Theory of Composite Shells

Dr. Poonam Kumari

Department of Mechanical Engineering

Indian Institute of Technology, Guwahati

Week - 08

Lecture - 26

Development of three-dimensional buckling solution

Dear learners welcome to week-08, lecture-03. In this lecture, I shall explain briefly the development of a three-dimensional buckling solution for a cylindrical shell.

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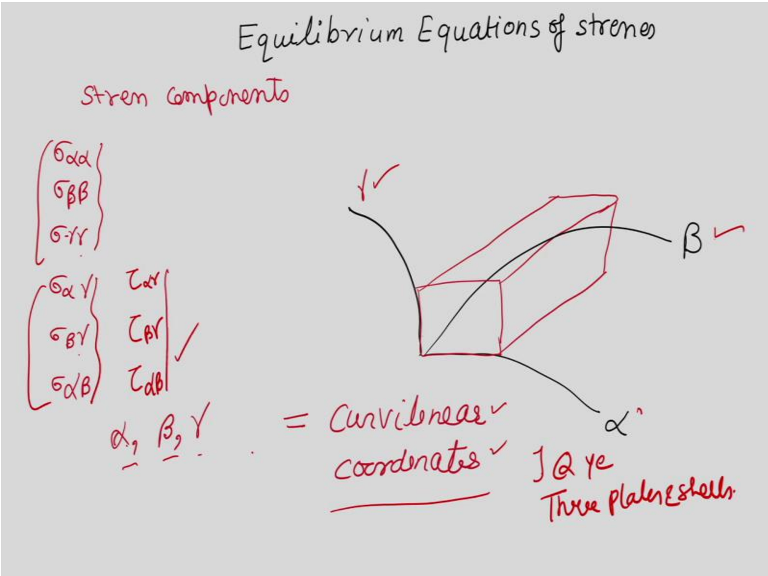

First of all, for the sake of completeness, we developed governing equations for two-dimensional shell theories and first-order shear deformation shell theories. We developed the strain displacement relations and the governing partial differential equations in the lame parameters form so that one can finally get the differential equation for their respective geometry, like if somebody wants to solve a spherical shell, ellipsoidal shell, or a conical shell.

But what I feel in the present case, one must also have some idea that what is the form of an equilibrium equation of stresses in curvilinear coordinate.

Because three-dimensional solutions, it is an essential part; of whether we are going to develop a three-dimensional solution for the case of a shell, plate, or beam. The three-dimensional equation of equilibrium governing equation, sometimes we call it a momentum equation is the variation shell part.

If we know that in the form of a curvilinear coordinate system, then we can develop a solution for the generalized shell, like a doubly curved shell or the conical shell which are not common as cylindrical shells and spherical shells.

In most of the books, the equilibrium equations are given in a spherical and cylindrical coordinate system; but suppose if you want to do for an elliptical shell or if you want to do for a conical shell, then these equations are not given in journal books.

For that purpose, I would like to explain or just for the sake of completeness going to put those equations here; these equilibrium equations are given in a book of JQ Ye three-dimensional plate and shell solutions.

In that book three curvilinear parameters α , β, and γ are taken and this is the volume element and α , β, and γ are the curvilinear coordinates. The corresponding stresses are $\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, and\ \sigma_{\gamma\gamma}$ and the shear stresses are $\sigma_{\alpha\gamma}$, $\sigma_{\beta\gamma}$, and $\sigma_{\alpha\beta}$.

You may also say that in terms of τ, because we used to say shear stress is represented in terms of τ. In some of the books generally for the consistency, they represent the shear stresses in terms of σ but in some of the books, it is written in terms of a τ.

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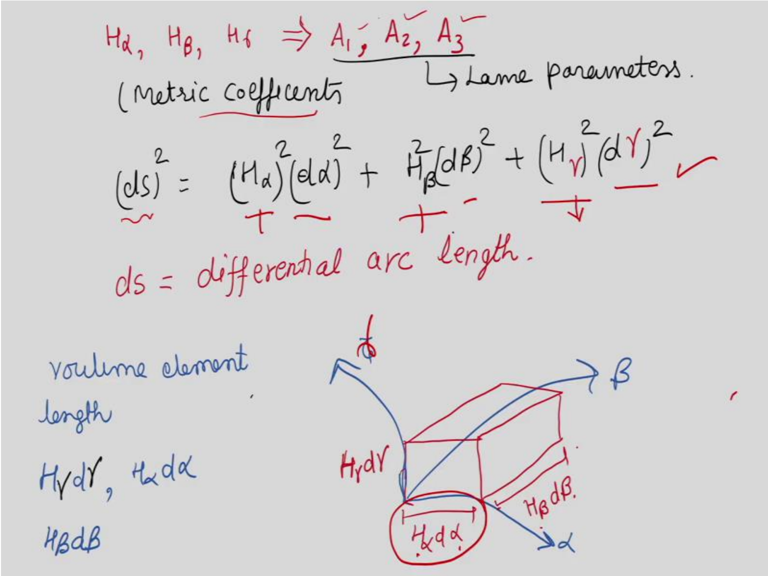

In that book, these are known as metric coefficients and if you remember in our theory, these $H_\alpha, H_\beta, and\ H_\gamma$ are lame's parameters $A_1, A_2, and\ A_3$, they are the same, just naming is different. Here, $H_\alpha, H_\beta, and\ H_\gamma$ are denoted as metric coefficients, which are similar to lame's parameter.

If you talk about two-dimensional, then it will be $A_1\ and\ A_2$ and if you talk about three-dimensional, then it will be $A_1, A_2, and\ A_3$. And the most important part is the differential arc length for a volume will be:

$$(ds)^2 = (H_\alpha)^2 (d\alpha)^2 + (H_\beta)^2 (d\beta)^2 + (H_\gamma)^2 (d\gamma)^2 \ .$$

Here you can see that these $H_\alpha^2, H_\beta^2, and\ H_\gamma^2$ are nothing, but they are the same as $A_1^2, A_2^2, and\ A_3^2$. Then the volume of an element along arc length:

Along $\alpha = (H_\alpha)^2 (d\alpha)^2$

Along $\beta = (H_\beta)^2 (d\beta)^2$

Along $\gamma = (H_\gamma)^2 (d\gamma)^2$

Sometimes we call it ς direction; if somebody wants to go and check in the book, then you can correlate.

That is why I kept the $\alpha,\ \beta, and\ \gamma$ same; but otherwise in our theory, we have used $\alpha,\ \beta, and\ \varsigma$ coordinates along the thickness direction.

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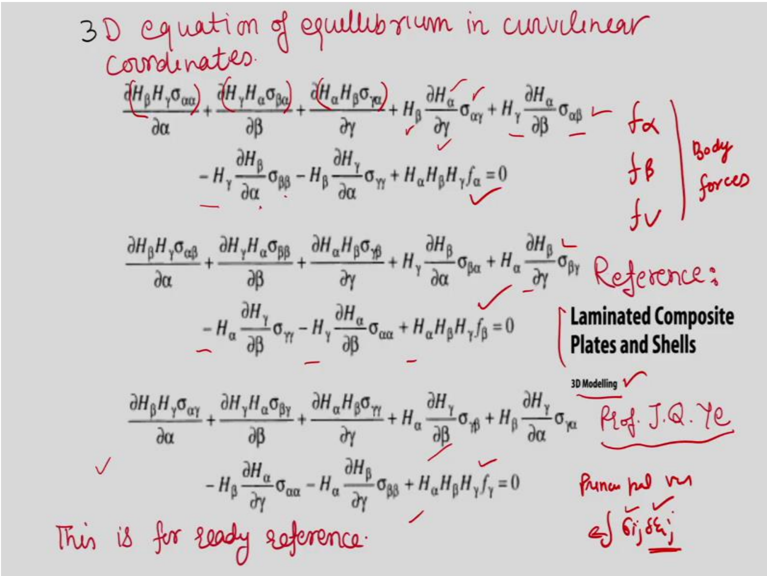

Now, we are interested to see the form of a 3-dimensional equation of equilibrium in curvilinear coordinate. Here you see:

$$\frac{\partial\left(H_\beta H_\gamma \sigma_{\alpha\alpha}\right)}{\partial\alpha}+\frac{\partial\left(H_\gamma H_\alpha \sigma_{\beta\alpha}\right)}{\partial\beta}+\frac{\partial\left(H_\alpha H_\beta \sigma_{\gamma\alpha}\right)}{\partial\gamma}+H_\beta\frac{\partial H_\alpha}{\partial\gamma}\sigma_{\alpha\gamma}+H_\gamma\frac{\partial H_\beta}{\partial\beta}\sigma_{\alpha\beta}$$

$$-H_\gamma\frac{\partial H_\beta}{\partial\alpha}\sigma_{\beta\beta}-H_\beta\frac{\partial H_\gamma}{\partial\alpha}\sigma_{\gamma\gamma}+H_\alpha H_\beta H_\gamma f_\alpha=0 \quad equation(1)$$

$$\frac{\partial\left(H_\beta H_\gamma \sigma_{\alpha\beta}\right)}{\partial\alpha}+\frac{\partial\left(H_\gamma H_\alpha \sigma_{\beta\beta}\right)}{\partial\beta}+\frac{\partial\left(H_\alpha H_\beta \sigma_{\gamma\beta}\right)}{\partial\gamma}+H_\gamma\frac{\partial H_\beta}{\partial\alpha}\sigma_{\beta\alpha}+H_\alpha\frac{\partial H_\beta}{\partial\gamma}\sigma_{\beta\gamma}$$

$$-H_\alpha\frac{\partial H_\gamma}{\partial\beta}\sigma_{\gamma\gamma}-H_\gamma\frac{\partial H_\alpha}{\partial\beta}\sigma_{\alpha\alpha}+H_\alpha H_\beta H_\gamma f_\beta=0 \quad equation(2)$$

$$\frac{\partial\left(H_\beta H_\gamma \sigma_{\alpha\gamma}\right)}{\partial\alpha}+\frac{\partial\left(H_\gamma H_\alpha \sigma_{\beta\gamma}\right)}{\partial\beta}+\frac{\partial\left(H_\alpha H_\beta \sigma_{\gamma\gamma}\right)}{\partial\gamma}+H_\alpha\frac{\partial H_\gamma}{\partial\beta}\sigma_{\gamma\beta}+H_\beta\frac{\partial H_\gamma}{\partial\alpha}\sigma_{\gamma\alpha}$$

$$-H_\beta\frac{\partial H_\alpha}{\partial\gamma}\sigma_{\alpha\alpha}-H_\alpha\frac{\partial H_\beta}{\partial\gamma}\sigma_{\beta\beta}+H_\alpha H_\beta H_\gamma f_\gamma=0 \quad equation(3)$$

Here, f_α, f_β, and f_γ are the body forces.

This set of equations is given in the book of laminated composite plates and shells 3D modeling by professor JQ Ye. I have taken these governing equations from this book.

Just for the sake of completeness, we should be aware that if we want to write a 3-dimensional equation of equilibrium in curvilinear coordinates, they will look like this, even the derivation is not part of this course. Whenever comes the theory of elasticity, then the derivation may be part of that.

You can get these 3-dimensional governing equations using the concept of principal virtual work done or the Hamilton principle. There you can write that $\sigma_{ij}\partial\varepsilon_{ij}$ and ultimately referring the ε_{ij} strain in terms of lame parameters or you can say the metric coefficient $H_\alpha, H_\beta, and\ H_\gamma$ and then you can derive this.

Finally, they will lead to 3-dimensional equations of equilibrium in a curvilinear coordinate system. One need not remember this set of equations; you must be aware that the set of systems is given in which book and where, because the problem is if you miss any plus sign, minus sign, or anything, then you will lead to the wrong governing equations.

Then how do you find that whatever is written here is right or wrong? It states that we get special cases like rectangular coordinate systems, cylindrical coordinate systems, and the spherical coordinate system. You find that we can deduce from this to a cylindrical or a spherical coordinate system 3-dimensional equation of equilibrium, which is generally given in most of the advanced solid mechanics' books.

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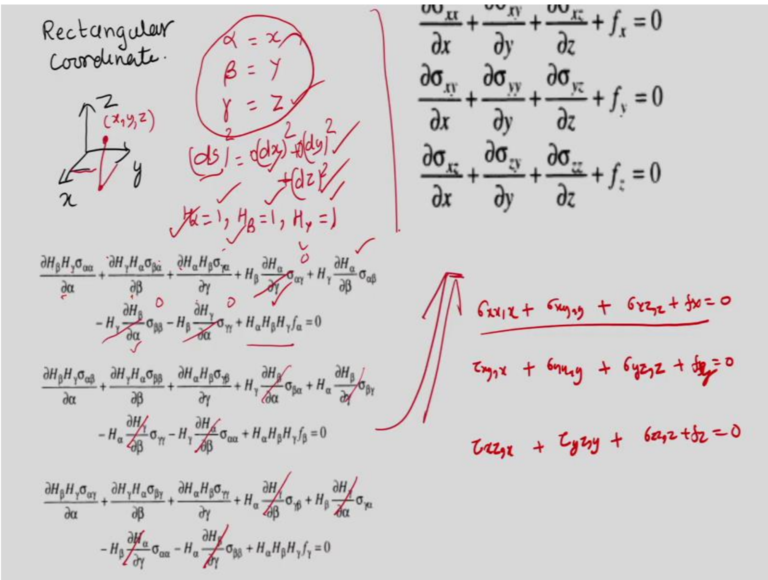

For a rectangular coordinate system, if we choose:

$\alpha = x$, $\beta = y$, and $\gamma = z$

The length of a curve $(ds)^2$ can be written as:

$(dx)^2 + (dy)^2 + (dz)^2$.

You can see that the coefficients are just 1 and we can directly convert. Let us say, in the first equation, $\beta = 1$, $\gamma = 1$

It will be:

$$\frac{\partial \sigma_{\alpha\gamma}}{\partial \alpha} \text{plus} \frac{\partial \sigma_{\beta\gamma}}{\partial \beta} + \frac{\partial \sigma_{\gamma\gamma}}{\partial \gamma}$$

Because H_α is 1, the derivative will not exist.

Similarly, one derivative will not exist; then H_β derivative is again 1, and it will not exist.

In this way, you can see that these terms will be 0 and it reduces to:

$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + f_x = 0$

It is the 3-dimensional equation of equilibrium in Cartesian coordinate.

In the same way, the second equation reduces to:

$$\sigma_{xy,x} + \sigma_{xx,y} + \sigma_{yz,z} + f_y = 0$$

The third equation will be:

$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} + f_z = 0.$$

We can see that, if we choose a lame metric coefficient properly and α, β, and γ; then definitely we can reduce to our form.

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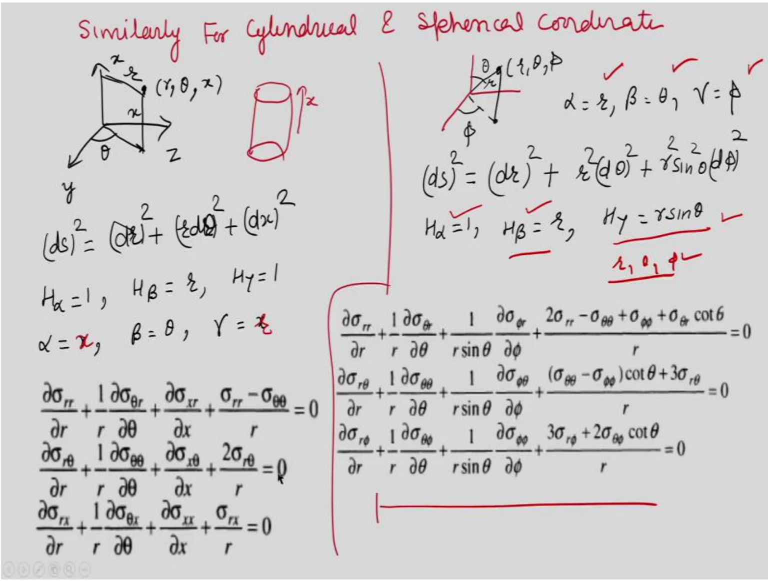

Then for the cylindrical case:

$H_\alpha = 1$, $H_\beta = r$, and $H_\gamma = 1$.

Whenever we have $\alpha = x$, $\beta = \theta$, and $\gamma = r$; I am going to tell with the help of one equation that if it is x, it will be σ_{xx}, $\alpha = x$, $\beta = \theta$.

From here, we can find the final governing equation in the cylindrical coordinate system:

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{\partial \sigma_{xr}}{\partial x}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{x\theta}}{\partial x}+\frac{2\sigma_{r\theta}}{r}=0$$

$$\frac{\partial \sigma_{rx}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta x}}{\partial \theta}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\sigma_{rx}}{r}=0$$

And for the spherical coordinate system:

$H_\alpha = 1$, H_β = r, and $H_\gamma = r\sin\theta$,

And the coordinate is r, θ, and φ.

Substituting here, one can find the following set of governing equations:

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi}+\frac{2\sigma_{rr}-\sigma_{\theta\theta}+\sigma_{\phi\phi}+\sigma_{\theta r}\cot\theta}{r}=0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi}+\frac{\left(\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)+\cot\theta+3\sigma_{r\theta}}{r}=0$$

$$\frac{\partial \sigma_{r\phi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi}+\frac{3\sigma_{r\phi}-2\sigma_{\theta\phi}\cot\theta}{r}=0$$

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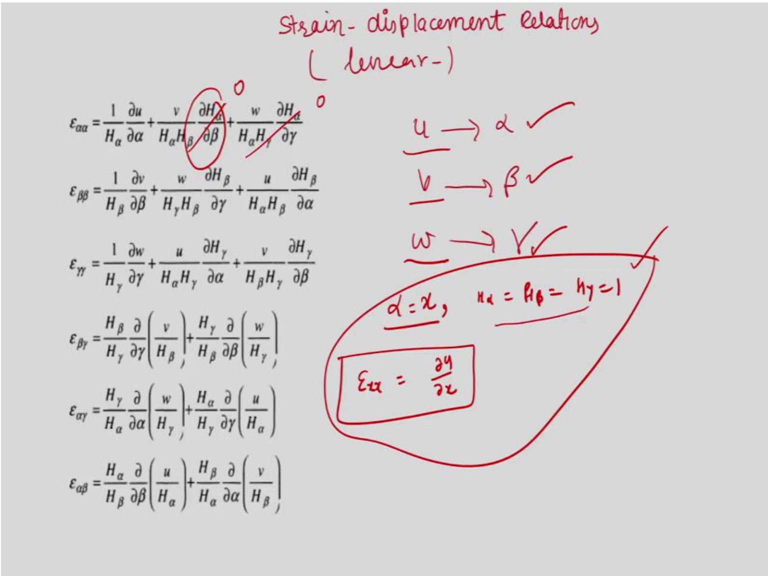

And the linear strain displacement relations:

$$\varepsilon_{\alpha\alpha} = \frac{1}{H_\alpha}\frac{\partial u}{\partial \alpha} + \frac{v}{H_\alpha H_\beta}\frac{\partial H_\alpha}{\partial \beta} + \frac{w}{H_\alpha H_\gamma}\frac{\partial H_\alpha}{\partial \gamma}$$

$$\varepsilon_{\beta\beta} = \frac{1}{H_\beta}\frac{\partial v}{\partial \beta} + \frac{w}{H_\gamma H_\beta}\frac{\partial H_\beta}{\partial \gamma} + \frac{u}{H_\alpha H_\beta}\frac{\partial H_\beta}{\partial \alpha}$$

$$\varepsilon_{\gamma\gamma} = \frac{1}{H_\gamma}\frac{\partial w}{\partial \gamma} + \frac{u}{H_\alpha H_\gamma}\frac{\partial H_\gamma}{\partial \alpha} + \frac{u}{H_\alpha H_\beta}\frac{\partial H_\beta}{\partial \alpha}$$

$$\varepsilon_{\beta\gamma} = \frac{H_\beta}{H_\gamma}\frac{\partial}{\partial \gamma}\left(\frac{v}{H_\beta}\right) + \frac{H_\gamma}{H_\beta}\frac{\partial}{\partial \beta}\left(\frac{w}{H_\gamma}\right)$$

$$\varepsilon_{\alpha\gamma} = \frac{H_\gamma}{H_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{w}{H_\gamma}\right) + \frac{H_\alpha}{H_\gamma}\frac{\partial}{\partial \gamma}\left(\frac{u}{H_\alpha}\right)$$

$$\varepsilon_{\alpha\beta} = \frac{H_\alpha}{H_\beta}\frac{\partial}{\partial \beta}\left(\frac{u}{H_\alpha}\right) + \frac{H_\beta}{H_\alpha}\frac{\partial}{\partial \alpha}\left(\frac{v}{H_\beta}\right)$$

If you remember we have already developed this set of relations using the lame parameters A_1, A_2, and A_3 and then we converted it into our shell case in terms of H_α, H_β, and H_γ; the linear definition of $\varepsilon_{\alpha\alpha}$, $\varepsilon_{\beta\beta}$, $\varepsilon_{\gamma\gamma}$, $\varepsilon_{\beta\gamma}$, $\varepsilon_{\alpha\gamma}$, and $\varepsilon_{\alpha\beta}$ is given here, where u means displacement along α direction, v means displacement along β direction, and w is displacement along γ direction.

Let us say, α = x, for a rectangular coordinate system and $H_\alpha = H_\beta = H_\gamma = 1$.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

Because $\frac{\partial H_\alpha}{\partial \beta} = 0$ and $\frac{\partial H_\alpha}{\partial \gamma} = 0$.

It reduces to:

$\varepsilon_{xx} = \frac{\partial u}{\partial x}$ in the rectangular coordinate system.

We can also reduce the linear strain displacement for different geometry cases; like it may be circular, non-circular, conical, spherical, ellipsoidal, and all that kind of things.

Sometimes in some of the books that exact derivation is not given; suppose you want to study a different kind of a shell and for that, you want to know what is your α, β, and

γ and you know the value of H_α, H_β, and H_γ then you can directly derive. You need not worry about exactly what is the strain displacement relation or equation of equilibrium for a particular shell-like conical, spherical, some different ellipsoidal, paraboidal, or any kind of a shell because in some of the books these even may not be given, but their parameters are available because this is the part of the mathematics that differential geometry. In the differential geometry these α, β, γ and lame's parameters will be there definitely, but we are interested to apply as a structure.

For a structure, we need to have the strain equations of equilibrium; in those cases, if we know this thing, we can derive the special cases.

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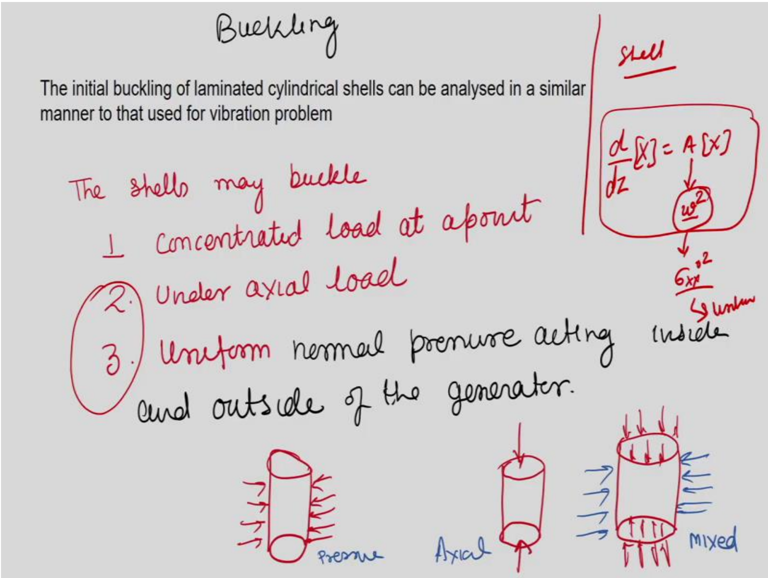

Now, we are going to study the buckling of a composite cylindrical shell. In the shell cases, the buckling may occur due to a concentrated load or due to a uniaxial under axial load, or due to normal pressure, which means inside pressure or sometimes the external pressure or internal pressure. And sometimes may be a combined effect of these. Therefore, buckling may occur.

The important part is the way we have solved the equation of free vibration. Let us say it is:

$$\frac{d}{dz}[X] = \mathrm{A}[X]$$

In the case of free vibration, A contains the function of ω^2, the same way, instead of A ω^2, it may contain $\sigma_{xx}^{\circ 2}$ which is unknown, critical buckling and we can sort it out.

I am going to explain first the buckling of a thin cylinder and then the buckling of an angle ply cylindrical shell so that you will get some idea that how to start. The buckling solutions required extensive mathematics; initially, we have to assume that, there may be unknown loading, due to that there may be some strains or it is only axial loading or combined loading, based on that we can proceed further.

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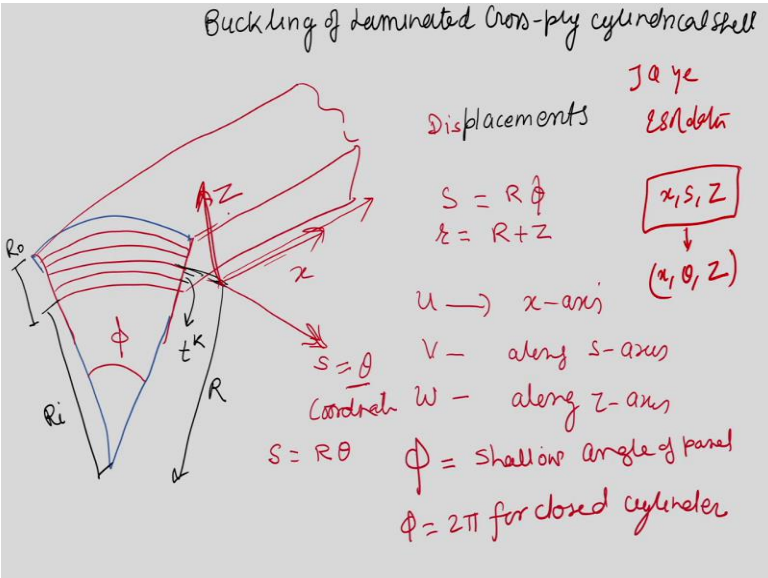

The first paper was the buckling of a laminated cross-ply for a cylindrical shell, it came around in 1990 or 1988 or 89. But for the 3-dimensional buckling of a laminated cross-ply shell, J Q Ye and Saldater had given the solution.

in that paper they said that the length along the longitudinal axis is taken as x, along the circumferential axis is taken as S and along the radial direction is taken as Z.

That paper is for angle ply buckling of a laminated shell and ϕ is the total angle or the shallow angle of a panel; if you talk about a complete cylindrical shell, then ϕ will be equal to 2π. In this way, the governing equations are written in terms of x, S, and Z. It is just a coordinate that is changed, sometimes it is written as x, θ, and Z. If you remember that our equations of equilibriums are written in terms of x, θ, and Z.

You have to convert using S parameters, so the other things remain the same.

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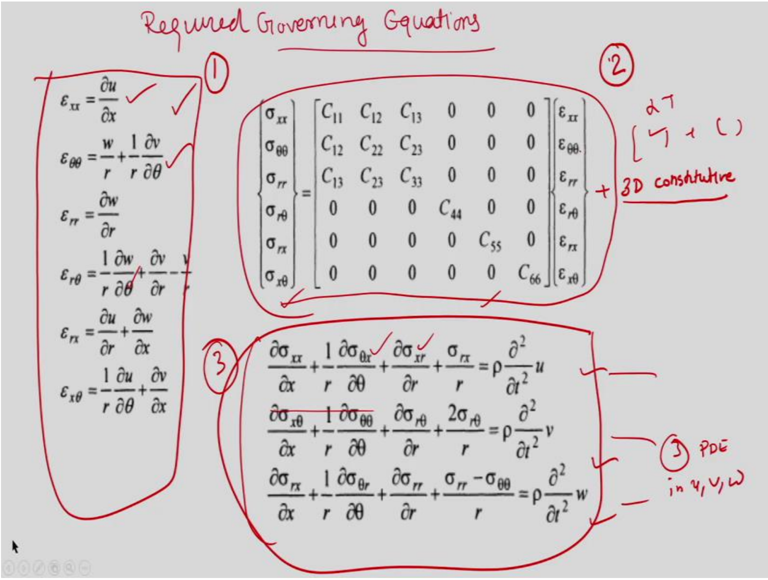

What are the basic equations required; if you are interested to do buckling, free vibration, or the bending of a cylindrical shell?

Here, I would like to say that, first of all, for any type of shell, you must know what is strain displacement relations; you may say that the ingredient for this recipe. For developing a governing equation, we need the strain displacement relations.

Here, you see the linear strain displacement relations:

$$\varepsilon_{xx}=\frac{\partial u}{\partial x};$$
$$\varepsilon_{\theta\theta}=\frac{w}{r}+\frac{1}{r}\frac{\partial v}{\partial \theta};$$
$$\varepsilon_{rr}=\frac{\partial w}{\partial r};$$
$$\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial w}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r};$$
$$\varepsilon_{rx}=\frac{\partial u}{\partial r}+\frac{\partial w}{\partial x};$$
$$\varepsilon_{x\theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}$$

Then, we need the constitutive relations for a material; if it is an orthotropic material or an isotropic material, then we need to know the 3 D constitutive relations.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{rx} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{rr} \\ \varepsilon_{r\theta} \\ \varepsilon_{rx} \\ \varepsilon_{x\theta} \end{Bmatrix}$$

Since these materials are considered elastic materials, then this relation is like this if we say that the material is a piezoelectric material, then maybe some more terms will be there. If you say that I am interested to study thermal buckling, then the temperature effect may be there. If somebody is interested in the hygroscopic behavior of a cylindrical shell, then some moisture effect will be there. Therefore, we must know what are the 3 D constitutive relations for our case.

The third thing is the equation of equilibrium for a particular geometry; whether it is a cylindrical shell, spherical shell, a plate, a conical shell, or an ellipsoidal shell. For such a kind you must know what type of equation of equilibrium you are going to use.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{r}\frac{\partial \sigma_{\theta x}}{\partial \theta} + \frac{\partial \sigma_{xr}}{\partial r} + \frac{\sigma_{rx}}{r} = \rho \frac{\partial^2}{\partial t^2} u$$
$$\frac{\partial \sigma_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = \rho \frac{\partial^2}{\partial t^2} v$$
$$\frac{\partial \sigma_{rx}}{\partial x} + \frac{1}{r}\frac{\partial \sigma_{\theta x}}{\partial \theta} + \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2}{\partial t^2} w$$

This is the general 3-dimensional equation of equilibrium, 3 D constitutive relations, and linear strain displacement relations in a cylindrical coordinate system.

Now, depending upon the cases, if you say that, I am going to study angle ply shell; then one of the directions is very very long in that case, along the x-direction, the x derivative may vanish. Wherever x derivative may come up, they may vanish or if you say that I want to do a complete shell under axis symmetry loading, then the θ derivative may go up. Depending upon your requirement, some terms may vanish and some terms may retain. But for a journal these terms are looking like this.

I am just going to tell you that the basic displacement-based 3-dimensional solution; σ_{xx}

can be expressed as C_{11}, C_{12}, C_{13} into ε_{xx}, then you substitute the value of strains here in terms of displacement and take the derivative along x, along θ and substitute it here.

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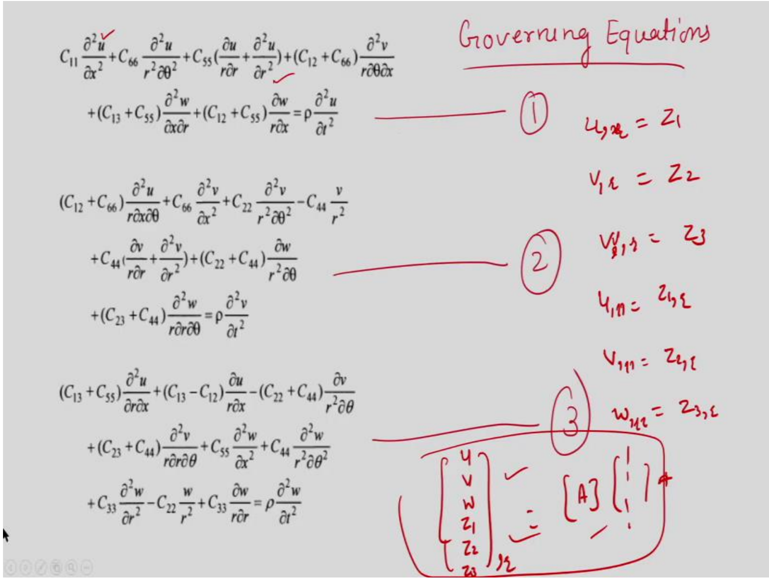

Ultimately, it leads to 3 partial differential equations in the form of u, v, and w, which look like this:

$$C_{11}\frac{\partial^2 u}{\partial x^2}+C_{66}\frac{\partial^2 u}{r^2\partial\theta^2}+C_{55}\left(\frac{\partial u}{r\partial r}+\frac{\partial^2 u}{\partial r^2}\right)+\left(C_{12}+C_{66}\right)\frac{\partial^2 v}{r\partial\theta\partial x}+\left(C_{13}+C_{55}\right)\frac{\partial^2 w}{\partial x\partial r}+$$

$$\left(C_{12}+C_{55}\right)\frac{\partial w}{r\partial x}=\rho\frac{\partial^2 u}{\partial t^2}\qquad equation(1)$$

$$\left(C_{12}+C_{66}\right)\frac{\partial^2 u}{r\partial x\partial\theta}+C_{66}\frac{\partial^2 v}{\partial x^2}+C_{22}\frac{\partial^2 v}{r^2\partial\theta^2}-C_{44}\frac{v}{r^2}+C_{44}\left(\frac{\partial v}{r\partial r}+\frac{\partial^2 v}{\partial r^2}\right)+$$

$$\left(C_{22}+C_{44}\right)\frac{\partial w}{r^2\partial\theta}+\left(C_{23}+C_{44}\right)\frac{\partial^2 w}{r\partial r\partial\theta}=\rho\frac{\partial^2 v}{\partial t^2}\qquad equation(2)$$

$$\left(C_{13}+C_{55}\right)\frac{\partial^2 u}{\partial r\partial x}+\left(C_{13}-C_{12}\right)\frac{\partial u}{r\partial x}-\left(C_{22}+C_{44}\right)\frac{\partial v}{r^2\partial\theta}+\left(C_{23}+C_{44}\right)\frac{\partial^2 v}{r\partial r\partial\theta}+C_{55}\frac{\partial^2 w}{\partial x^2}+$$

$$C_{44}\frac{\partial^2 w}{r^2\partial\theta^2}+C_{33}\frac{\partial^2 w}{\partial r^2}-C_{22}\frac{w}{r^2}+C_{33}\frac{\partial w}{r\partial r}=\rho\frac{\partial^2 w}{\partial t^2}\qquad equation(3)$$

These are the three differential equations and the displacement form. Here you say that derivative of u with respect to x, with respect to θ, then with respect to r, u, v, and w.

We can solve these three partial differential equations. Our concept is that can we convert them into a first-order form like the derivative of $u,_x$ or sometimes we are interested in terms of a radial coordinate.

$$u,_r = z_1;\ v,_r = z_2;\ w,_r = z_3;\ u,_{rr} = z_{1,r};\ v,_{rr} = z_{2,r};\ and\ \ w,_{rr} = z_{3,r}$$

In this way, it can be reduced to u, v, and w and then z_1, z_2, z_3 and so on and derivative with respect to r, and here may be some matrix and u_1, u_2, and u_3. By setting this, we can solve bending problems and free vibration problems.

But now we are interested to find a buckling problem; let us see how to proceed for that pre-buckling state.

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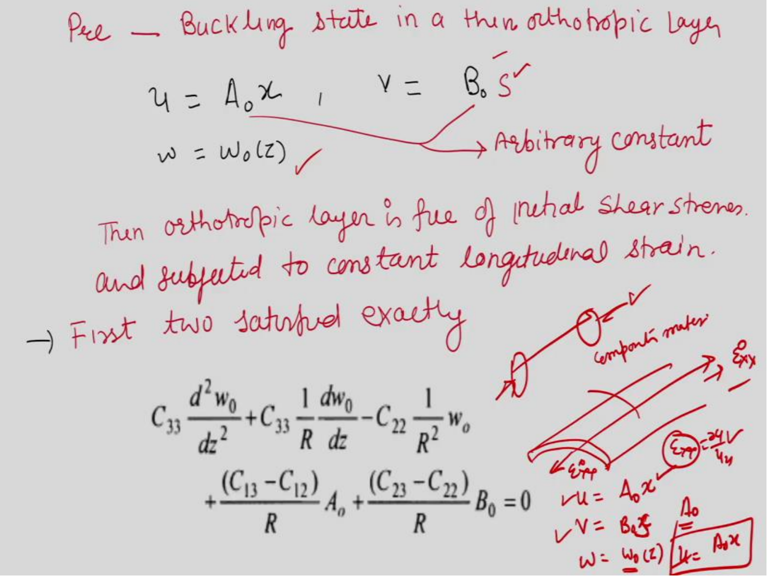

In this book the very first topic is the buckling state in a thin orthotropic layer; let us say, it is a very thin cylindrical shell made of composite material or orthotropic material and it is under the axial loading, the displacement can be any arbitrary constant:

$u = A_{\circ}x$, $\mathrm{v} = B_{\circ}s$, and $w = w_{\circ}(z)$; it is not changing, because it is a function of z only.

Here u is a function of x only, and v is a function of s only; because we assume that under axial loading, this displacement is substantial due to the axial stress. Therefore, thin and initial shear stresses are neglected, there are no shear stresses, and subjected to a constant longitudinal strain.

Sometimes we called it ε_{xx}°.

From there, $\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$.

When ε°_{xx} is constant, let us say $A_{\circ}$, we do not know how much, then u = $A_{\circ}x$.

It is subjected to an initial strain loading and constant strain ε°_{xx}, which is let us say a magnitude of $A_{\circ}$, from where it can be found:

u = $A_{\circ}x$

v = $B_{\circ}s$ as I have said that circumferential coordinate along the x-direction

$w = w_{\circ}(z)$

If we assume the displacement field like this and substitute it in the above three partial differential equations, then it leads to the first and second equations exactly satisfied, and the third equation will be:

$$C_{33}\frac{d^2 w_0}{dz^2} + C_{33}\frac{1}{R}\frac{dw_0}{dz} - C_{22}\frac{1}{R^2}w_0 + \frac{(C_{13} - C_{12})}{R}A_{\circ} + \frac{(C_{23} - C_{22})}{R}B_{\circ} = 0$$

Because there are no dynamic terms, therefore:

∂u, ∂v, ∂w, and ∂t = 0 and this equation becomes like this.

This is a second-order differential equation.

We can solve the second-order differential equation as it is or we can convert it into a first-order differential equation, then the solution will be easy for us.

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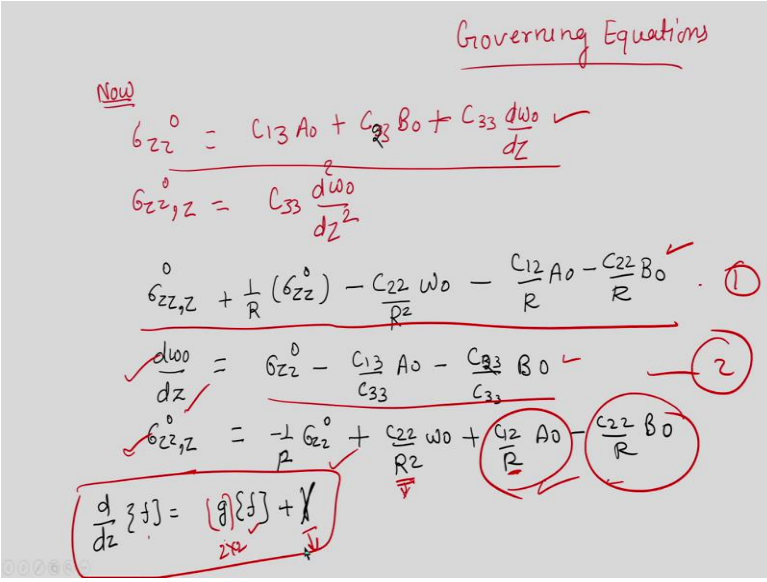

We are assuming that let us say from this equation, a parameter σ°_{zz} can be expressed as:

$C_{13}A_0 + C_{23}B_0 + C_{13}\dfrac{dw_0}{dz}$ and $\sigma^{\circ}_{zz,z}$ is $C_{33}\dfrac{d^2w_0}{dz^2}$.

If you substitute in this equation here:

$$C_{33}\frac{d^2w_0}{dz^2} + C_{33}\frac{1}{R}\frac{dw_0}{dz} - C_{22}\frac{1}{R^2}w_0 + \frac{(C_{13}-C_{12})}{R}A_{\circ} + \frac{(C_{23}-C_{22})}{R}B_{\circ} = 0$$

This equation will become like this:

$$\sigma^{\circ}_{zz,z} + \frac{1}{R}\left(\sigma^{0}_{zz}\right) - \frac{C_{22}}{R^2}w_0 - \frac{C_{12}}{R}A_0 - \frac{C_{22}}{R}B_0$$

This equation: $\dfrac{dw_0}{dz} = \sigma^{\circ}_{zz} - \dfrac{C_{13}}{C_{33}}A_0 - \dfrac{C_{23}}{C_{33}}B_0$.

Now, we have two equations; $\sigma^{\circ}_{zz,z}$ and $dw_{0,z}$, on the right-hand side you see that there is no derivative, only the basic variables, and some constants:

$$\sigma^{\circ}_{zz,z} = -\frac{1}{R}\sigma^{\circ}_{zz} + \frac{C_{22}}{R^2}w_0 + \frac{C_{12}}{R}A_0 - \frac{C_{22}}{R}B_0$$

We can express this set of equations as a 2 by 2 set of equations as:

$$\frac{d}{dz}\{f\} = [g]\{f\} + X \,.$$

This is the first-order non-homogeneous equation; one can solve it in different ways. One important part here is that we assumed that our cylindrical shell is thin, that is why we are taking this as R, the mean radius of the shells.

It is known to you in that case, this g will be a constant coefficient and the solution is very easy. But if you talk about a laminated cylindrical shell; in that case, we have to either use the successive layer approach or we have to use the modified Frobenius series to get the solution of such kind of equation.

From here by satisfying the boundary condition at x = 0 and x = a, we can find the governing solution.

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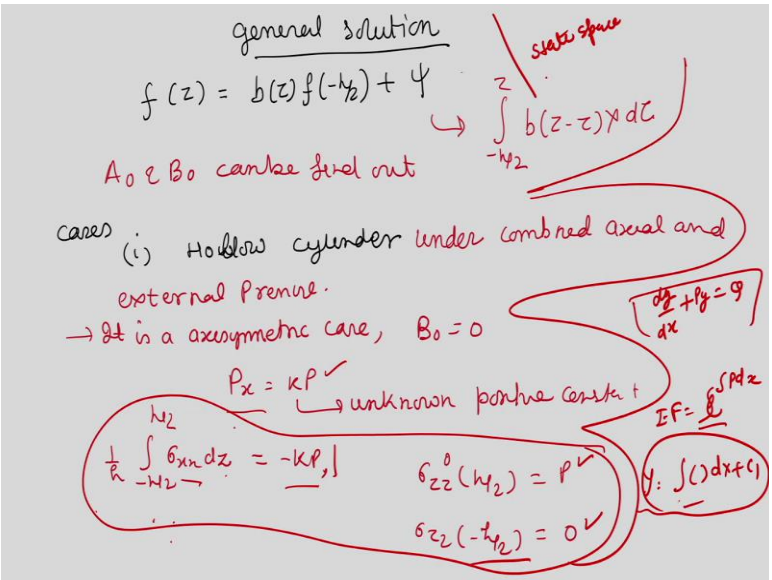

The solution can be written as:

$$\frac{dy}{dx} + py = Q$$

There is some equation like that, then we can have an integrated factor approach:

$$EF = e^{pdx} \,.$$

Ultimately, the solutions can be written as the integrating factor into dx plus C_1 constant and applying some boundary conditions.

In this approach:

$$f(z) = b(z) f\left(-h/2\right) + \psi$$

$$\psi = \int_{-h/2}^{z} b(z - z) X d\tau ,$$

Where, $\int_{-h/2}^{z}$ is the boundary condition.

This approach is known as the state space approach. When there is an ordinary differential equation with the constant coefficient, then we can get the solution using the state space approach. There are a number of approaches to getting the direct solution of a first-order differential equation.

One of the approaches is known as the state space approach and the solution is written like this.

A_0 and B_0 are the arbitrary constant that can be found. Depending upon the cases, let us say if a cylinder is hollow under combined axial and external pressure; At the boundary condition $\sigma^{\circ}_{zz}\left(h/2\right)$ there will be pressure, and at $\sigma^{\circ}_{zz}\left(-h/2\right)$, there will be no pressure and the axial load:

$$px = kP .$$

Ultimately,

$$\frac{1}{h} \int_{-h/2}^{h/2} \sigma_{xx} dz = -kP .$$

In this way, by satisfying all these things, we can find the constant A_0 and B_0. We can find the load at which the cylinder will pre-buckle.

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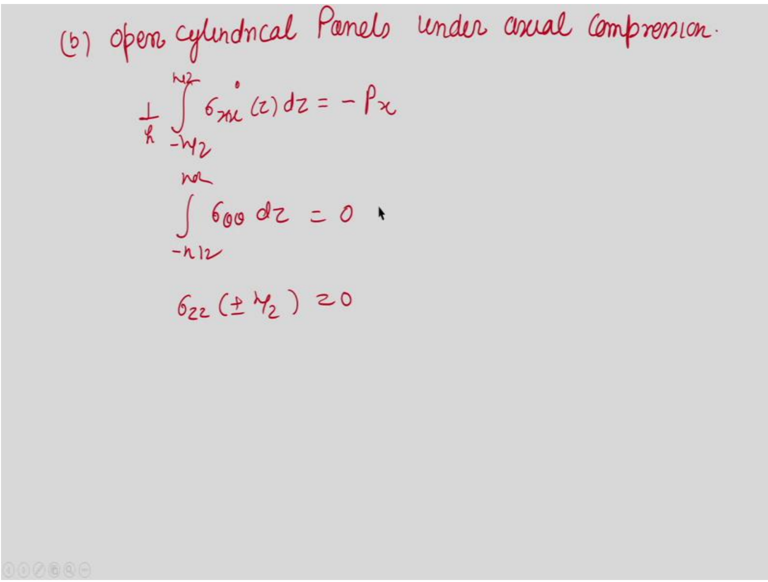

There will be the open cylinder case, in which only the axial compression is there; there $\sigma_{zz}\left(\pm h/2\right)=0$, there is no circumferential load.

Therefore, $\int_{-h/2}^{h/2} \sigma_{\theta\theta} dz = 0$.

And top and bottom are going to be 0 and we can find the solutions.

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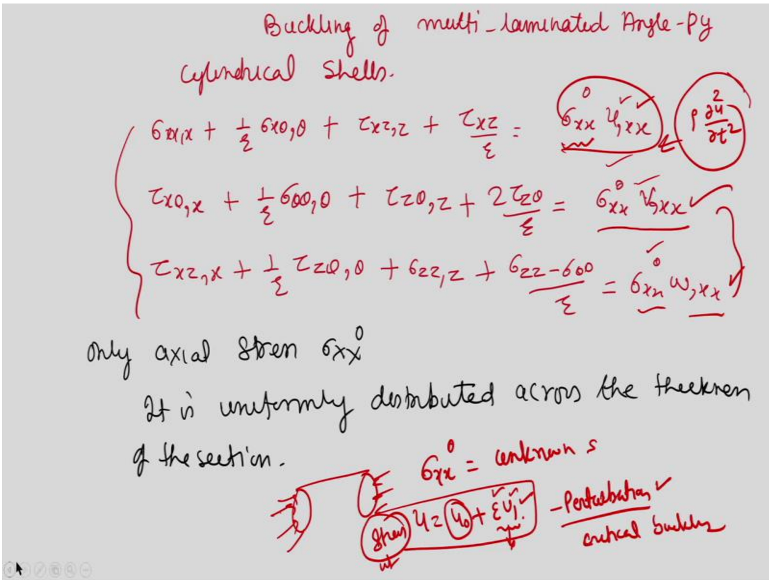

Now, we are talking about the buckling of a multi-laminated angle ply cylindrical shell. Previously we have discussed pre-buckling just before the buckling, what are the load or what is the extension required to satisfy the governing equation, but just after that, it starts to buckle. For that case, if you remember the following governing equations:

$$\sigma_{xx,x} + \frac{1}{r}\sigma_{x\theta,\theta} + \tau_{xz,z} + \frac{\tau_{xz}}{r} = \sigma_{xx}^{0} u,_{xx}$$
$$\sigma_{x\theta,x} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \tau_{z\theta,z} + \frac{2\tau_{z\theta}}{r} = \sigma_{xx}^{0} v,_{xx}$$
$$\sigma_{xz,x} + \frac{1}{r}\tau_{z\theta,\theta} + \sigma_{zz,z} + \frac{\sigma_{zz} - \sigma_{\theta\theta}}{r} = \sigma_{xx}^{0} w,_{xx}$$

$\rho \dfrac{\partial u^2}{\partial t^2}$ is replaced with $\sigma_{xx}^{0} u,_{xx}$.

Where, σ_{xx}^{0} is axial stress of unknown magnitude that we want to find, this is the unknown magnitude of axial stress, which is causing a buckling in the cylinder, and then it can be written as:

$\sigma_{xx}^{0} u,_{xx}$, $\sigma_{xx}^{0} v,_{xx}$, and $\sigma_{xx}^{0} w,_{xx}$.

There are many ways to set up the governing equations; the first way is to assume a perturbation technique let us say,

u $= u_0 + \varepsilon v_1$

Where, u_0 is your normal displacement field and that εv_1 is causing a buckling. The buckling means a small amount of load due to that small amount of deflection will be there. This technique is known as a perturbation technique.

Through that, we can also find the buckling and the critical buckling loads and through this technique, we assume that it is under axial stress and then dynamic terms are replaced like this.

Now, you can understand, that we have to substitute other forms here. As the boundary condition, when you applied axial stress let us say here; then the boundary conditions are known in terms of stresses and, displacement also. If we are able to form up the set of the equation in terms of stresses that will be more beneficial to us. So, we used to write the set of governing equations in the mixed form.

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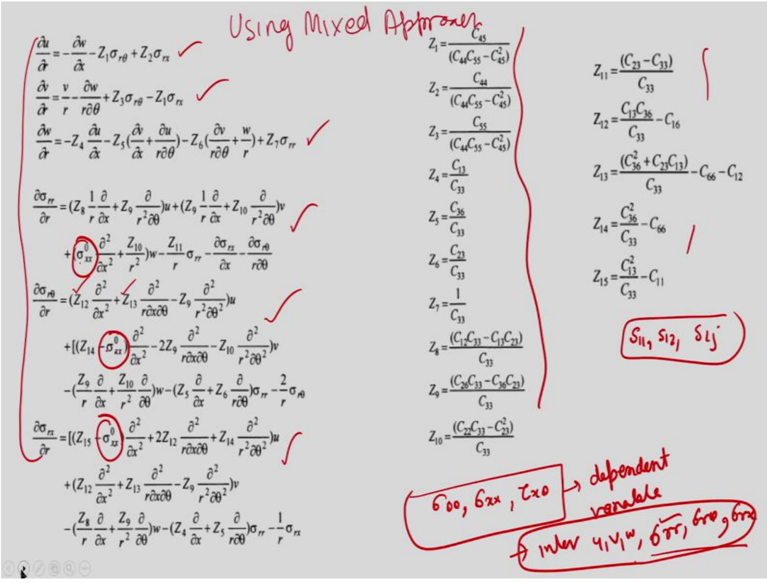

Using these strain displacement relations and constitutive relations we can get the following equations along the thickness direction:

$$\frac{\partial u}{\partial r} = -\frac{\partial w}{\partial x} - Z_1\sigma_{r\theta} + Z_2\sigma_{rx} \quad equation(1)$$

$$\frac{\partial v}{\partial r} = \frac{v}{r} - \frac{\partial w}{r\partial\theta} + Z_3\sigma_{r\theta} - Z_1\sigma_{rx} \quad equation(2)$$

$$\frac{\partial w}{\partial r} = -Z_4\frac{\partial u}{\partial x} - Z_5\left(\frac{\partial v}{\partial x} + \frac{\partial u}{r\partial\theta}\right) - Z_6\left(\frac{\partial v}{r\partial\theta} + \frac{w}{r}\right) + Z_7\sigma_{rr} \quad equation(3)$$

Then through the equation of equilibrium we get the following equations:

$$\frac{\partial\sigma_{rr}}{\partial r}=\left(Z_8\frac{1}{r}\frac{\partial}{\partial x}+Z_9\frac{\partial}{r^2\partial\theta}\right)u+\left(Z_9\frac{1}{r}\frac{\partial}{\partial x}+Z_{10}\frac{\partial}{r^2\partial\theta}\right)v+\left(\sigma_{xx}^0\frac{\partial^2}{\partial x^2}+\frac{Z_{10}}{r^2}\right)w-\frac{Z_{11}}{r}\sigma_{rr}$$
$$-\frac{\partial\sigma_{xx}}{\partial x}-\frac{\partial\sigma_{r\theta}}{r\partial\theta}\quad equation(4)$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}=\left(Z_{12}\frac{\partial^2}{\partial x^2}+Z_{13}\frac{\partial^2}{r\partial x\partial\theta}-Z_9\frac{\partial^2}{r^2\partial\theta^2}\right)u+\left[\left(Z_{14}-\sigma_{xx}^0\right)\frac{\partial^2}{\partial x^2}-2Z_9\frac{\partial^2}{r\partial x\partial\theta}-Z_{10}\frac{\partial^2}{r^2\partial\theta^2}\right]v-$$
$$\left(\frac{Z_9}{r}\frac{\partial}{\partial x}+\frac{Z_{10}}{r^2}\frac{\partial}{\partial\theta}\right)w-\left(Z_5\frac{\partial}{\partial x}+Z_6\frac{\partial}{r\partial\theta}\right)\sigma_{rr}-\frac{2}{r}\sigma_{r\theta}\quad equation(5)$$

$$\frac{\partial\sigma_{rx}}{\partial r}=\left[\left(Z_{15}-\sigma_{xx}^0\right)\frac{\partial^2}{\partial x^2}+2Z_{12}\frac{\partial^2}{r\partial x\partial\theta}+Z_{14}\frac{\partial^2}{r^2\partial\theta^2}\right]u+\left(Z_{12}\frac{\partial^2}{\partial x^2}+Z_{13}\frac{\partial^2}{r\partial x\partial\theta}-Z_9\frac{\partial^2}{r^2\partial\theta^2}\right)v-$$
$$\left(\frac{Z_8}{r}\frac{\partial}{\partial x}+\frac{Z_9}{r^2}\frac{\partial}{\partial\theta}\right)w-\left(Z_4\frac{\partial}{\partial x}+Z_5\frac{\partial}{r\partial\theta}\right)\sigma_{rr}-\frac{1}{r}\sigma_{rx}\quad equation(6)$$

Ultimately, we will get those six equations.

$\sigma_{\theta\theta}$, σ_{xx}, and $\tau_{x\theta}$ can be expressed in terms of u, v, w, and three transverse variables. These transverse variables are σ_{rr}, $\sigma_{r\theta}$, and σ_{rx} .

If you remember that in the lecture- 02 of week -08, I explained to develop a mixed formulation for angle ply shell, and there I explained that we can express these dependent variables using some mathematics like this, just by the transformation of a matrix. Here, the same approach is used, that is why some coefficients are coming differently.

In the previous form, I expressed the governing equation in terms of S_{11}, S_{12}, basically in S_{ij} compliance but here these are expressed in terms of a stiffness:

$$Z_1=\frac{C_{45}}{\left(C_{44}C_{55}-C_{45}^2\right)};\ Z_2=\frac{C_{44}}{\left(C_{44}C_{55}-C_{45}^2\right)};\quad Z_3=\frac{C_{55}}{\left(C_{44}C_{55}-C_{45}^2\right)};\ Z_4=\frac{C_{13}}{C_{33}};$$
$$Z_5=\frac{C_{36}}{C_{33}};\ Z_6=\frac{C_{23}}{C_{33}};\ Z_7=\frac{1}{C_{33}};\ Z_8=\frac{\left(C_{12}C_{33}-C_{13}C_{23}\right)}{C_{33}};\ Z_9=\frac{\left(C_{26}C_{33}-C_{36}C_{23}\right)}{C_{33}};$$
$$Z_{10}=\frac{\left(C_{22}C_{33}-C_{23}^2\right)}{C_{33}};\ Z_{11}=\frac{\left(C_{23}-C_{33}\right)}{C_{33}};\ Z_{12}=\frac{\left(C_{13}-C_{36}\right)}{C_{33}}-C_{16};$$
$$Z_{13}=\frac{\left(C_{36}^2+C_{23}C_{13}\right)}{C_{33}}-C_{66}-C_{12};\ Z_{14}=\frac{C_{36}^2}{C_{33}}-C_{66};\ and\ Z_{15}=\frac{C_{13}^2}{C_{33}}-C_{11}$$

This is the main difference. One other difference there is the bending to this, here σ_{xx}^0 is coming into the picture in every equation and which is unknown to us. We arrange the equations like this and we can form a setup of governing equations.

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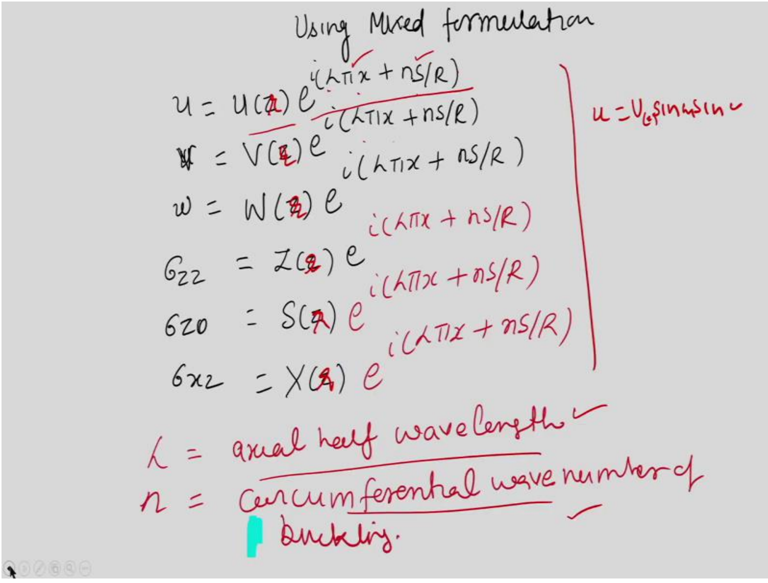

For the case of a cylindrical shell, we can assume:

$$u = u(r) e^{i\left(\lambda \pi x + ns/R\right)};$$
$$v = v(r) e^{i\left(\lambda \pi x + ns/R\right)};$$
$$w = w(r) e^{i\left(\lambda \pi x + ns/R\right)};$$
$$\sigma_{zz} = Z(r) e^{i\left(\lambda \pi x + ns/R\right)};$$
$$\sigma_{z\theta} = S(r) e^{i\left(\lambda \pi x + ns/R\right)};$$
$$\sigma_{xz} = X(r) e^{i\left(\lambda \pi x + ns/R\right)}$$

We can say that when it is radius coordinate, we can write that ns in sub cylindrical coordinate along the longitudinal directions.

Where, λ and n are mode shapes or buckling wave numbers. If you remember for the case of a simply supported, I assumed the radial deflection can be:

$$U_r \sin \sin$$

From there, it becomes a derivative of r only.

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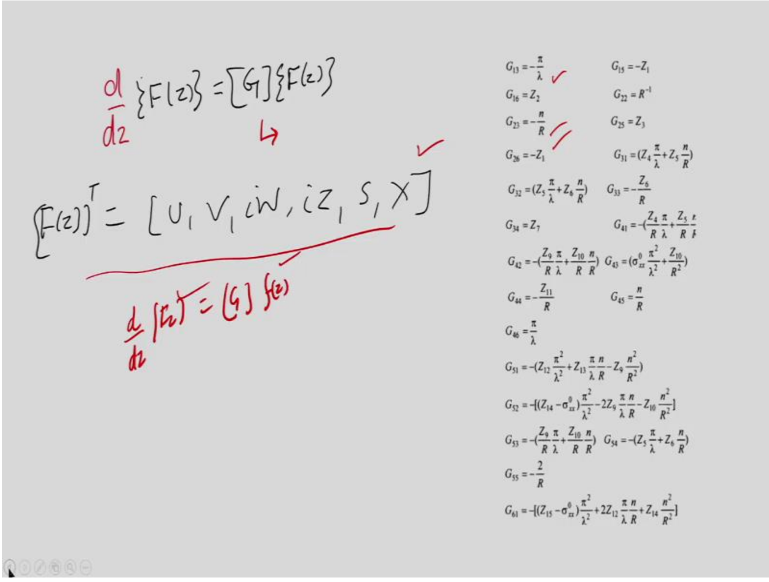

If we assume all this, substitute these in the above six equations; which leads to an equation like this:

$$\frac{d}{dz}\{F(z)\}=[G]\{F(z)\}.$$

Here, the important part is that this G, if you say that we are going to solve like a successive layer approach, that a layer is divided mathematically into infinite sub-layers, then the mean radius of that layer, in that case, it will be a constant.

The mean radius of that layer can be taken as R. In that way it will be a constant. Therefore, the solution is known to you. We can get the solution as:

$$[F(z)]^T=[U,V,iW,iZ,S,X].$$

And the solution is similar to that I explained with the free vibration, σ_{xx} is not known to us. We will say that the eigenvector solution is the state space approach; it becomes eigenvalue and then it is the solution, then the determinant must be 0. We are giving a guess. Through iterative technique, we can find the critical buckling load.

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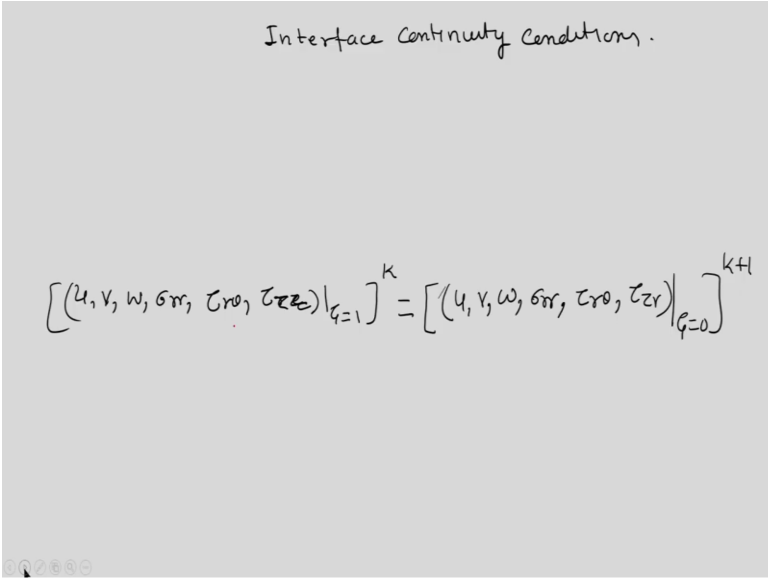

And ultimately for satisfying the boundary conditions top and bottom, we have to satisfy the interface continuity conditions for a laminated shell. If it is a single layer, then this equation is not required:

$$\left[\left(u, v, w, \sigma_{rr}, \sigma_{r\theta}, \sigma_{zr}\right)\Big|_{\varsigma=1}\right]^{K} = \left[\left(u, v, w, \sigma_{rr}, \sigma_{r\theta}, \sigma_{zr}\right)\Big|_{\varsigma=0}\right]^{K+1}.$$

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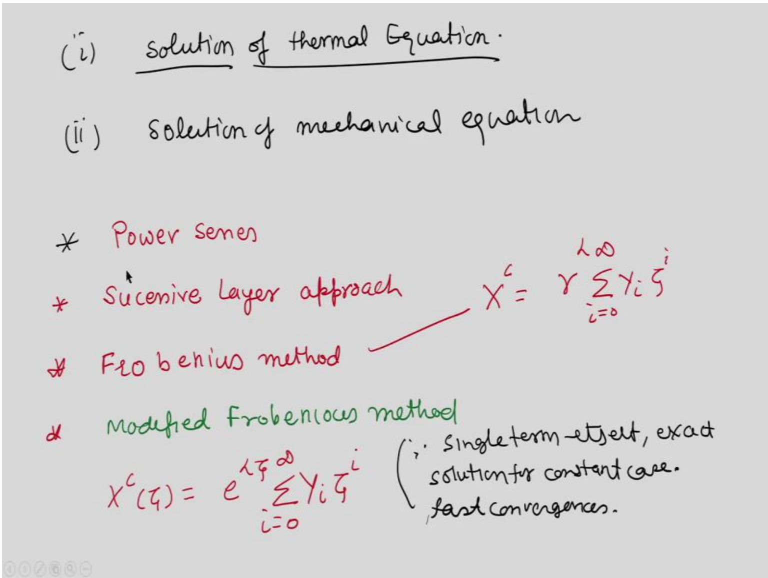

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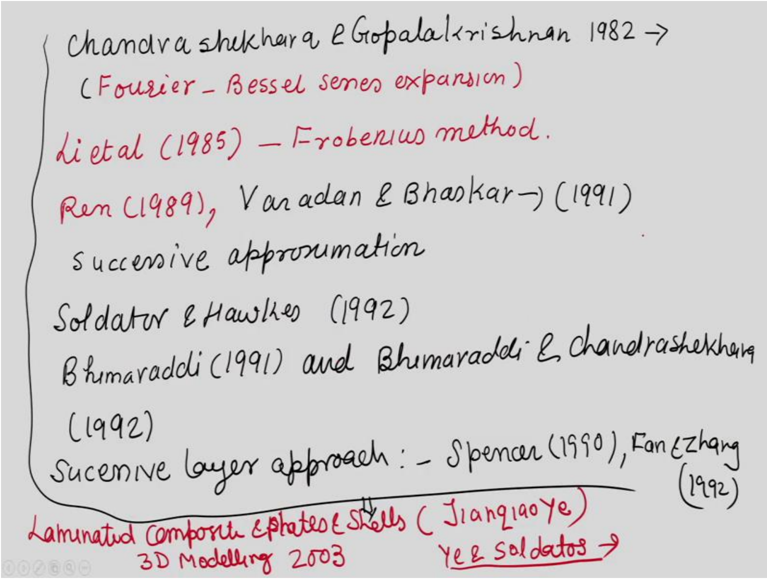

Solutions can be found by any technique; in this book, the successive layer approach is used. It is slightly computationally intensive; but accuracy remains, which means there is no issue with the accuracy, accuracy is fine.

This approach is first proposed by Soldators and then Varadan and Bhaskar and then it has been used by JQ Ye and their coworkers. Till here in this course, I explained to develop the basic idea of a composite; then I explained the basic theorem of surfaces in differential geometry and I also explained the strain displacement relations and different principles.

And then using the first-order shear deformation theory, I developed the basic differential equations. Then I presented the Navier solution for a simply supported case; then a levy solution and I also explained the approximate solution and the extended Kantorovich technique for a cylindrical shell.

And in the last week, I explained how to develop a 3-dimensional solution for the case of the cylindrical shell. Definitely in the literature, solutions are available for the case of a doubly curved shell, spherical shell, or conical shell.

Once you have understood the technique; then definitely you can understand the articles given in the literature and you can proceed with your type of problem. These days in research the problems are like a cylindrical shell with the cut-outs or functionally graded shell or shell having some other kind of loading under a blast loading. These kinds of problems are solved or we talk about the nano shells or the shells having carbon nanotubes.

There will be some more terms required and more complex mathematics is required. But if you are aware of the basic formulation, then definitely anybody can understand the complex formulation for any kind of special case.

With this, I would like to thank you very much. And hopefully, it will help you in your research project or in understanding the shell equations.

Thank you very much.