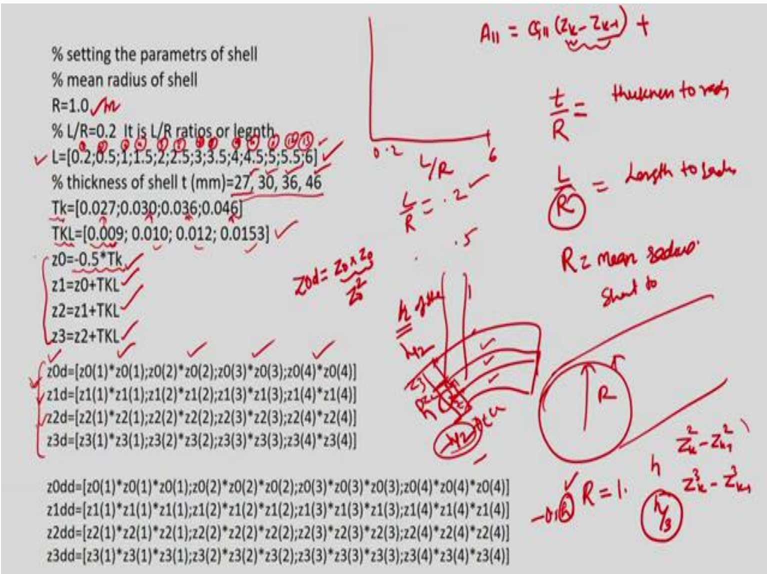

In the next step, we have to set the material parameters and radius to thickness. There are two parameters one is $\frac{t}{R}$ and another is $\frac{L}{R}$. A shell is defined basically in thickness to radius ratio and in length to radius ratio. R is the mean radius. Let us say, for any given R, we can formulate for a thickness that is one way.

Another way in a parametric form let us say that R = 1, we can say L/ R maybe 0.2 a very short cylinder and 0.5, 1, 2, 2.5, and up to 6. In this way, L can vary. Because R = 1, therefore, L is changing. Similarly, we can also vary the thickness of the shell. We have taken here 4 thicknesses of the shell 27mm, 30mm, 36mm, and 46mm.

Let us say, R = 1meter then, this thickness = 1 mm and the length to radius ratio is like this. We can define a column vector, Tk the thickness of the shell in meter. The total thickness where we say the h of the shell if it is a 3- layer, this is h. It goes from $\frac{-h}{2}$ to $\frac{h}{2}$. In each layer, our thickness is the same.

In most of the studies, the thickness of each layer is considered the same. If h is known then each layer thickness will be $\frac{h}{3}$.

The thickness of a Lth layer or any layer = 0.009 for if the thickness total thickness = 0.027

If it is 0.030, TKL = 0.010

for Tk = 0.036, TKL = 0.012, and

for Tk = 0.046, TKL = 0.0153.

Depending upon the total thickness of the shell each layer thickness can be found.

We are going to study a shell having 4 different thicknesses and 13, $\frac{L}{R}$ ratios. The reason for choosing 13 is we can set that only one $\frac{L}{R}$ ratio but, I have seen that the results are plotted on a $\frac{L}{R}$ from 0.2 to 6.

From a short cylinder to a long cylinder, the buckling parameters vary, or even in the case of a bending the stresses vary, if $\frac{L}{R}$ ratio changes. Then, we have to calculate A_{11},

the definition of $A_{11} = Q_{11}\left(Z_k - Z_{k-1}\right)$.

The thickness of the top - 1 because it is a composite layer then, in the first case sometimes we need Z_k^2 , for the case of bending stiffness, we need Z_k^3, we need all these types of variables.

Here, we are developing for the case of 4 thickness, z0d =- 0.5 Tk.

This $\frac{-h}{2}$ = 0.5 h.

What will be the total thickness? The second coordinate if we add $\frac{-h}{2}$ + Tk that will be z1d and if we add again, it will be z2d and the total Tk = z3d.

In this way, we can evaluate the coordinates of each layer along the thickness direction z0 bottom coordinate, first interface, second interface, and the top coordinates.

z0= - 0.5*tk

z1=z0+TKL

z2=z1+TKL

z3=z2+TKL.

Here, TKL is the thickness of each layer. In this way we have evaluated, but in the matrix, we sometimes need the Z_k^2 or Z_k^3, therefore, we are creating another setup of system, where we can say:

z0d=[z0(1)* z0(1); z0(2)* z0(2); z0(3)* z0(3); z0(4)* z0(4)]

z1d=[z1(1)* z1(1); z1(2)* z1(2); z1(3)* z1(3); z1(4)* z1(4)]

z2d=[z2(1)* z2(1); z2(2)* z2(2); z2(3)* z2(3); z2(4)* z2(4)]

z3d=[z3(1)* z3(1); z3(2)* z3(2); z3(3)* z3(3); z3(4)* z3(4)]

$$Z_{0d} = \frac{Z_0 + Z_p}{Z_0^2}$$

In this way, the first value is corresponding in that matrix will be Z_0^2, the second value corresponding will be the second thickness, the third value will be the third thickness and the fourth value will be the fourth thickness.

Similarly,

z0dd=[z0(1)* z0(1)*z0(1); z0(2)* z0(2)* z0(2); z0(3)* z0(3)* z0(3); z0(4)* z0(4)*z0(4)]

z1dd=[z1(1)* z1(1)*z1(1); z1(2)* z1(2)* z1(2); z1(3)* z1(3)* z1(3); z1(4)* z1(4)*z1(4)]

z2dd=[z2(1)* z2(1)*z2(1); z2(2)* z2(2)* z2(2); z2(3)* z2(3)* z2(3); z2(4)* z2(4)*z2(4)]

z3dd=[z3(1)* z3(1)*z3(1); z3(2)* z3(2)* z3(2); z3(3)* z3(3)* z3(3); z3(4)* z3(4)*z3(4)]

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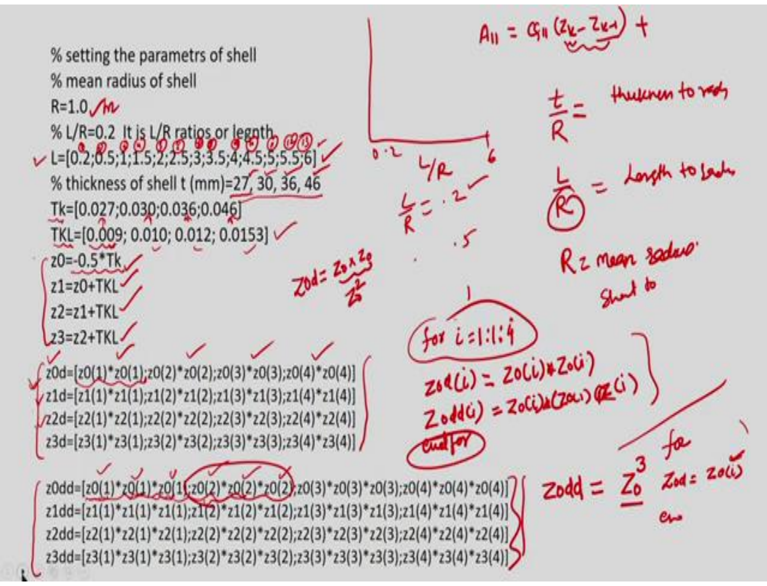

I have written in a very simple form; one can write in a better way for a programming point of view. You can write a loop also.

I want to give you confidence that any student, Ph.D. student, or master student can try themselves no need to worry that you cannot develop all these. If you try you can do it, but one has to be very much careful like, I have written z03 times, you can say that this is z02 times.

These parameters need to be like z2, if there is a problem then you can write in terms of a loop say, z0d(i) times 1, 2, 3, 4 it will go like that.

These four lines can be written in various ways, the very simple way is this one that you can write manually that is why later on, for 7- layer, 10- layer for 3- layer we can do.

But, when say a multi-layer, i.e., 25 layers, then writing this will be very cumbersome. For that case, you have to write in terms of loops and you have to develop a better algorithm for that. It may look very easy for simple cases, but if you directly convert it for a complex case, it will be very difficult or cumbersome.

For 3-layer writing, this does not matter, but for 25 layers, you have to think to write in a better way. Using some algorithm or using some functions can we evaluate this?

In that way, one can think but the basic way is this. Your program tells you that it is evaluated in this way. I have already told you that this thing you can write I is 1:1:4 then, you can say:

z0d(i)=z0(i)*z0(i).

Similarly, z0dd(i) = z0(i)*z0(i)*z0(i).

This is a simple way, if you say 25 layers then it will go for 25 layers, if you say 10 layers it will go for 10 layers, you need not remember all these things. A program can be written in this way in MATLAB.

The only thing is this syntax may change depending upon the programming language. This is known as you are preparing the better algorithm, that it can be applied to a nth layer. We have written in this way.

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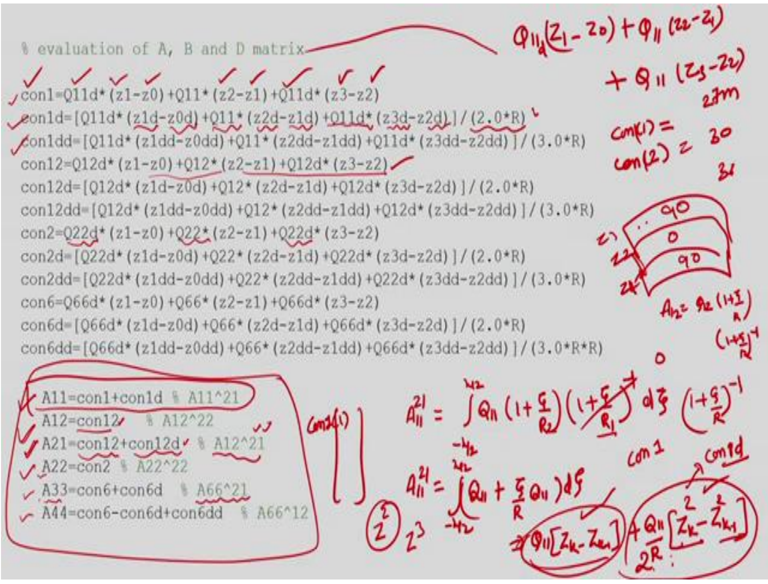

Now, we have to evaluate the matrix A, B and D.

If you remember the matrix $A_{11}^{21} = \int_{-h/2}^{h/2} Q_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma$.

Let us say, we have to write completely inverse of $d\varsigma$ or dz, $R_1 = \infty$ therefore,

$\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ term will not exist.

$$A_{11}^{21} = \int_{-h/2}^{h/2} \left(Q_{11} + \frac{\varsigma}{R} Q_{11}\right) d\varsigma \text{ .}$$

If you open it, for a composite, it becomes

$$Q_{11}\left[Z_k - Z_{k-1}\right] + \frac{Q_{11}}{2R}\left[Z_k^2 - Z_{k-1}^2\right].$$

Let us say, $Q_{11}\left[Z_k - Z_{k-1}\right]$ is constant 1 and this term $\frac{Q_{11}}{2R}\left[Z_k^2 - Z_{k-1}^2\right]$ is constant 2.

First, we have a 90˚ layer, then we have 0˚ layer, then we have 90˚ layer.

We can write:

con1=Q11d*(z1-z0)+Q11*(z2-z1)+Q11d*(z3-z2).

In this way, con 1 is evaluated.

It is very easy anybody can write that already we have evaluated the value of Z_{11}. In con1 we are going to have 4 constants. con1 is not just a constant, it is a column matrix number 1. We can say that con number 1 is corresponding to the shell thickness of 27mm.

con 12 second number is corresponding to 30 mm and so on, it will have 4 values.

Similarly, con1d=[Q11d*(z1d-z0d)+Q11*(z2d-z1d)+Q11d*(z3d-z2d)]/(2.0*R)

con1dd=[Q11d*(z1dd-z0dd)+Q11*(z2dd-z1dd)+Q11d*(z3dd-z2dd)]/(3.0*R)

con12=Q12d*(z1-z0)+Q12*(z2-z1)+Q12d*(z3-z2)

con12d=[Q12d*(z1d-z0d)+Q12*(z2d-z1d)+Q12d*(z3d-z2d)]/(2.0*R)

con12dd=[Q12d*(z1dd-z0dd)+Q12*(z2dd-z1dd)+Q12d*(z3dd-z2dd)]/(3.0*R)

con2=Q22d*(z1-z0)+Q22*(z2-z1)+Q22d*(z3-z2)

con12d=[Q22d*(z1d-z0d)+Q22*(z2d-z1d)+Q22d*(z3d-z2d)]/(2.0*R)

con12dd=[Q22d*(z1dd-z0dd)+Q22*(z2dd-z1dd)+Q22d*(z3dd-z2dd)]/(3.0*R)

con6=Q66d*(z1-z0)+Q66*(z2-z1)+Q66d*(z3-z2)

con12d=[Q66d*(z1d-z0d)+Q66*(z2d-z1d)+Q66d*(z3d-z2d)]/(2.0*R)

con12dd=[Q66d*(z1dd-z0dd)+Q66*(z2dd-z1dd)+Q66d*(z3dd-z2dd)]/(3.0*R)

In this way, one can calculate the second constant if we add all these things that will give you the A11 for the present case: A11=con1+con1d % A11^21.

This is an important point. Most of the time the students are not aware of developing all these things and they mess up. I think with this explanation one can try. we have a definition of A12=con12 % A12 ^22.

$$A_{12} = Q_{12}\left(1+\frac{\varsigma}{R}\right)\left(1+\frac{\varsigma}{R}\right)^{-1} d\varsigma \,.$$

A21=con12+con12d % A12^21.

A12 =A21, in week-05, I discussed that we will integrate only up to Z^2 term, we do not consider Z^3. Therefore, $\left(1+\frac{\varsigma}{R}\right)^{-1}$ will get canceled, it is very simple that only con12 will be there.

Similarly, A22=con2 % A22^22, in the A22 also $\left(1+\frac{\varsigma}{R}\right)^{-1}$ will get canceled.

A33=con6+con6d % A66^21.

A44=con6-con6d+con6dd % A66^12.

Here, I have written the comments also, what is A22, A33, A44 in this program. In this way, we can evaluate the matrix A11, A12, A21, A22, A33, and A44.

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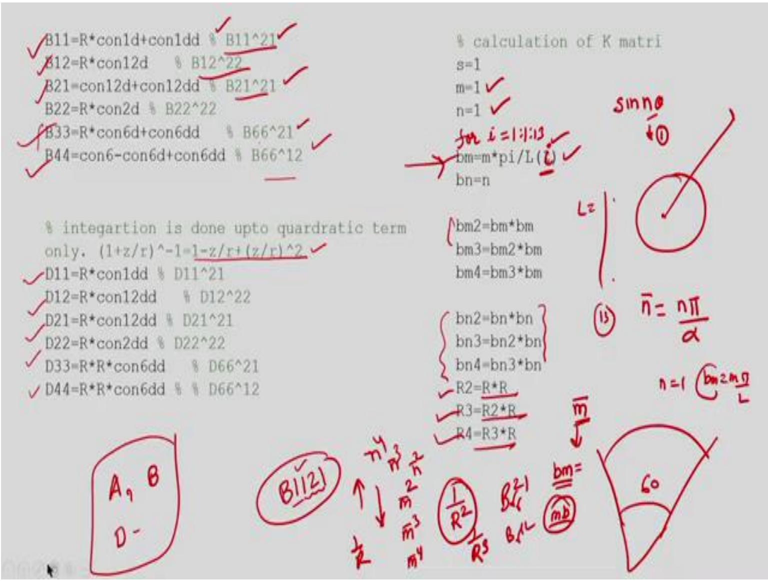

Similarly, using these constants we can write the matrix B and matrix D. Already I told you that we have taken only the quadratic term in the integration, this is a slightly reduced expression, if you take more terms then it will be having some bigger expression.

The definition of B11= R*con1d+con1dd, in this way, if we write a comment, it helps us to find the meaning of B11. Either we have to write B11^21 because in the concept of a program when we do write a program there is no concept of superscript or subscript.

We have to think in that way.

Then B12=R*con12d % B12^22

B21=con12d+con12dd % B21^21

B22=R*con2d %B22^22

B33=R*con6d+con6dd % B66^21

B44=con6-con6d+con6dd % B66^12.

There is a difference, whenever B66^ 21 comes, we will use B33 whenever B66^12 comes, we will use this one.

D11=R*con1dd % D11^21

D12=R*con12dd % D11^21

D21=R*con12dd % D11^21

D22=R*con2dd % D11^21

D33=R*R*con6dd % D11^21

D44=R*R*con6dd % D11^21

In this way, matrix A, matrix B, and matrix D is evaluated.

Now, we have to find, let us say m = 1 and n = 1. When we talk about a closed cylinder, we have only boundary conditions along the x-direction and there if $\sin n\theta$, n is taken as 1, consider that simply supported. We are not varying along θ direction, we can say $\bar{n}$, for a finite shell panel will be $n\pi/\alpha$, if you say that my span angle is up to here 60˚, 120˚, but for a closed shell, we take n = 1.

$\bar{m}$ = $m\pi$ /L. In a program, we say that $\bar{m}$ = bm=m*pi/L(i).

Here I have written 1, this parameter I am going to change every time.

There are many ways or you do it manually, let us say length ratio is 1, we can evaluate buckling parameter then we change it to 3, 4, 5 any time because, we have that L matrix having 13 variables 1 - 13, differential $\frac{L}{R}$ ratio. One way is this, another way is let us say, i=1:1:13.

It will evaluate for a particular $\frac{L}{R}$ ratio for let us say, i = 1, then all the parameters will be calculated.

Then, we have bm2=bm*bm, sometimes we need $\bar{m}^2$

bm3=bm2*bm, sometimes we need $\bar{m}^3$

Similarly, bm4=bm3*bm.

Same way, bn2=bn*bn

bn3=bn2*bn

bn4=bn3*bn.

If you remember in the formulation, sometimes R^2, R^3 are required, every time writing $\frac{1}{R}$ is difficult, therefore, we said that R2=R*R; R3=R2*R; R4=R3*R.

In this way, one can set the parameters.

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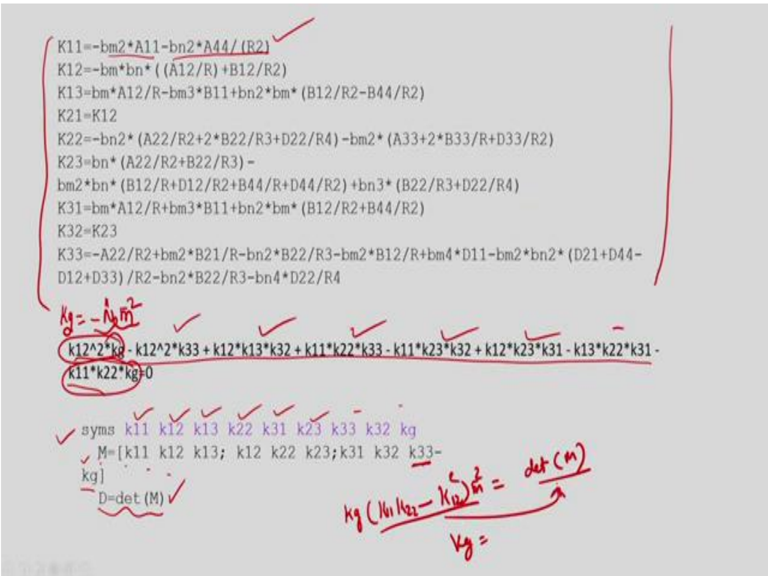

Following are the elements of K matrix:

K11=-bm2*A11-bn2*A44/(R2)

K12=-bm*bn*((A12/R)+(B12/R2))

K13=bm*A12/R-bm3*B11+bn2*bm*(B12/R2-B44/R2)

K21=K12

K22=-bn2*(A22/R2+2*B22/R3+D22/R4)-bm2*(A33+2*B33/R+D33/R2)

K23=bn*(A22/R2+B22/R3)bm2*bn*(B12/R+D12/R2+B44/R+D44/R2)+bn3*(B22/R3+D22/R4)

K31=bm*A12/R-bm3*B11+bn2*bm*(B12/R2+B44/R2)

K32=K23

K33=-A22/R2+bm2+B21/R-bn2*B22/R3-bm2*B12/R+bm4+D11bm2*bn2*(D21+D44-D12+D33)/R2-bn2*B22/R3-bn4*D22/R4.

K12^2*kg - k12^2*33 + k12*k13*k32 + k11*k22*k33 – k11*k23*k32 + k12*k23*k31 – k13*k22*k31 – k11*k22*kg =0

I would like to say that I have not written the matrix of kg, here kg = $N_{11}^{0}\overline{m}^{2}$, for that purpose what will be the formula of this?

I have done a symbolic test that let us say k11, k12, k13, k22, k31, k23, k33, k32, kg all these things and it is written in a matrix form:

M= [k11 k12 k13; k12 k22 k23; k31 k32 k33-kg],

Here k33- kg term is written, evaluate the determinant of that.

In the symbolic form, the determinant will be D=det (M).

There are 2 terms in which kg is involved, and other things can be put on the right-hand side.

We can say that $K_g(N_{11}N_{22} - K_{12}^2)\overline{m}^2 = \det(M)$.

In this way, the critical load can be evaluated.

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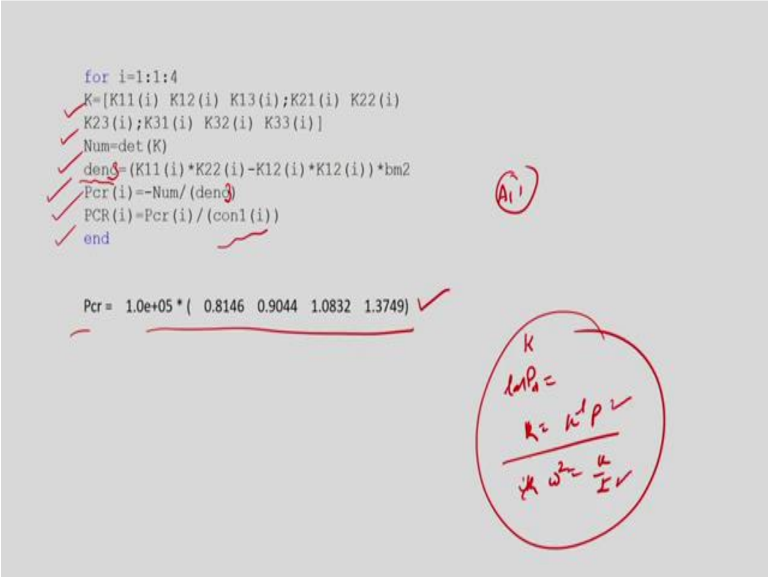

Then

for i=1:1:4;

K=[K11(i) K12(i) K13(i); K21(i) K22(i) K23(i); K31(i) K32(i) K33(i)]

Num=det(K)

deno3 = K11(i)* K22(i) - K12(i)* K12(i)*bm2.

We can calculate the Pcr(i)=-Num/ deno3

PCR(i)=Pcr(i)/(con1(i)).

PCR is saying that you have to divide with the A11.

In this way the values are can be calculated.

Anybody can try like this for any case at least for simply supported it is very easy whether you talk about bending, or free vibration or buckling one can develop a solution like this.

Pcr = 1.0e+05* (0.8146 0.9044 1.0832 1.3749)

I have not discussed that even for the bending case I left for the reader, one can write a program. In this week, I have written a small basic program.

If it is a problem of static bending, we have to find the P loading also.

And $U = K^{-1}P$.

If we are interested in a free vibration then P = 0 and $\omega^2 = \frac{K}{I}$.

We can write all these things; K is known to us and we have to set the value of P and inertia matrix and we can solve the problem of bending for free vibration and so on.

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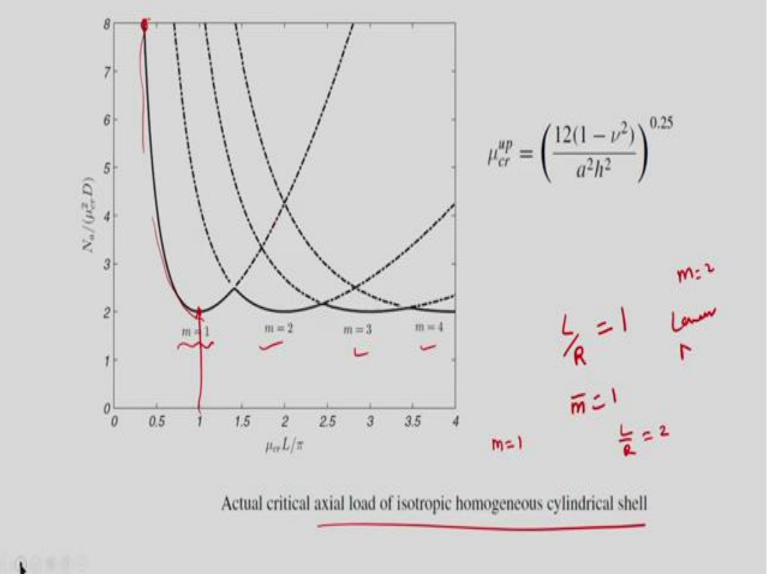

The next is this critical axial load: $\mu_{cr}^{up} = \left(\frac{12(1-v^2)}{a^2h^2}\right)^{0.25}$.

I have just got for an isotropic homogeneous cylindrical shell, when you say that m =1 for a short cylinder, it is high, it is suddenly down and near to 1 that 4L/ R = 1. The critical buckling load for m = 1 will take place here.

When you have a $\frac{L}{R}$ = 2 then, for the case of m = 2, you will get the lower load because in the case of a buckling we want the lowest minimum load. It is not always possible that it will be lowest for the case of m = 1 only depending upon the various geometric parameters it may be lowest or may not be lowest you can see here that corresponding m2 m3 and m4, and it is coming down and then shooting again. One can find the critical buckling load.

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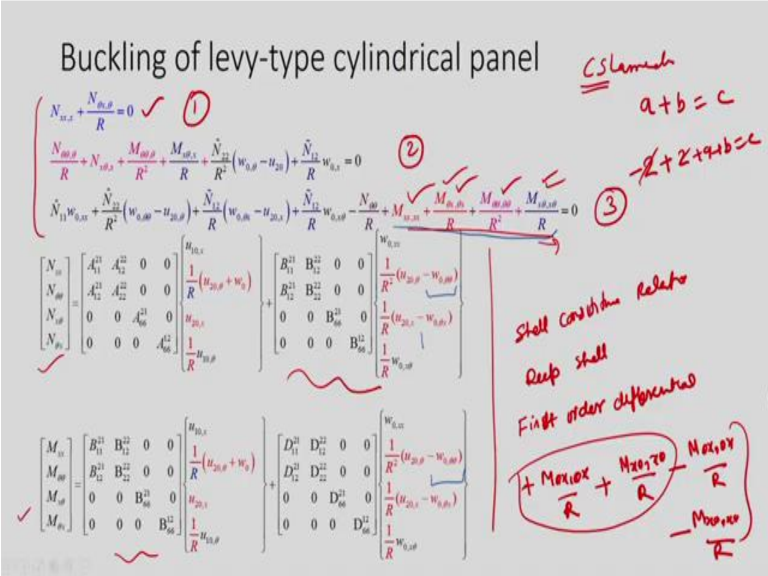

Now, the next concept is that can we develop a solution for a Levy-type cylindrical panel. I am here going to give you the basic concept that how to develop a buckling solution for a Levy-type cylindrical panel. And foremost thing is that we started with 3 equations for a classical shell laminate because in most of the books isotropic shell is studied mostly but for the case of a composite either they are in a research journal article or they may be in different books.

I am going to give you the basic concept to develop a Levy-type solution based on classical shell theory. For a classical shell theory, for a circular cylinder the governing equations are like that:

$$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} + q_1 = 0 \quad equation(1)$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + \frac{M_{\theta\theta,\theta}}{R^2} + \frac{M_{x\theta,\theta}}{R} + \frac{N_{22}}{R^2}\left(w_0,_{\theta} - u_{20}\right) + \frac{N_{12}}{R}\left(w_0,_{x}\right) = 0 \quad equation(2)$$

$$N_{11} w_0,_{xx} + \frac{N_{22}}{R^2}\left(w_0,_{\theta\theta} - u_{20,\theta}\right) + \frac{N_{12}}{R}\left(w_0,_{\theta x} - u_{20,x}\right) + \frac{N_{12}}{R} w_0,_{x\theta} - \frac{N_{\theta\theta}}{R} + M_{xx,xx}$$
$$+ \frac{M_{\theta x,\theta x}}{R} + \frac{M_{\theta\theta,\theta\theta}}{R^2} + \frac{M_{x\theta,x\theta}}{R} = 0 \quad equation(3)$$

And following are the shell constitutive relations:

$$
\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \begin{bmatrix} A_{11}^{21} & A_{12}^{22} & o & o \\ A_{12}^{21} & A_{22}^{22} & o & o \\ o & o & A_{66}^{21} & o \\ o & o & o & A_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(u_{20,\theta} + w_0\right) \\ u_{20,x} \\ \frac{1}{R} u_{10,\theta} \end{bmatrix} + \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} w_{0,xx} \\ \frac{1}{R^2}\left(u_{20,\theta} - w_{0,\theta\theta}\right) \\ \frac{1}{R}\left(u_{20,x} - w_{0,\theta x}\right) \\ \frac{1}{R} w_{0,x\theta} \end{bmatrix}
$$

$$
\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(u_{20,\theta} + w_0\right) \\ u_{20,x} \\ \frac{1}{R} u_{10,\theta} \end{bmatrix} + \begin{bmatrix} D_{11}^{21} & D_{12}^{22} & o & o \\ D_{12}^{21} & D_{22}^{22} & o & o \\ o & o & D_{66}^{21} & o \\ o & o & o & D_{66}^{12} \end{bmatrix} \begin{bmatrix} w_{0,xx} \\ \frac{1}{R^2}\left(u_{20,\theta} - w_{0,\theta\theta}\right) \\ \frac{1}{R}\left(u_{20,x} - w_{0,\theta x}\right) \\ \frac{1}{R} w_{0,x\theta} \end{bmatrix}
$$

Here, I would like to say that these are the shell constitutive relation and valid for a deep shell because, when you talk about a shallow or thin shell, then this B does not participate in the matrix. If you remove this B, this formulation will be valid for a thin shell or a deep shallow shell, but if it contains these things then definitely it gives you a result for a deep shell. For the case of a mixed formulation, we aim to develop in terms of a first-order differential equation, because, that is easy and we can get the exact solution for that case.

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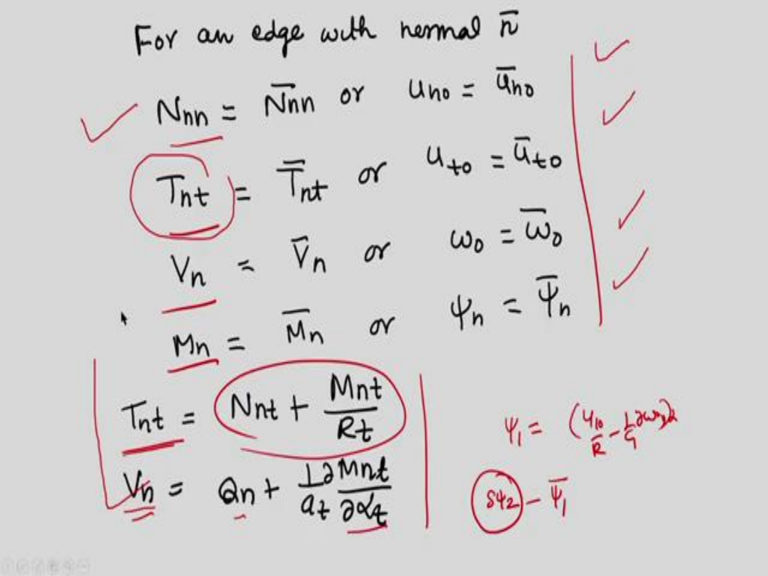

The boundary conditions can be expressed for an edge, let us say, ‘n’ is normal and ‘T’ is the tangent:

$$N_{nn} = \bar{N}_{nn} \;\; or \;\; u_{no} = \bar{u}_{no}; \quad T_{nt} = \bar{T}_{nt} \;\; or \;\; u_{to} = \bar{u}_{to}$$
$$V_n = \bar{V}_n \;\; or \;\; w_0 = \bar{w}_0; \quad M_{nn} = \bar{M}_{nn} \;\; or \;\; \psi_n = \bar{\psi}_n$$

If it is the case of an FSDT, the development of a Levy type solution is comparatively easy, but if you are going for a classical shell then it is slightly difficult because we have parameters T_{nt} and V_n

Where,

$$T_{nt} = N_{nt} + \frac{M_{nt}}{R_t}$$

$$V_n = Q_n + \frac{1}{a_t}\frac{\partial M_{nt}}{\partial \alpha_t} .$$

We have these composite variables T_{nt} and V_n, these contain some more things. First of all, we have to find these variables for the present case.

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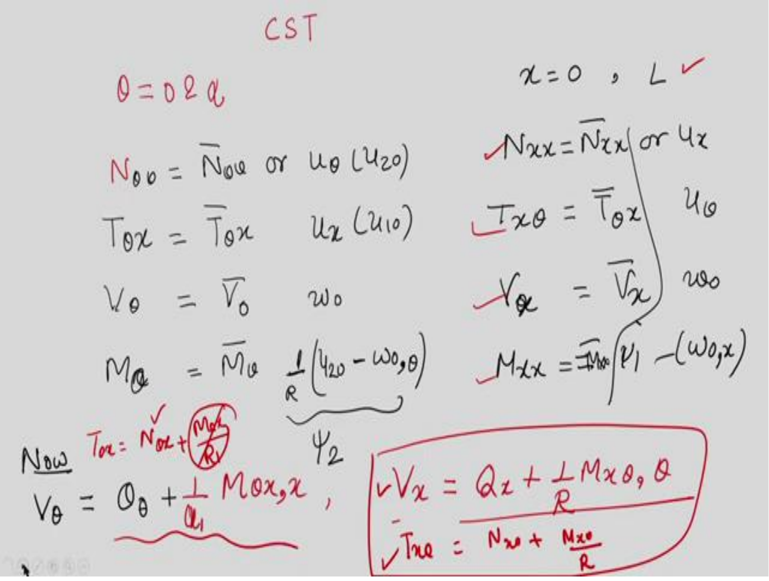

Let us say, for θ = 0 and α , the boundary conditions will be:

$$N_{\theta\theta} = \bar{N}_{\theta\theta} \quad or \quad u_\theta(u_{20}); \quad T_{\theta x} = \bar{T}_{\theta x} \quad or \quad u_x(u_{10})$$

$$V_\theta = \bar{V}_0 \quad or \quad w_0; \quad M_\theta = \bar{M}_\theta \, \frac{1}{R}\left(u_{20} - w_{0,\theta}\right) \; or \; \psi_2$$

Here, the definition of $V_\theta = \; Q_\theta + \frac{1}{a_1} M_{\theta x,x}$.

Similarly, the definition of $T_{\theta x} = \; N_{\theta x} + \frac{M_{\theta x}}{R_1}$, R_1 is the radius of curvature along the tangent direction which is 1, therefore, R_1 = infinity it will not contribute here.

For x = 0 and L:

$$N_{xx} = \bar{N}_{xx} \; or \quad u_x; \quad T_{x\theta} = \bar{T}_{\theta x} \quad or \; u_\theta$$

$$V_x = \bar{V}_x \quad or \quad w_0; \; M_{xx} = \bar{M}_{xx} \;\; \psi_1 - \left(w_{0,x}\right)$$

These are the variables we need to consider because we cannot say that we are only specifying Q_x or $M_{x\theta}$, the boundaries will be specified in terms of

$$V_x \; = \; Q_x + \; \frac{1}{R} M_{x\theta,\theta}$$

$$T_{\theta x} = N_{x\theta} + \frac{M_{x\theta}}{R} \quad .$$

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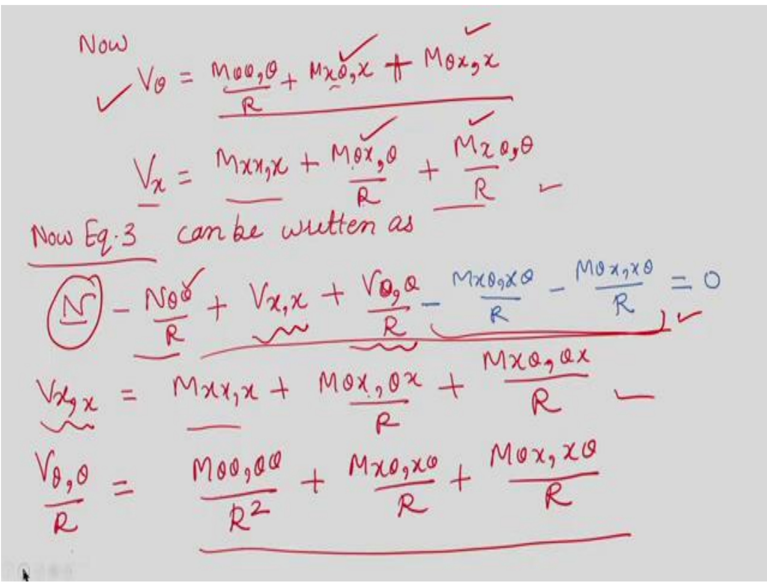

Finally, what is the meaning of V_θ and V_x.

$$V_\theta = \frac{M_{\theta\theta,\theta}}{R} + M_{x\theta,x} + M_{\theta x,x}$$

$$V_x = M_{xx,x} + \frac{M_{\theta x,\theta}}{R} + \frac{M_{x\theta,\theta}}{R}.$$

In this regard the formulation contains the third equation;

$$N_{11}w_0,_{xx} + \frac{N_{22}}{R^2}\left(w_0,_{\theta\theta} - u_{20,\theta}\right) + \frac{N_{12}}{R}\left(w_0,_{\theta x} - u_{20,x}\right) + \frac{N_{12}}{R}w_0,_{x\theta} - \frac{N_{\theta\theta}}{R} + M_{xx,xx}$$
$$+ \frac{M_{\theta x,\theta x}}{R} + \frac{M_{\theta\theta,\theta\theta}}{R^2} + \frac{M_{x\theta,x\theta}}{R} = 0 \;\; equation(3)$$

This equation contains a double derivative of a moment that we want to get rid of. This equation we can rearrange in terms of V_θ and V_x, if we want to add some variables.

Now equation (3) can be written as:

$\bar{N} - \frac{N_{\theta\theta}}{R} + V_{x,x} + \frac{V_{x,x}}{R} - \frac{M_{x\theta,x\theta}}{R} - \frac{M_{\theta x,\theta x}}{R} = 0$. These are the additional terms because, if you see in this, we need 4 terms, but in the equation, we have only the corresponding 2 terms. Therefore, we can say that adding 2 terms more and $-\frac{M_{x\theta,x\theta}}{R}$ $-\frac{M_{\theta x,\theta x}}{R}$.

The terms $M_{xx,x} + \frac{M_{\theta x,\theta x}}{R} + \frac{M_{x\theta,\theta x}}{R} = V_x \; or V_\theta$.

In this way, the equation (3) equilibrium will not change.

Let us say, a + b = c. If you add -2 + 2 + a + b = c = 0.

It is only giving you a mathematical advantage. Adding something and subtracting the same thing is not going to change that equation.

It gives you a mathematical advantage that this equation (3) can be written in terms of $V_x \; and \, V_\theta$. That is why I said that developing a mixed formulation using the classical shell is slightly difficult.

Derivative of $V_{x,x}$ = $M_{xx,x} + \frac{M_{\theta x,\theta x}}{R} + \frac{M_{x\theta,\theta x}}{R}$

And derivative of $\frac{V_{\theta,\theta}}{R}$ = $\frac{M_{\theta\theta,\theta\theta}}{R^2} + \frac{M_{x\theta,x\theta}}{R} + \frac{M_{\theta x,\theta x}}{R}$.

In this way equation (3) is fine.

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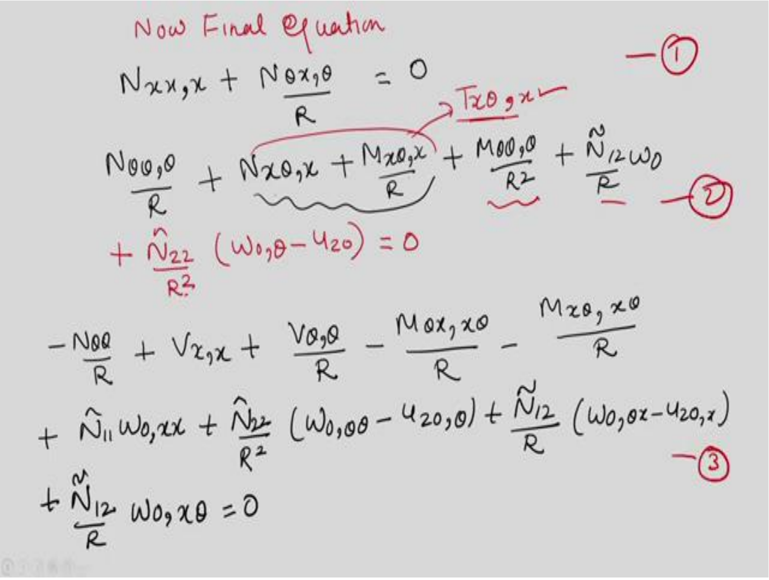

Then we have equation (2) containing these variables. $\frac{M_{x\theta,x}}{R} + \frac{M_{\theta\theta,\theta}}{R^2}$ is nothing but we can club together under the head $T_{x\theta,x}$. This equation is again modified and it contains these terms. Now we have modified equations 1, 2, and 3.

$$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} = 0 \;\; equation(1)$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + \frac{M_{x\theta,x}}{R} + \frac{M_{\theta\theta,\theta}}{R^2} + \frac{\tilde{N}_{12}}{R} w_0 + \frac{\tilde{N}_{22}}{R^2}\left(w_{0,\theta} - u_{20}\right) = 0 \;\; equation(2)$$

$$-\frac{N_{\theta\theta}}{R} + V_{x,x} + \frac{V_{\theta,\theta}}{R} - \frac{M_{\theta x,x\theta}}{R} - \frac{M_{x\theta,x\theta}}{R} + \hat{N}_{11} w_{0,xx} + \frac{\hat{N}_{22}}{R^2}\left(w_{0,\theta\theta} - u_{20,\theta}\right) +$$
$$\frac{\tilde{N}_{12}}{R}\left(w_{0,\theta x} - u_{20,x}\right) + \frac{\tilde{N}_{12}}{R} w_{0,x\theta} = 0 \;\; equation(3)$$

We are going to develop further.

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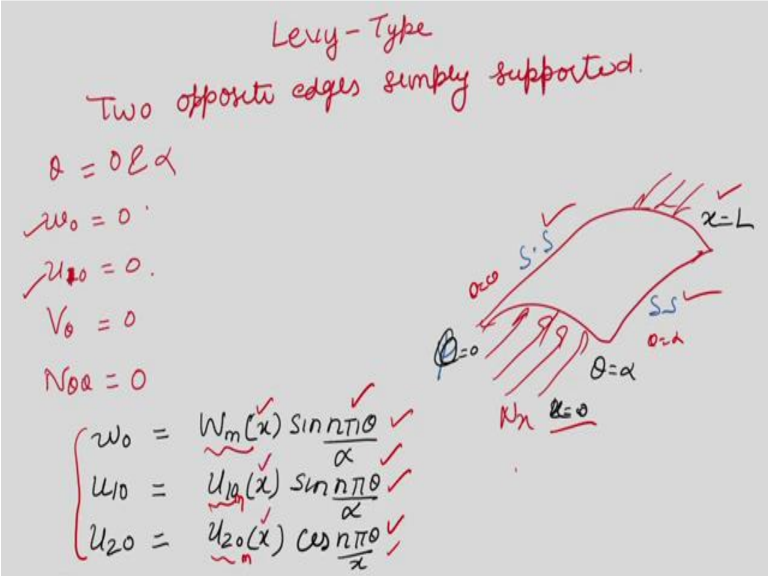

Now, two opposite edges are simply supported; for that case, we are assuming that $\theta = 0$ and $\theta = \alpha$ is simply supported and x = 0 and x = L can have any boundary condition. By keeping this in view we can assume the displacement variable along θ direction in sine and cosine series.

Here, $w_0 = u_{10} = v_\theta = N_{\theta\theta} = 0$.

Along θ:

$$w_0 = w_m(x)\frac{\sin n\pi\theta}{\alpha}$$

$$u_{10} = u_{10}(x)\frac{\sin n\pi\theta}{\alpha}$$

$$u_{20} = u_{20}(x)\frac{\cos n\pi\theta}{x}\ .$$

Here, w_m, u_{10}, and u_{20} are the function of x. Previously, these were constant, but now these are a function of x.

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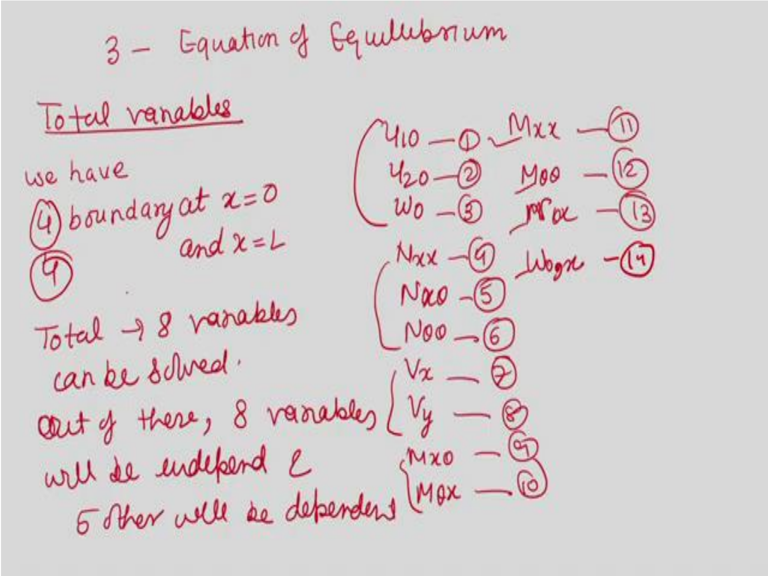

We have 3 displacements and stresses $v_x, v_y, M_{x\theta}, M_{\theta x}$. At a time, the maximum we can solve 4 variables at x = 0 and 4 variables at x = L. Total we can solve maximum 8 variables, out of $u_{10}, u_{20}, w_0, N_{xx}, N_{x\theta}, N_{\theta\theta}, v_x, v_y, M_{x\theta}, M_{\theta x}, M_{xx}, M_{\theta\theta}$,

$N_{\theta x}$, *and* $w_{0,x}$ variables.

We can solve only 8 variables and the rest of the variables will be dependent. What will be those variables which we are going to put in the state-space equation means the independent equations, we have 3 displacements.

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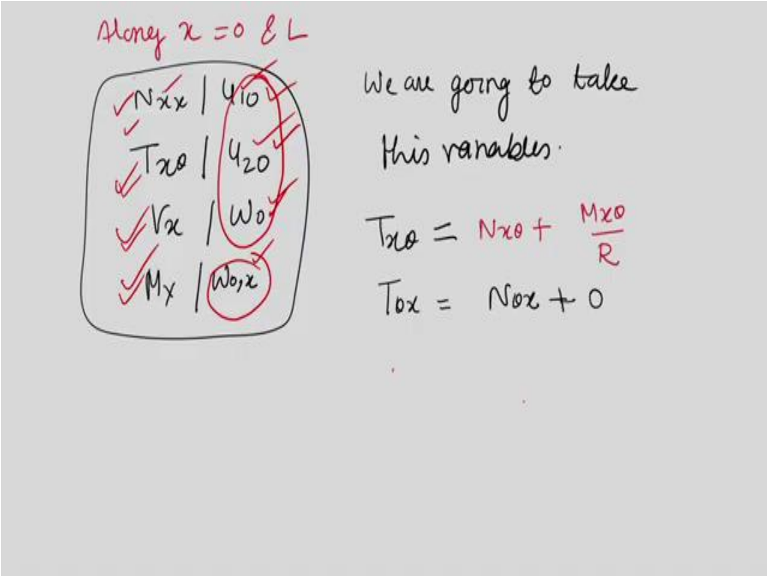

Next, at the boundary x = 0 and x = L,

Either N_{xx} or u_{10} ; $T_{x\theta} / u_{20}$; V_x / w_0 ; and $M_x / w_{0,x}$.

u_{10} , u_{20} , w_0 we have already taken, we know displacement will be there then the derivative along x-direction will also be a variable.

Now, we are going to take N_{xx} , $T_{x\theta}$, V_x , and M_x as the variables.

The reason for choosing a mixed form is that we can satisfy the boundary conditions exactly because these are our variables.

$$T_{x\theta} = N_{x\theta} + \frac{M_{x\theta}}{R}$$

$$T_{\theta x} = N_{\theta x} + 0 .$$

If it is chosen for displacement then we have to satisfy in terms of a further average sense.

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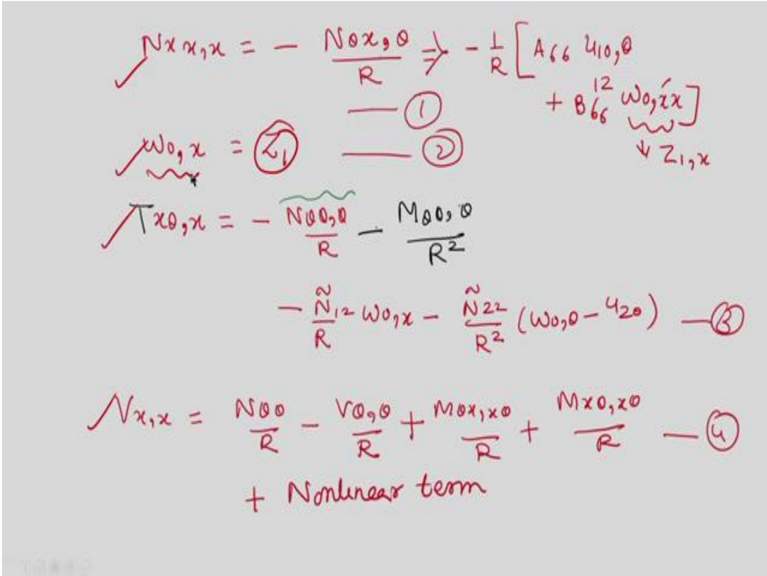

The very first equation will be:

$$N_{xx,x} = -\frac{N_{\theta x,\theta}}{R}.$$

Here, $\frac{N_{\theta x,\theta}}{R} = \frac{1}{R}\left[A_{66}u_{10,\theta} + B_{66}^{12}w_{0,xx}\right]$.

Similarly,

$T_{x\theta,x} = \frac{N_{\theta\theta,\theta}}{R} - \frac{M_{\theta\theta,\theta}}{R^2} - \frac{\tilde{N}_{12}}{R}w_{0,x} - \frac{\hat{N}_{22}}{R^2}\left(w_{0,\theta\theta} - u_{20,\theta}\right)$, using the shell constitutive relations A, B, and D matrices.

$$V_{x,x} = \frac{N_{\theta\theta}}{R} - \frac{V_{\theta,\theta}}{R} + \frac{M_{\theta x,x\theta}}{R} + \frac{M_{x\theta,x\theta}}{R} + Nonlinear\,term,$$

In this way, we will get 3 equations, and then we have chosen $w_{0,x}$ in a separate variable let us say it is z_1.

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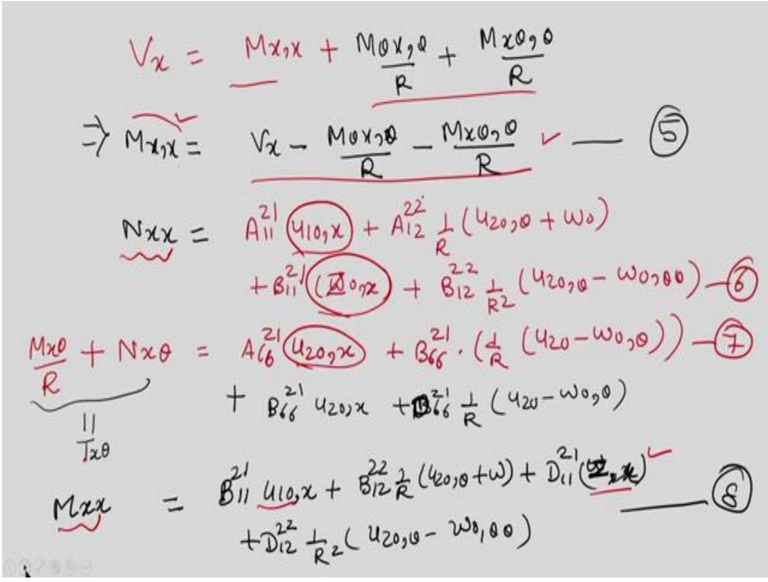

We get four equations, now, we need four more equations. For that purpose, let us say:

$$V_x = M_{xx,x} + \frac{M_{\theta x,\theta}}{R} + \frac{M_{x\theta,\theta}}{R}.$$

From here, $M_{xx,x} = V_x - \frac{M_{\theta x,\theta}}{R} - \frac{M_{x\theta,\theta}}{R}$.

Here, $N_{xx} = A_{11}^{21} u_{10,x} + A_{11}^{22} \frac{1}{R}\left(u_{20,\theta} + w_0\right) + B_{11}^{21} z_{0,x} + B_{12}^{22} \frac{1}{R^2}\left(u_{20,\theta} + w_{0,\theta\theta}\right)$

$u_{10,x}$, $z_{0,x}$ are the variables.

Similarly,

$$T_{x\theta} = \frac{M_{x\theta}}{R} + N_{x\theta} = A_{66}^{21} u_{20,x} + B_{66}^{21} \frac{1}{R}\left(u_{20} + w_{0,\theta}\right) + B_{66}^{21} u_{20,x} + D_{66}^{21} \frac{1}{R}\left(u_{20} + w_{0,\theta}\right)$$

And $M_{xx} = B_{11}^{21} u_{10,x} + B_{12}^{22} \frac{1}{R}\left(u_{20,\theta} + w_0\right) + D_{11}^{21} z_{1,x} + D_{12}^{22} \frac{1}{R^2}\left(u_{20,\theta} + w_{0,\theta\theta}\right)$.

In this way, these equations are prepared.

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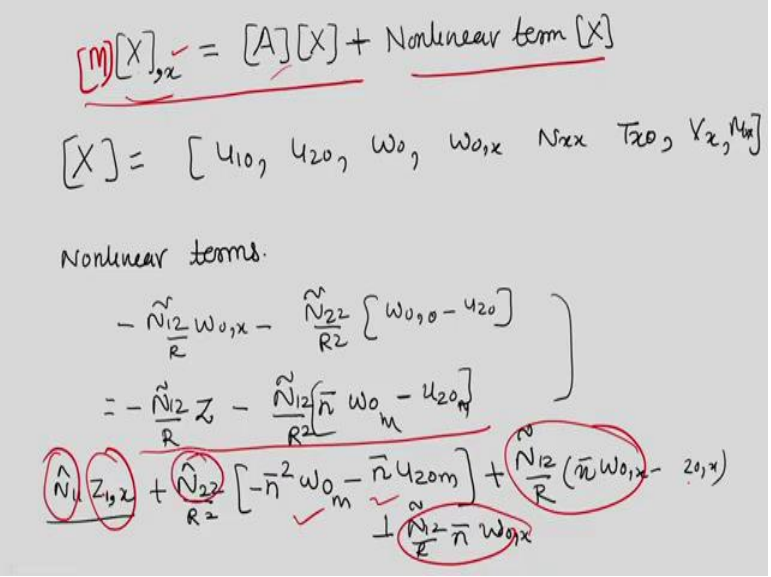

Now, we can properly arrange these equations and we can write $\left[X\right]$ is a column vector that contains the variables $\left[u_{10},\ u_{20},\ w_0, N_{xx},\ T_{x\theta}, V_x,\ M_{xx},\ w_{0,x}\right]$.

It will have some matrix $[M][X]_{,x} = [A][X] + Nonlinear\,terms[X]$.

What are those non-linear terms?

I have written the in the second equation:

$-\frac{\tilde{N}_{12}}{R}w_{0,x} - \frac{\tilde{N}_{22}}{R^2}\left(w_{0,\theta} - u_{20}\right) \quad -\frac{\tilde{N}_{12}}{R}z - \frac{\tilde{N}_{12}}{R^2}\left(\bar{n}w_{0m} - u_{20m}\right)$ these are the non-linear terms. They will be correspondingly together either they will be in a matrix or when we have a derivative, they can go to the left-hand side. Similarly, from the third equation:

$$\hat{N}_{11}z_{1,x} + \frac{\tilde{N}_{22}}{R^2}\left(-\bar{n}^2 w_{0m} - \bar{n}u_{20m}\right) + \frac{\tilde{N}_{12}}{R}\left(\bar{n}w_{0,x} - \bar{n}u_{20,x}\right) + \frac{\tilde{N}_{12}}{R}\bar{n}w_{0,x}$$

You can see the derivative of this term $\hat{N}_{11}z_{1,x}$ will go to the left-hand side. Similarly, $\frac{\tilde{N}_{12}}{R}\left(\bar{n}w_{0,x} - \bar{n}u_{20,x}\right)$ will go to the right-hand side, $\frac{\tilde{N}_{12}}{R}\bar{n}w_{0,x}$ will go to the left-hand side, and the rest of the terms will go to the right-hand side.

In this, a matrix M coefficient of that and matrix A will be calculated. Till now we do not know the value of $\hat{N}_{11}$ or $\tilde{N}_{22}$, depending upon the situation anyone will be nonzero and others will be zero.

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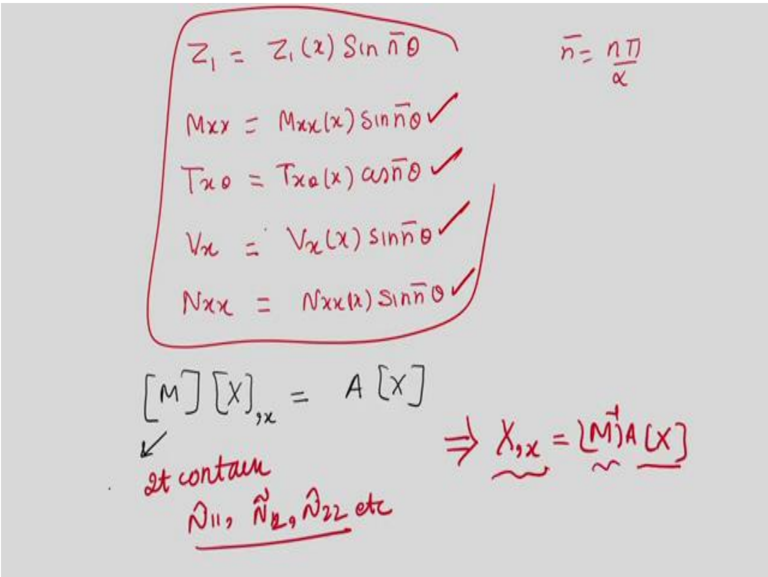

Next, we have only assumed the solution of $u_{10},\ u_{20},\ and\ w_0$. The same way using the constitutive relations, we can find:

$$
\begin{aligned}
z_1 &= z_1(x)\sin\bar{n}\theta \\
M_{xx} &= M_{xx}(x)\sin\bar{n}\theta \\
T_{x\theta} &= T_{x\theta}(x)\cos\bar{n}\theta \\
V_x &= V_x(x)\sin\bar{n}\theta \\
N_{xx} &= N_{xx}(x)\sin\bar{n}\theta
\end{aligned}
$$

Ultimately, the solution can be written as:

$$[X]_{,x} = [M]^{-1} A[X].$$

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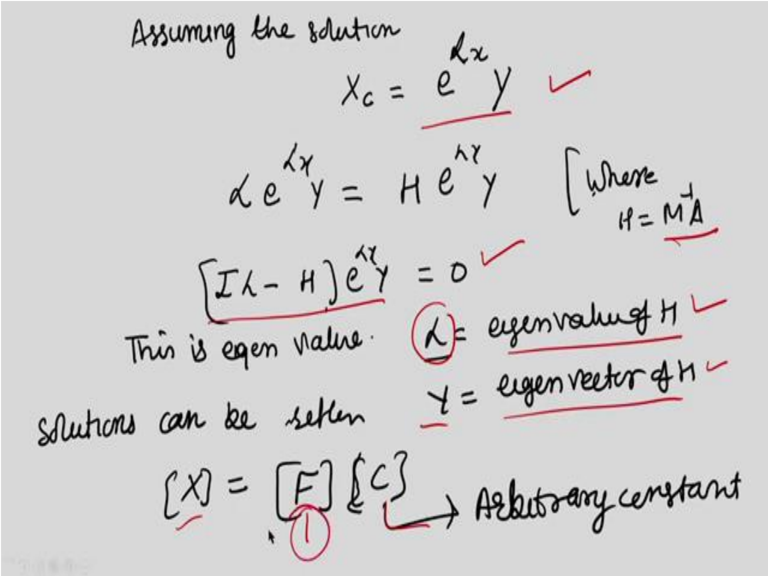

This is the standard equation and the complementary solution can be represented $X_c = e^{\lambda x} y$ like this.

And if we substitute into the previous equation $\alpha e^{\lambda x} y = He^{\lambda y} y$ that leads to an eigenvalue problem $[I\lambda - H]e^{\lambda y} y = 0$.

Where, λ is the eigenvalue of a matrix H, H = $M^{-1}A$, and Y is the eigenvector of H. We can write the solution:

$$[X] = [F][C]$$

Here, F may contain eigenvalues and eigenvectors.

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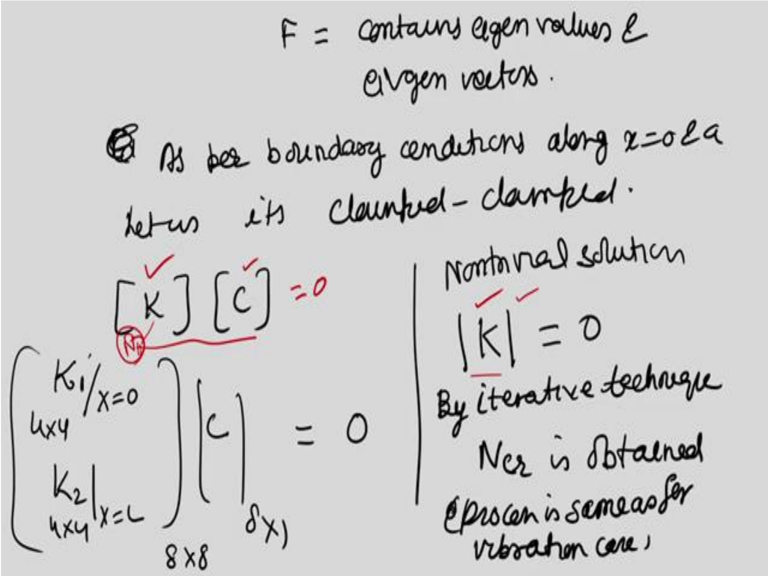

Finally, as per boundary conditions along x = 0 & a, you can separate a matrix $[K][C]$ we form a matrix and equate it to 0. This is a homogeneous matrix so the nontrivial solution will be possible when the determinant $|K|$ = 0.

This K contains $\hat{N}_{11}$ or $\hat{N}_0$ means the critical buckling load information but, we do not know the value how can we set the determinant equal to 0.

A similar expression we have obtained when we were doing the free vibration of a cylinder. In that case, using the iterative technique. In week -06, I already explained in detail that using the bisection method we can find the frequency ω , and the same way we can find the uncritical

for this case, a 3 by 3 matrix, somebody can also develop a closed-form solution by hand that what are the actual values of eigenvalues or eigenvectors are coming, it leads to the determinant matrix and can solve in MATLAB.

With this, the buckling of a cylindrical shell topic is over. In week-08 I am going to solve the 3-dimensional problem of a shell, i.e., to develop a three-dimensional solution for different cases like bending, free vibration, and buckling cases.

Thank you very much.