Theory of Composite Shells

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Week - 08

Lecture - 02

Development of three-dimensional solution

Dear learners welcome to week- 08 lecture- 02. In the lecture- 01 of the week- 08, I developed a bending solution using the three-dimensional elasticity. The solution for governing equations for the developed and all-around simply supported case was done. Initially, I tried to develop the solution for angle ply shell panels where that x is equal to 0 which means it can be of infinite length along the x-direction. Along the θ direction panel is simply supported and the governing differential equations are obtained.

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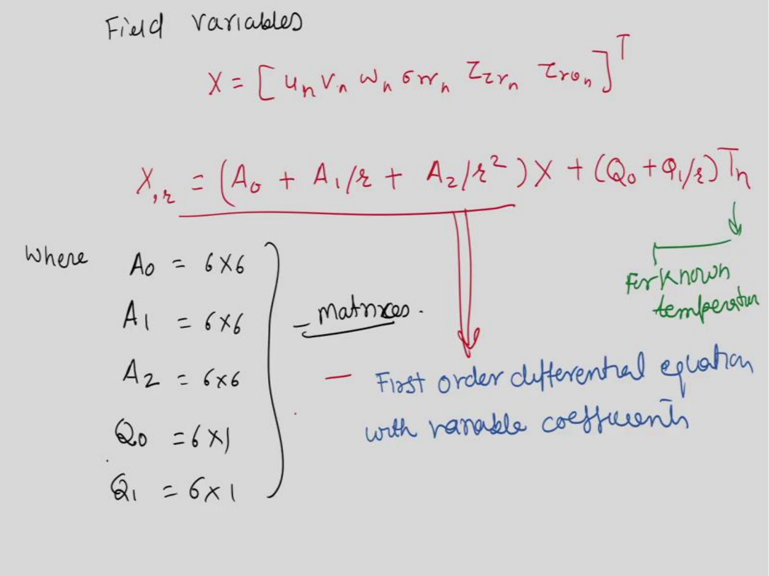

Those equations were like this:

Where, $X = \left[u_n, v_n, w_n, \sigma_{rr_n}, \tau_{zr_n}, \tau_{r\theta_n}\right]^T$ and

$X,_r = \left(A_0 + \frac{A_1}{r} + \frac{A_2}{r^2}\right)X + \left(Q_0 + \frac{Q_1}{r}\right)T_n$ is the differential equation

Where $A_0, A_1, and\ A_2$ are the matrices and Q_0 and Q_1 are the temperature thermal matrices that mean loading vector for temperature case.

Here, we see that this is the first-order differential equation with a variable coefficient.

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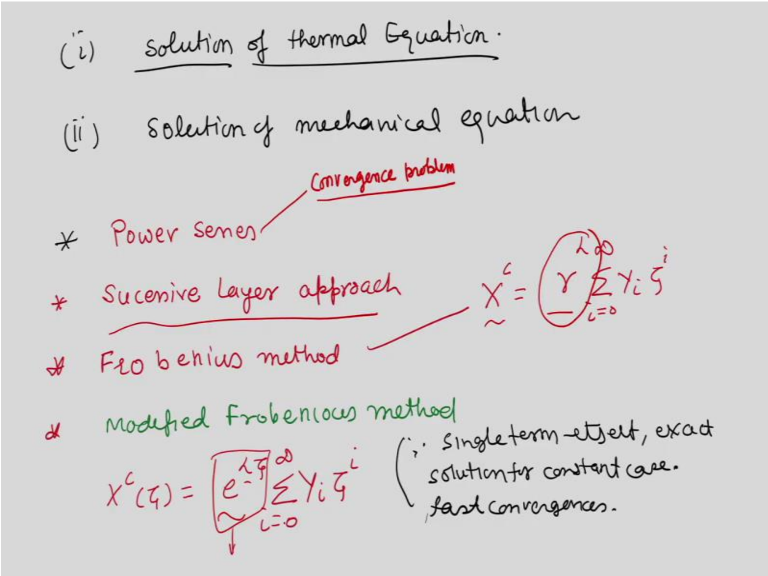

The solution to this equation can be done in a number of ways. In the literature, this equation can be solved using the power series method, but in the power series method convergence is an issue i.e., sometimes for a particular stacking sequence or geometric configuration, the power series does not converge very fast.

Then an approach is developed that is known as the successive layer approach. In this approach cylindrical shell or each lamina is considered that it is made of n or p number of fictitious layers and this layer thickness is very very small so that the governing equation in that regime can be considered as a constant.

And, we know that the ordinary differential equation with a constant coefficient can be solved easily for the case of a plate using the Pagano solution. The Frobenius method was applied to solve this equation.

In the original Frobenius method:

The complementary of the solution $X^C = r^{\lambda} \sum_{i=0}^{\infty} Y_i \varsigma^i$.

Recently, the modified Frobenius method was applied i.e., instead of r^{λ} here $e^{\lambda\varsigma}$ is taken.

The reason behind taking an exponential instead of r is let us say i = 0, which gives the complete solution for an ordinary differential equation with the constant coefficient. These series converge fast.

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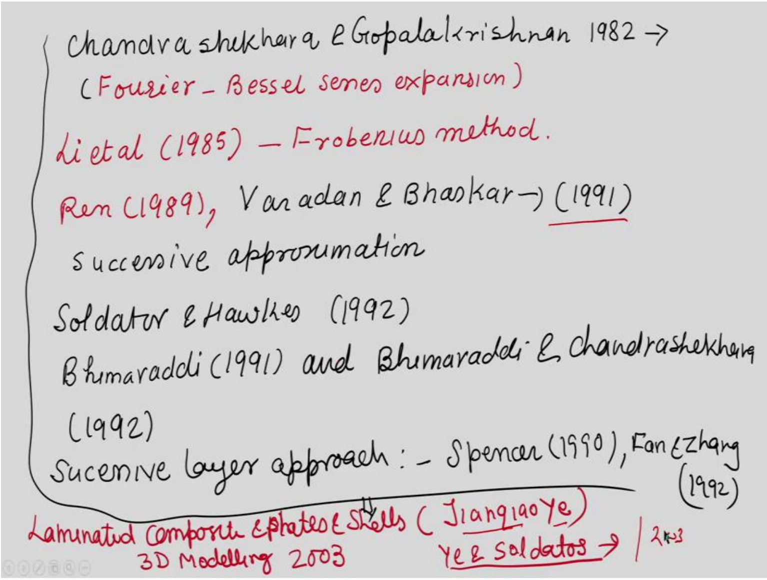

In this field, the three-dimensional solutions are very rare or I would like to say that for a special case, special boundaries we can get the three-dimensional solutions.

In that direction, the first solution was reported by Chandrashekhar and Gopalkrishnan in 1982 using the Fourier and Bessel series expansion. They solved the three-dimensional problem of a cylindrical shell. Then Renetal, Varadan, and Bhaskar solved the cylindrical composite laminated shell.

Soldator and his group Soldator and Hawkes using the successive approximation technique have solved a number of problems of cylindrical shells with simply supported boundary conditions free vibration of cylindrical laminated shells and buckling of laminated cylindrical shells and they also tried for the clamped boundary condition cases. Bhimaraddi and Chandrashekhara in the same 1991 and 1992 solve the problem of free vibration and static of cylindrical shells.

In this direction handful of papers are available. A very famous book “Laminated composites and plates and shells 3D modeling" by Jiangqoye was published in 2003. In this book the three-dimensional solutions of plate and shell made up of composites started from bending, free vibration, and buckling have been discussed and it is one of the good books for the case of three-dimensional solutions.

The other issues are also resolved, such as how to develop an algorithm for solving differential equations specifically for first-order differential equations in with state-space technique, and then the direct calculation of exponential of a matrix and various techniques are explained.

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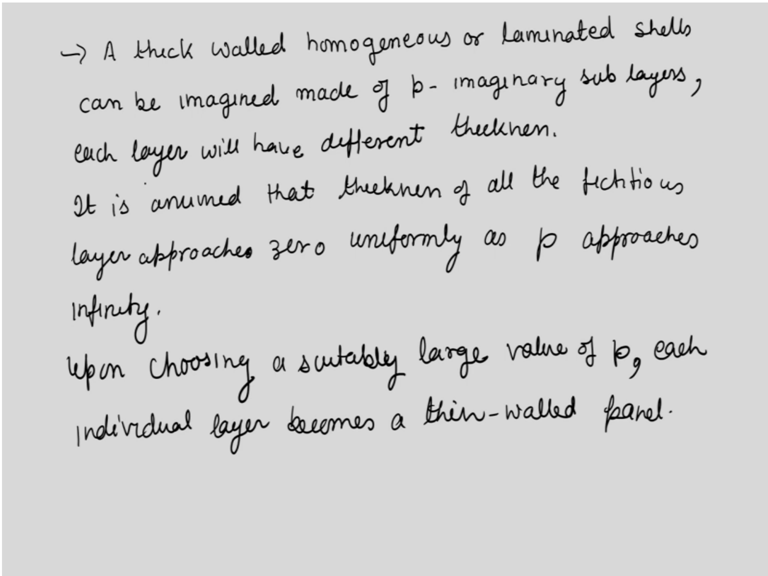

In the case of a successive layer approach; let us assume a layer is made of p-imaginary sub-layers and each layer will have some different thickness. It is assumed that the thickness of all fictitious layers approaches zero uniformly as p approaches infinity. If you divide it into an infinite set of layers then definitely the thickness of that imaginary layer is going to be 0.

Upon choosing a suitably large value of a p, even that in thickness, thickness is small, if you divide it into further 10 layers or 20 layers then it is very very small. In that case, it becomes an individual layer and a thin-walled panel and the governing differential equation can be solved easily.

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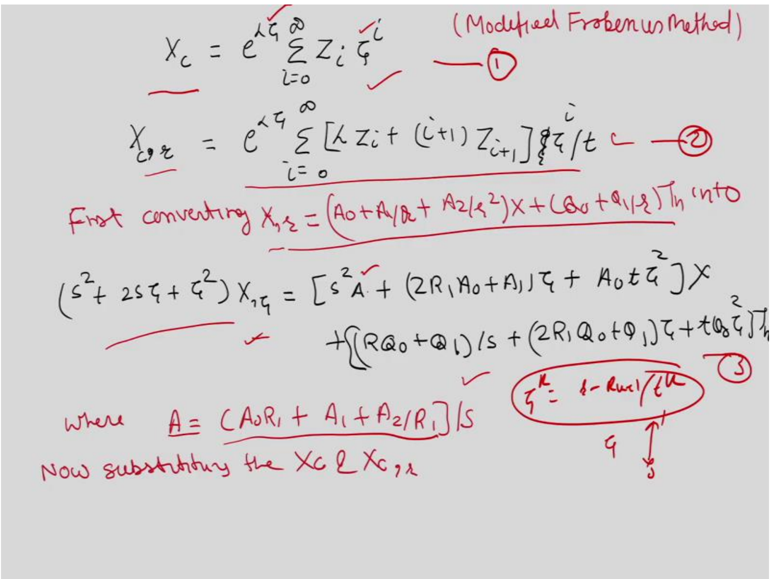

There is another approach which is the modified Frobenius method. Using this method professor Kapuria and Dumir in 1997 or 1996 developed three-dimensional solutions for piezoelectric shell panels and finite shells in which some layers are made of composite and some layers are made of piezoelectrics.

The modified Frobenius series method is one of the most advantageous methods compared to other methods here equations are directly solved. Therefore, the complementary solution can be expressed like this:

$$X^{C}\left(\varsigma\right)=e^{\lambda\varsigma}\sum_{i=0}^{\infty}Z_{i}\varsigma^{i}\quad equation(1).$$

If we take the derivative with respect to r or with respect to ς, then we can find the second equation:

$$X_{c,r}=e^{\lambda\varsigma}\sum_{i=0}^{\infty}[\lambda Z_{i}+(i+1)Z_{i+1}]\varsigma^{i}/t^{c}\quad equation(2).$$

The governing equation was $X,_{r}=\left(A_{0}+\frac{A_{1}}{r}+\frac{A_{2}}{r^{2}}\right)X+\left(Q_{0}+\frac{Q_{1}}{r}\right)T_{n}$.

First, we have to convert it into ς because we have taken a non-dimensionalization coordinate ς. ς varies in each layer from 0 to 1.

First of all, we have to convert $X,_r$ into ς.

$\varsigma^K = r - \frac{R_{k+1}}{t_k}$. Using those concepts:

$$\left(s^2 + 2s\varsigma + \varsigma^2\right)X,_\varsigma = \begin{matrix} \left[s^2 A + \left(2R_1A_0 + A_1\right)\varsigma + A_0 t\varsigma^2\right]X + \\ \left[\left(RQ_0 + Q_1\right)/s + \left(2R_1Q_0 + Q_1\right)\varsigma + tQ_0\varsigma^2\right]T_n \quad equation(3) \end{matrix}.$$

Where, A contains $\left(A_0R_1 + A_1 + A_2/R_1\right)/s$ these matrices.

Now, substituting equation 1, equation 2, and equation 3.

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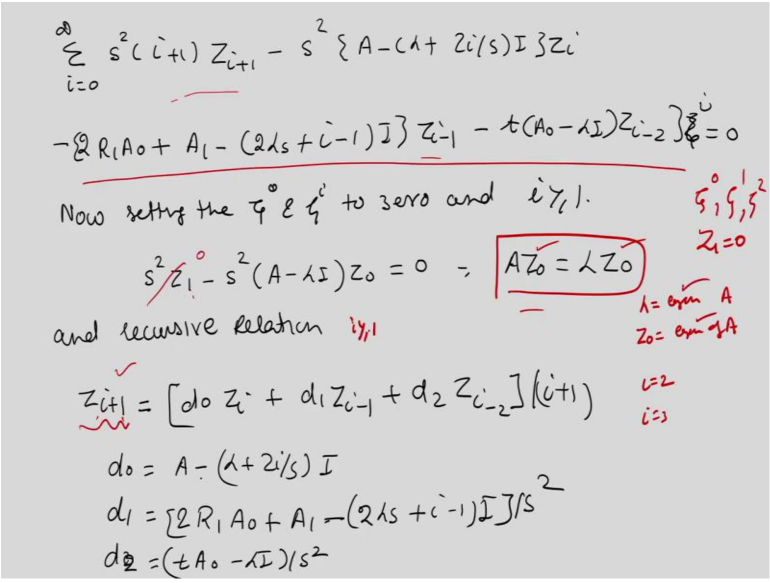

Finally, that equation becomes like this:

$$\sum_{i=0}^{\infty} s^2(i+1)Z_{i+1} - s^2\left\{A - C^\lambda + Z_i(s)I\right\}Z_i -$$
$$\left\{2R_1A_0 + A_1 - \left(2\lambda_s + i - 1\right)I\right\}\varsigma - 1 - t\left(A_0 - \lambda I\right)\varsigma_{i-2}\right\}\varsigma^i = 0$$

Here you can see that these are the coefficients of ς. Some are the coefficient of ς^0, some are the coefficient of ς^1, ς^2 and ς^3. Now setting the coefficients of ς^0 and ς^i, $i \geq 1$, and setting it to 0 leads to an eigenvalue problem:

$s^2 Z_1 - s^2\left(A - \lambda I\right)Z_0 = 0 = AZ_0 = \lambda Z_0$.

Where lambda is the eigenvalue of a matrix A and Z_0 is the eigenvector of matrix A, where, matrix A is $\left(A_0 R_1 + A_1 + A_2 / R_1\right)$, it contains all the terms it is not just simply A and then we can find the recursive relations when we go for setting i = 2 = 0, i = 3= 0.

From there we can find:

$$Z_{i+1} = \left[\alpha_0 Z_i + \alpha_1 Z_{i-1} + \alpha_2 Z_{i-2}\right] / \left(i+1\right)$$

Z_0 is taken out, we assumed, in this case, Z_1 = 0, it is valid for $i \geq 1$. Because when i is put 1 it becomes 2. Therefore, we can find the recursive relations.

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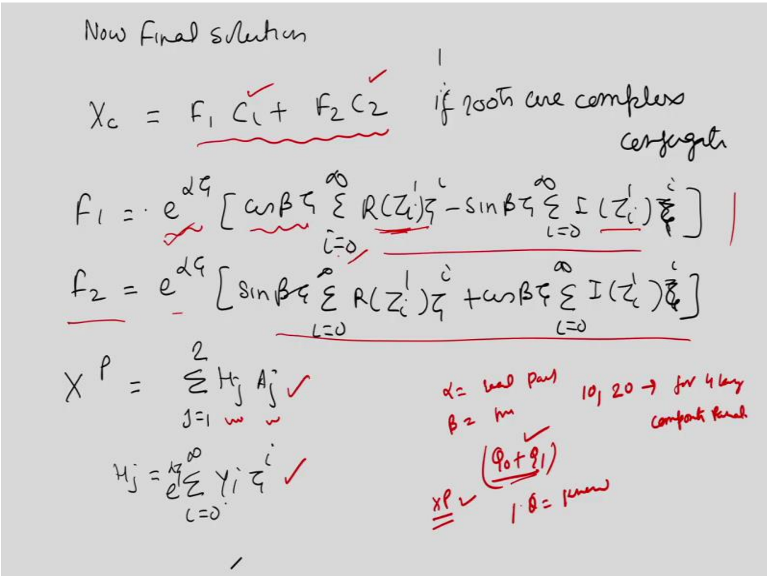

Finally, the solutions can be written in terms of a series. Now, we have found the eigenvector and an eigenvalue.

We can write the solution the same way as I discussed previously if roots are real or roots are complex conjugate, then we can write:

$$X_c = F_1 C_1 + F_2 C_2 .$$

$$F_1 = e^{\lambda \varsigma}\left[\cos \beta\varsigma \sum_{i=0}^{\infty} R\left(z_1^i\right)\varsigma^i - \sin \beta\varsigma \sum_{i=0}^{\infty} I\left(z_1^i\right)\varsigma^i\right].$$

We found out that for the case of composite laminates not more than ten terms are required in a series. For the case of a constant, it was just a real part of the eigenvector, but now it is a summation of a series from i to ∞. One can find the convergence term and add all these terms together and multiply with $e^{\alpha\varsigma}$.

Similarly, F_2 can be obtained as $e^{\lambda\varsigma}\left[\sin\beta\varsigma\sum_{i=0}^{\infty}R\left(z_i^1\right)\varsigma^i+\cos\beta\varsigma\sum_{i=0}^{\infty}I\left(z_i^1\right)\varsigma^i\right]$.

Sometimes 10 terms or 20 terms are required for a four-layer composite panel. In this way, we can write the complementary solutions where α is the real part of the root and β is the imaginary part of the root. Since we have Q_0 *and* Q_1 temperature loading due to that, we can write the particular solution.

The particular solution $X^P = \sum_{i=0}^{2} H_j A_j$

$$H_j = e^{\lambda\varsigma}\sum_{i=0}^{\infty}Y_i\varsigma^i.$$

As I have said that for a known temperature let us say θ is known to you, we can find a particular solution like this. As the mechanical and thermal equations are not coupled, we can solve the temperature equation independently and use the results here for getting the particular solution under thermal loading. If there is no thermal loading then this particular part will not be there.

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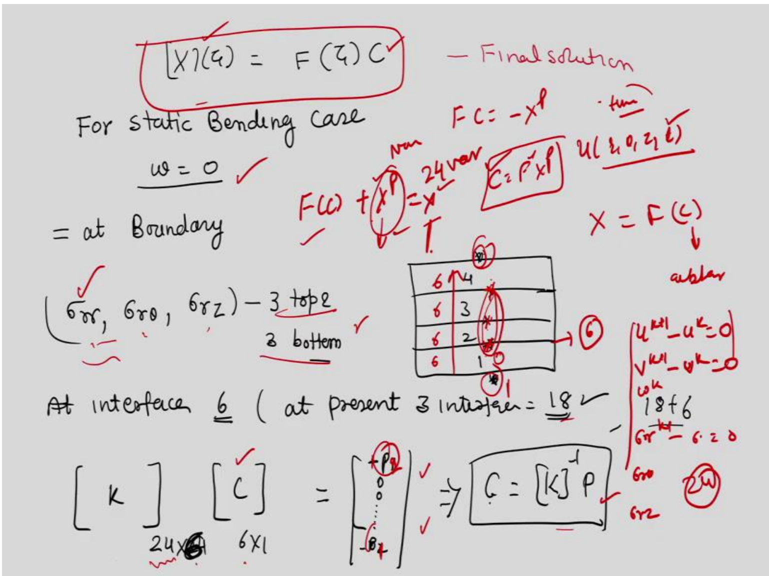

The final expression is $[X](\varsigma) = F(\varsigma)C$, in which we add the particular solution as well as the complementary solutions.

As we have started from taking time like u is the function of r, θ, z and time. For the case of a static bending ω = 0 and has no thermal loading.

The equation will be X = FC, where C is the arbitrary constant.

Now, we are going to solve for 4-layer, 5- layer, or 10-layer composite laminates. Each layer will have the variables u, v, w, σ_{rr}, $\sigma_{r\theta}$, σ_{rz}. How many variables we are going to have?

If we take an example of a 4-layer composite shell panel then at each interface we need to satisfy 6 boundary conditions let us say $u^{k+1} - u^k$ = 0.

Similarly, $v^{k+1} - v^k$ = 0.

The same way we are going to satisfy σ_{rr}, $\sigma_{r\theta}$, and σ_{rz}. 6 variables in each layer, we are now having 24 variables.

Can we set up the 24 governing equations? Yes, 18 from these three interfaces and 3 from the bottom, and 3 from the top. We can have 24 boundary conditions. We can apply that let us say three σ_{rr}, $\sigma_{r\theta}$, σ_{rz}. Three at the top and three at the bottom and six at the interfaces.

Another matrix is formed which is K matrix, then $[K]$ = 24 X 6 and $[C]$ = 6 X 1 and the load that boundary conditions at σ_{rr} = $+P_2$ at the top and $-P_1$ at the bottom. This is a linear algebraic equation.

We can find C = $[K]^{-1} P$.

In this way, a cylindrical shell under mechanical loading can be solved. If we say that under the thermal loading, then:

$$F(C) + X^P = \mathrm{X}$$

Then again, we have to apply the temperature.

There will be no pressure, only the temperature terms will be there. Temperature loading terms come from these particular solutions.

$$FC = -X^P$$

$$C = F^{-1}X^P.$$

In this way the constants can be found and substituting back gives the variables u, v, w at any ς location.

Along the thickness, we can find all the variables under pressure loading under thermal loading.

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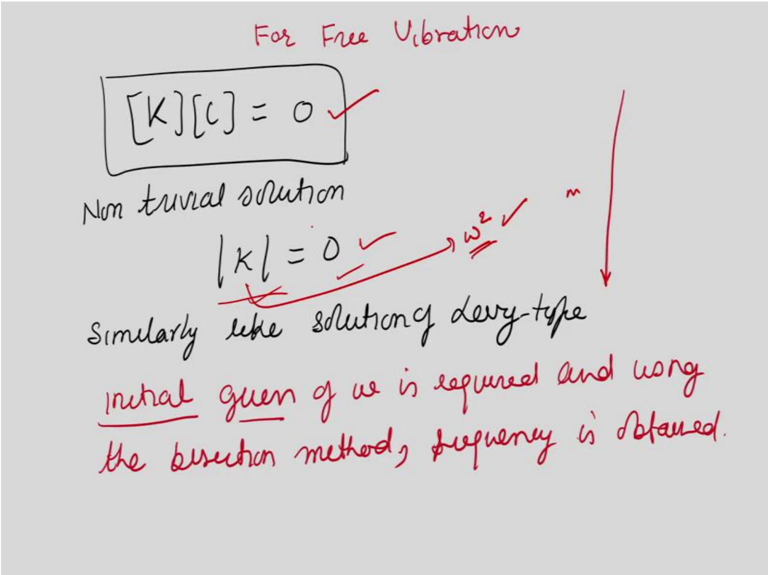

If it is a case of a free vibration then the pressure is going to be 0, there will be no temperature then the right-hand side $[K][C] = 0$.

For that case, the non-trivial solution will exist only when its determinant $|K| = 0$.

The procedure I already explained is how to get the natural frequency of such a system in the Levy solution of a cylindrical shell. For this system, there is ω^2 if this is the natural frequency then the determinant $|K| = 0$.

By assuming an initial guess hit and trial procedure then we can proceed and find the bounds and ultimately, we can find the frequency of the system. In this way, an infinite set of frequencies can be obtained whether in the case of two-dimensional shell theory we can get only five natural frequencies at a time for a particular m or n, but here for a particular m or n we will get an infinite set of frequency.

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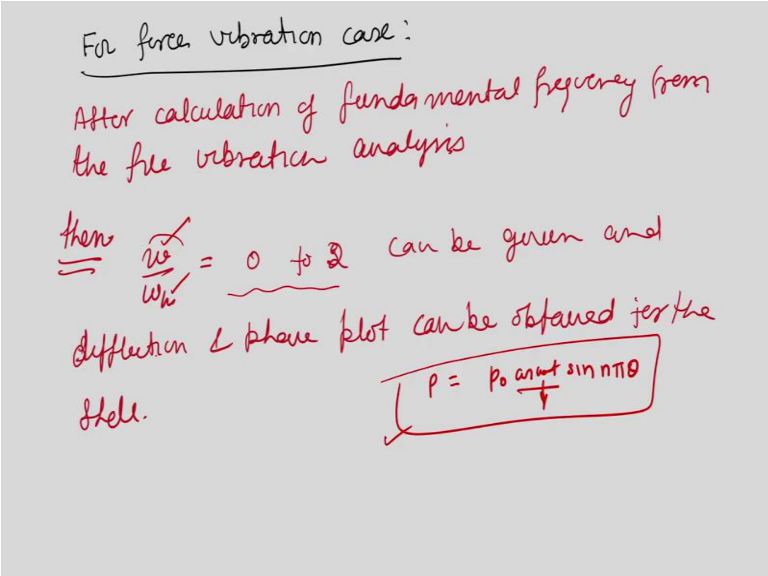

Now, force vibration case: after calculating the fundamental frequencies, we can take in the ratio of force upon the natural frequency, and then we can proceed from 0 to 2, 0 to 3 and in the loading case:

P = $p_0 \cos \omega t \sin n\pi\theta$.

Here, putting the value load function and deflection and phase plots can be obtained and in this way, free force vibration of the system can be obtained. For that we must know the forcing frequency; if we know the forcing frequency, the same way as pressure loading, we can obtain the results.

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Modelling of cylindrical shell in ABAQUS & its static analysis tutorial is uploaded. You can go through it.

Modeling of a cylindrical shell in an abacus for static analysis is uploaded as a small tutorial.

You can go through in that you will understand how to model a cylindrical shell in commercial software and how to get the results. Ultimately, whenever you develop your theory or develop a theoretical model you want to verify or compare, in commercial software you can similarly model that and can analyze.

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Finite Shell

$$\left(\begin{array}{l} \varepsilon_{\theta\theta} = (u + v_{,\theta})/r \\ \varepsilon_{zz} = w_{,z} \\ \varepsilon_{rr} = u_{,r} \\ \gamma_{zr} = w_{,r} + u_{,z} \\ \gamma_{r\theta} = (u_{,\theta} - v)/r + v_{,r} \\ \gamma_{\theta z} = v_{,z} + w_{,\theta}/r \end{array}\right)$$

Now, I will briefly explain the finite shell model. In the case of an infinite shell, we assumed that $w_{,z}$ is going to be 0, where w is just a function of r and theta. But, for the case of a linear finite shell, this type of strain displacement in cylindrical coordinate is considered.

$$\varepsilon_{\theta\theta} = \frac{(u + v,_{\theta})}{r}$$
$$\varepsilon_{zz} \neq 0$$
$$\varepsilon_{rr} = u,_r$$
$$\gamma_{zr} = w,_r$$
$$\gamma_{r\theta} = \frac{(u,_{\theta} - v)}{r} + v,_r$$
$$\gamma_{\theta z} = v,_z + \frac{w,_{\theta}}{r}$$

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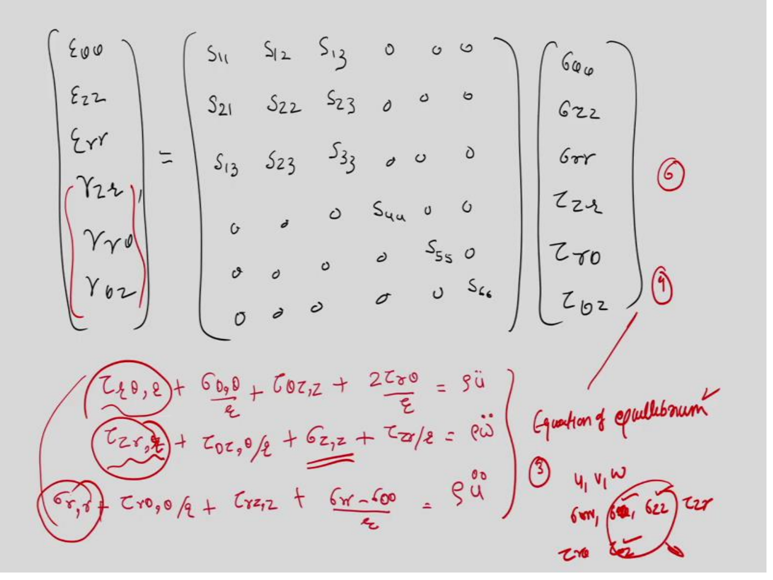

And the following constitutive relations are considered:

$$\begin{bmatrix} \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \varepsilon_{rr} \\ \gamma_{zr} \\ \gamma_{r\theta} \\ \gamma_{\theta z} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rr} \\ \tau_{zr} \\ \tau_{r\theta} \\ \tau_{\theta z} \end{bmatrix}$$

And following equations of equilibrium are considered:

$$\tau_{r\theta,r} + \frac{\sigma_{\theta,\theta}}{r} + \tau_{\theta z,z} + \frac{2\tau_{r\theta}}{r} = \delta\ddot{u}$$

$$\tau_{zr,r} + \frac{\tau_{\theta z,\theta}}{r} + \sigma_{z,z} + \frac{\tau_{r\theta}}{r} = \delta\ddot{w}$$

$$\sigma_{r,r} + \frac{\tau_{r\theta,\theta}}{r} + \tau_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \delta\ddot{u}$$

You see that we have 3 equations of equilibrium and 6 constitutive relations. Ultimately, we have 9 equations, variables u, v, w and 6 stresses σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, τ_{zr}, $\tau_{r\theta}$, and $\tau_{\theta z}$. Similarly, here also $\sigma_{\theta\theta}$, σ_{zz} and $\sigma_{\theta z}$ can be expressed as a dependent variable.

If we ultimately convert these three equations and from here $\sigma_{r\theta,r}$, there must be some $\sigma_{\theta,\theta}$, $\sigma_{r,r}$ and $\sigma_{rz,r}$. From these three equations, we can develop six differential equations.

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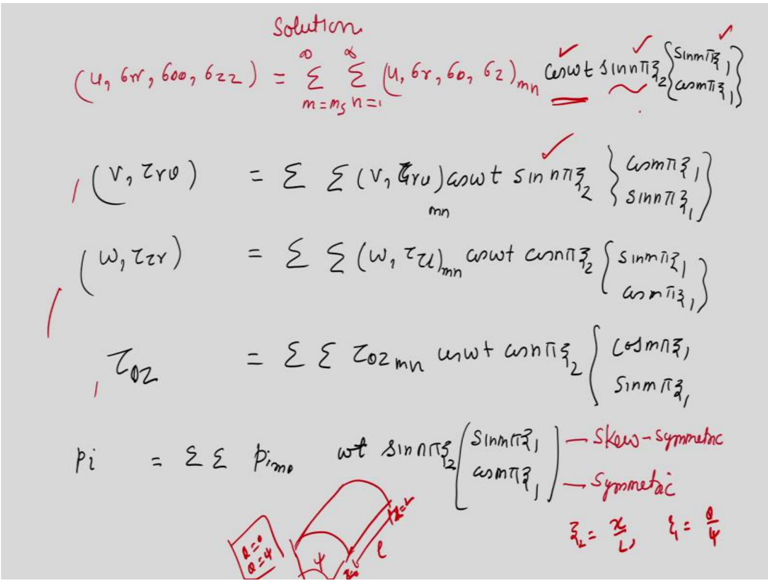

And, we assume that this is a finite shell along the length L and this total variation is let us say ψ. We can have a boundary condition at x = 0 & L or we can say Z = 0 & L and then we have a boundary condition θ = 0 & ψ.

If we choose that all of these are simply supported Navier solution.

We can assume the solutions like $\sin n\pi\xi_2$ along the $\xi_2 = \frac{x}{L}$.

$\xi_1 = \dfrac{\theta}{\psi}$ along θ direction, along x direction and the time function can be expressed $\cos\omega t$.

We can express the variables like this:

$$(u, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}) = \sum_{m=ns}^{\infty}\sum_{n=1}^{\infty}(u, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz})_{mn} \cos\omega t \sin n\pi\xi_2 \begin{Bmatrix} \sin m\pi\xi_1 \\ \cos m\pi\xi_1 \end{Bmatrix}$$

$$(v, \tau_{r\theta}) = \sum_{m=ns}^{\infty}\sum_{n=1}^{\infty}(v, \tau_{r\theta})_{mn} \cos\omega t \sin n\pi\xi_2 \begin{Bmatrix} \cos m\pi\xi_1 \\ \sin m\pi\xi_1 \end{Bmatrix}$$

$$(w, \tau_{zr}) = \sum_{m=ns}^{\infty}\sum_{n=1}^{\infty}(w, \tau_{zr})_{mn} \cos\omega t \cos n\pi\xi_2 \begin{Bmatrix} \sin m\pi\xi_1 \\ \cos m\pi\xi_1 \end{Bmatrix}$$

$$\tau_{\theta z} = \sum_{m=ns}^{\infty}\sum_{n=1}^{\infty}(\tau_{\theta z})_{mn} \cos\omega t \cos n\pi\xi_2 \begin{Bmatrix} \cos m\pi\xi_1 \\ \sin m\pi\xi_1 \end{Bmatrix}$$

$$p_i = \sum_{m=ns}^{\infty}\sum_{n=1}^{\infty}(p_i)_{mn} \cos\omega t \sin n\pi\xi_2 \begin{Bmatrix} \sin m\pi\xi_1 \\ \cos m\pi\xi_1 \end{Bmatrix}$$

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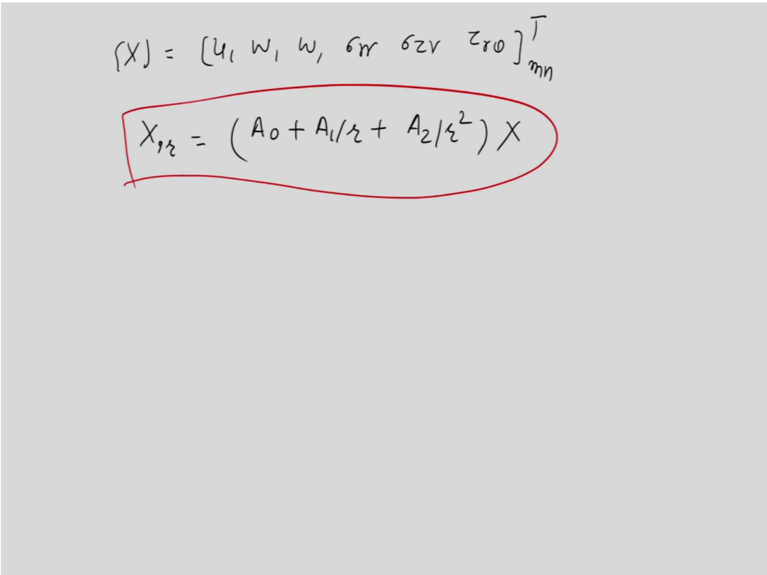

If we assume solutions along x directions or z-direction and θ direction then the set of governing equations reduces to only differential equations along the radial coordinate. And finally, the governing equation is expressed the same way and it can be solved similarly to the previous case.

$$[X] = [u, v, w, \sigma_{rr}, \sigma_{zr}, \sigma_{r\theta}]$$

$$X,_r = \left(A_0 + \frac{A_1}{r} + \frac{A_2}{r^2} \right) X$$

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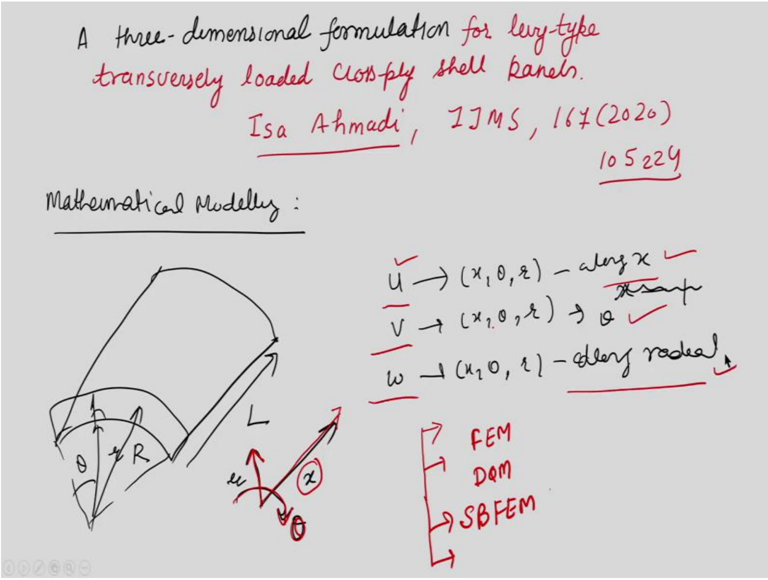

In this lecture, I am going to discuss a very recent article that is the three-dimensional formulation for levy-type transversely loaded cross-ply shell panels. I would like to say that even till date the solutions for a levy-type cylindrical shell panel are not very much reported. Only three or four papers are reported in these directions specifically for the analytical solutions we are interested.

We can do the finite element solutions; we can do some differential quadrature methods or scale boundary finite element method SBFEM. There is a number of techniques by which we can solve the cylindrical shell panel or any kind of shell panel subjected to different boundary conditions and loading, but the development of an analytical solution is difficult.

It is the very recent year 2020 paper by Isa Ahmadi and I would like to discuss that in this paper beautifully the governing equations are converted into only x and θ by using the concept of Lagrangian interpolation. I am going to explain this.

Let us first go with that paper; in that paper, the longitudinal axis is considered as x, where the radial axis is considered as r and the circumferential axis is considered as θ where u displacement is taken along x-axis, v taken along θ axis and w is taken along the radial axis.

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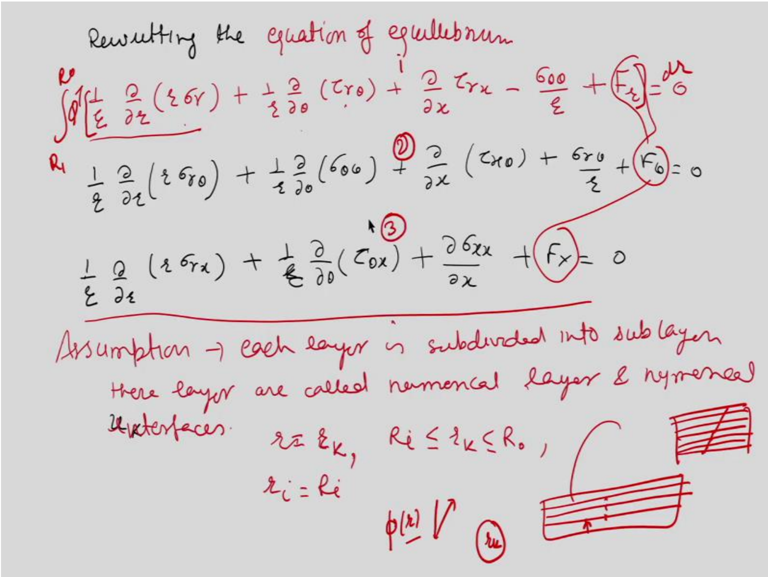

Here, I would like to point out that in this paper the equation of equilibrium is written slightly differently not as I have written previously or given in some other books. If we write the equations like this then the governing equations can be developed very smoothly.

$$\int_{R_1}^{R_0} \phi^T \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta}) + \frac{\partial}{\partial x}\tau_{rx} - \frac{\sigma_{\theta\theta}}{r} + F_r \right] = 0 \;\; equation(1)$$

$$\int_{R_1}^{R_0} \phi^T \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) + \frac{\partial}{\partial x}\tau_{x\theta} + \frac{\sigma_{r\theta}}{r} + F_\theta \right] = 0 \; equation(2)$$

$$\int_{R_1}^{R_0} \phi^T \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rx}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta x}) + \frac{\partial}{\partial x}\sigma_{xx} + \frac{\sigma_{r\theta}}{r} + F_x \right] = 0 \;\; equation(3).$$

F_r, F_θ, and F_x are the body forces.

This is the very first step the equation of equilibrium is written slightly differently. If you open all these things, it becomes the same, but now they have been written in such a way. The next assumption is that in each layer it is sub assume that this is made of some imaginary layers. These imaginary layers are called numerical layers and these interfaces are called numerical interfaces.

In these layers a radial coordinate varies and let us say r_k will be inner of that layer and outer of that layer radius. We can say that we can divide a layer into sub-layers. The reason behind dividing mathematically or imaginary sub-layers is that they assumed a function ϕ that varies along the radial thickness direction. It varies linearly from one layer to another layer.

In an actual sense if you divide a single layer into two, let us say from here, it will go linearly from here to here, but if you divide it into some small layer, it may take a quadratic or any kind of a variation. The accuracy of inter-laminar stresses will be very high for such cases.

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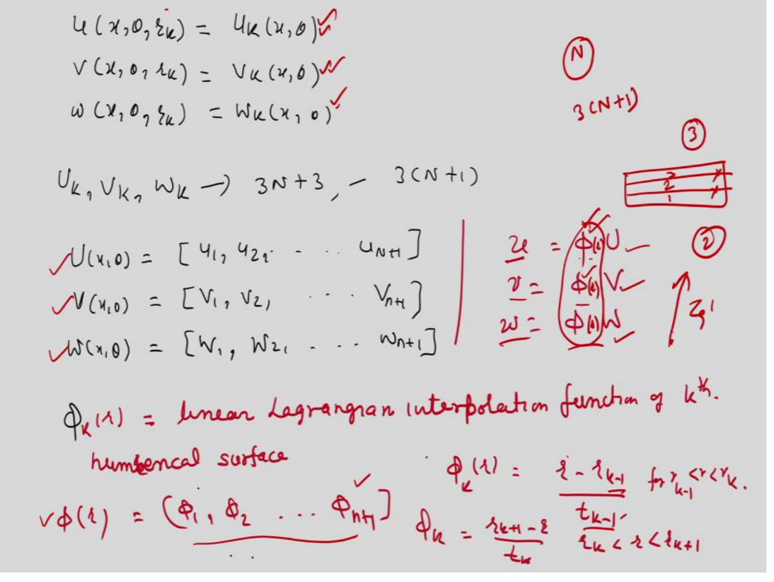

Let us say that u, v, w are a function of (x, θ) and r_k is the radius at K-th layer and U_k, V_k, and W_k is a function of (x,θ) only.

Each layer will have three displacements. We will be going to have 3N + 1 variable; 3 at the top and 3 at the bottom. In this way, 3N + 1 variables are required.

$$U(x,\theta)\,[u_1, u_2, ...u_{n+1}]$$

$$V(x,\theta) = [v_1, v_2, ...v_{n+1}]$$

$$W(x,\theta) = [w_1, w_2, ...w_{n+1}]$$

These layers we can easily divide into n numerical layers. Let us say if we divide it into two numerical layers 1, 2, and 3, we will have 3 layers and 2 interfaces.

u_k (x, θ). And what about r_k? Along the radial direction, he has chosen a function ϕ like ς varies from 0 to 1.

Here ς varies from $\varphi_k(r) = \dfrac{r - r_{k-1}}{t_{k-1}}$, the bottom layer coordinates upon the thickness of that bottom layer.

This φ is totally a function of r. In each layer, this φ will also be different.

We will have n + 1.

Ultimately, u = $\phi(i)U$

v = $\phi(i)V$

w = $\phi(i)W$

Now, from here this φ is varying from each layer.

If you remember in the case of first-order shear deformation theory from bottom to top linearly varying, we have taken a function ς, but here a function is taken in each layer it varies linearly and we have as many layers and variables. This is the known coordinate system ultimately.

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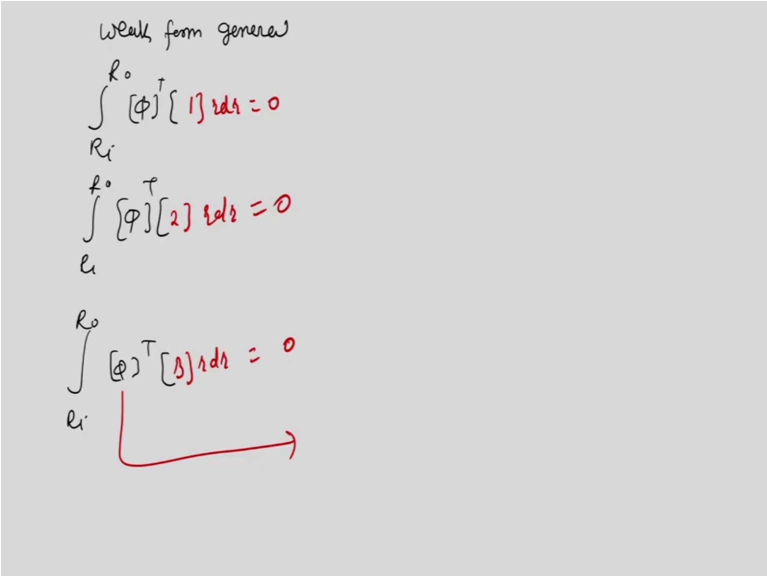

Now, we can develop the weak form of this. This is the equation of equilibrium if we multiply to develop a weak form of this:

Let us say, $\int_{R_i}^{R_0} [\varphi]^T \, rdr$.

Ultimately,

$$\int_{R_i}^{R_0} [\varphi]^T \left[equation(1)\right] rdr \;= 0$$

$$\int_{R_i}^{R_0} [\varphi]^T \left[equation(2)\right] rdr = 0$$

$$\int_{R_i}^{R_0} [\varphi]^T \left[equation(3)\right] rdr \;= 0.$$

Where φ is the Lagrange interpolation function. This is the weak form generated.

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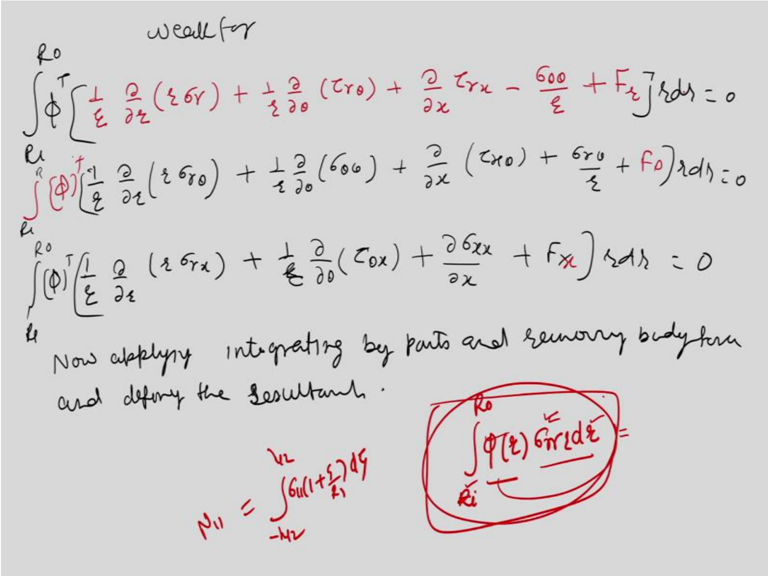

Now, using the integrating by parts we can develop the finally equations as following:

$$\int_{R_i}^{R_0}[\varphi]^T\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{r\theta})+\frac{\partial}{\partial x}\tau_{rx}-\frac{\sigma_{\theta\theta}}{r}+F_r\right]rdr=0$$
$$\int_{R_i}^{R_0}[\varphi]^T\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{r\theta})+\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta\theta})+\frac{\partial}{\partial x}\tau_{x\theta}+\frac{\sigma_{r\theta}}{r}+F_\theta\right]rdr=0 \quad .$$
$$\int_{R_i}^{R_0}[\varphi]^T\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rx})+\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{\theta x})+\frac{\partial}{\partial x}\sigma_{xx}+\frac{\sigma_{r\theta}}{r}+F_x\right]rdr=0$$

Now, we can apply the integration by part and sum using this concept. Here, φ (r), R_i to R_0. What is the other variable rdr?

This is only a function of r. If you remember:

$$N_{11} = \int_{-h/2}^{h/2}\sigma_{11}\left(1+\frac{\varsigma}{R_1}\right)d\varsigma$$

The same way, this $\int_{R_i}^{R_0}\varphi(r)\sigma_{rr}rdr$ can be defined as some coefficients, we can obtain analytically the value of those functions.

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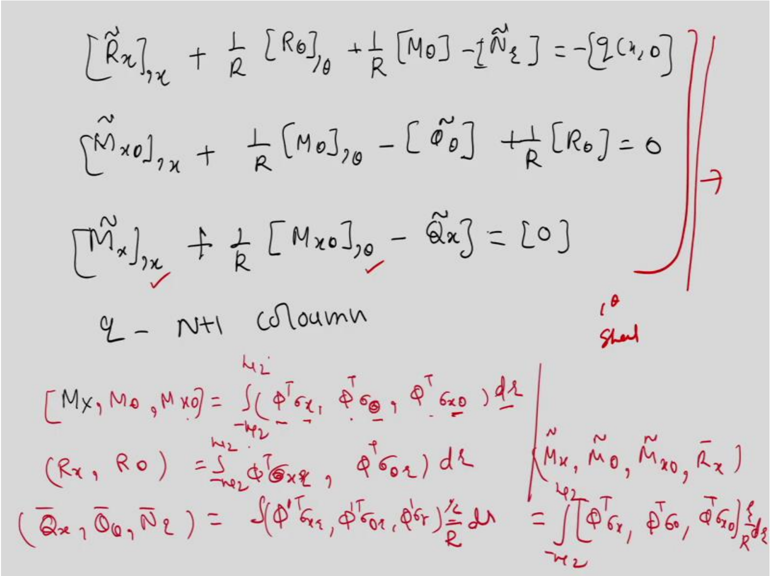

If we do so it gives you three partial differential equations in x and θ.

$$\left[\tilde{R}_x\right]_{,x} + \frac{1}{R}\left[R_\theta\right]_{,\theta} + \frac{1}{R}\left[M_\theta\right] - \left[\tilde{N}_r\right] = -q(x,\theta)$$

$$\left[\tilde{M}_{x\theta}\right]_{,x} + \frac{1}{R}\left[M_\theta\right]_{,\theta} - \left[\tilde{Q}_\theta\right] + \frac{1}{R}\left[R_\theta\right] = 0$$

$$\left[\tilde{M}_x\right]_{,x} + \frac{1}{R}\left[M_{x\theta}\right]_{,\theta} - \left[\tilde{Q}_x\right] = 0$$

By doing so, it reduces equations into x and θ

Where, $\left[M_x, M_\theta, M_{x\theta}\right] = \int_{-h/2}^{h/2} \left(\varphi^T\sigma_x, \varphi^T\sigma_\theta, \varphi^T\sigma_{x\theta}\right) dr$.

The same way, $\left[R_x, R_\theta\right] = \int_{-h/2}^{h/2} \left(\varphi^T\sigma_{xr}, \varphi^T\sigma_{\theta r}\right) dr$

$$\left[\bar{Q}_x, \bar{Q}_\theta, \bar{N}_r\right] = \int_{-h/2}^{h/2} \left(\varphi^T\sigma_{xr}, \varphi^T\sigma_{\theta r}, \varphi^T\sigma_r\right)\frac{r}{R}dr$$

$$\left[\tilde{M}_x, \tilde{M}_\theta, \tilde{M}_{x\theta}, \bar{R}_x\right] = \int_{-h/2}^{h/2} \left(\varphi^T\sigma_x, \varphi^T\sigma_\theta, \varphi^T\sigma_{x\theta}\right)\frac{r}{R}dr \ .$$

From here definitely, we can apply the levy-type boundary condition, we have only along x and θ.

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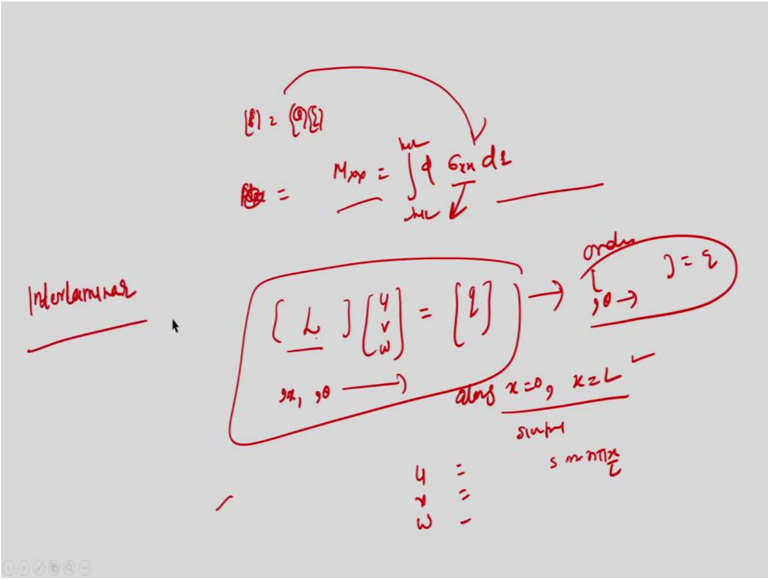

Using the shell constitutive relations, you can say:

$$[\sigma] = [Q][\varepsilon]$$

And let us say $M_{xx} = \int_{-h/2}^{h/2} \varphi \sigma_{xx} dr$.

From here substituting these things, it can be expressed in terms of shell constitutive relations, and substituting back here ultimately these three equations are converted into displacements u, v and w and let us say linear operator is equal to loading something.

Now, this linear operator may have a derivative with respect to x and a derivative with respect to θ and sometimes it may have a double derivative and so on. If we want to solve this type of problem, let us say along x = 0 & L is simply supported.

We can assume u, v, w accordingly whether it comes sine or cosine along x axis $n\frac{\pi x}{L}$, if we substitute into these equations. Ultimately, this equation is reduced to an ordinary differential equation in θ coordinate. Those ordinary differential equations can be solved analytically. Now, it does not remain in the partial differential equation.

And, the most important part is that these equations are with constant coefficients because the function of r is taken into the integration like in our previous cases. It will be just like an algebraic equation; we can solve it. I would like to say that is a slightly different approach, we have tried to develop using the Hamilton principle, but here by just using three equations of equilibrium those equations are obtained.

And, it has been found that results are in very good agreement, and most importantly the interlaminar stresses. What is that? Why we are interested to have all kinds of difficult formulations and so on?

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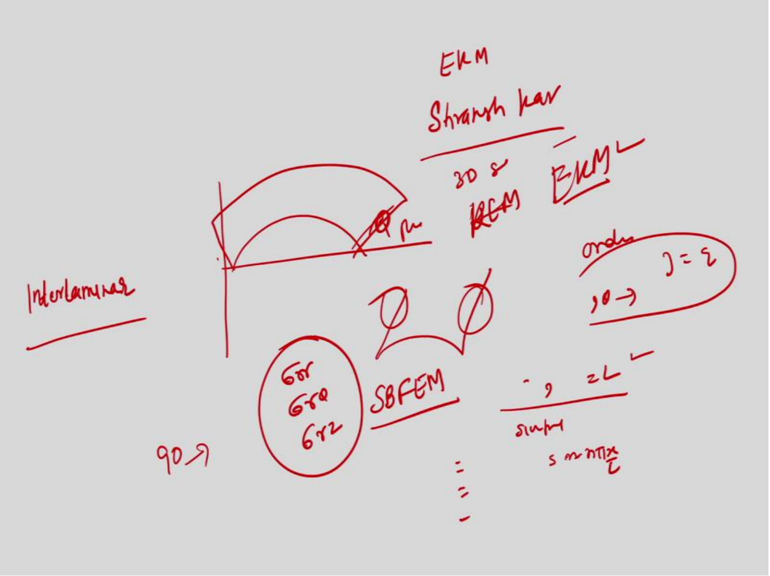

When you have a different boundary condition at the edges if this edge is free then the stress variation over here is very huge so that σ_{rr}, $\sigma_{r\theta}$ and σ_{rz} causes the delamination. We are interested to find an accurate estimation of σ_{rr}, $\sigma_{r\theta}$ and σ_{rz}. In this way, a levy-type solution is presented.

There is some more type of solutions available like using the extended kantorovich method recently one of my Ph.D. students Shransh Kar developed a three-dimensional solution for a cross-ply panel using the EKM approach. Where that is subjected to arbitrary boundary support conditions solved using the multi-term approach.

Recently, I have seen the free vibration analysis of a cylindrical shell using the scaled bound finite element method is solved. In this way, we can get the solutions of cylindrical shells, and sometimes the other kind of shells like, spherical shells and conical shells are also tried. These analytical solutions are most important.

We should try to develop the first analytical solution if not possible then we can go for a finite element solution. In this direction, I would like to say that in the nineties the first solution of static bending of composite laminate shells was proposed, but now it is 2020 that solutions of a levy type cylindrical shell are proposed.

It took 30 years in that direction of analytical solution whether you talk about EKM or a slightly new technique. One can try in this direction can we develop some series of analytical solutions or can we try to solve a more complex problem like cylinders with the hole or the buckling of the cylinders or the fatigue analysis of the cylinders.

I would like to say that even the cylinders under impact loading, composite cylinders under impact loading, and composite cylinders under blast loading under fatigue are still the area of research. Some have different configurations, and different cut-outs, some have some stiffness, and a variety of loading is possible in the case of a cylinder. We can try these things.

Thank you very much.