Theory of Composite Shells

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Week – 08
Lecture – 03
(Basic Theorems of differential geometry)

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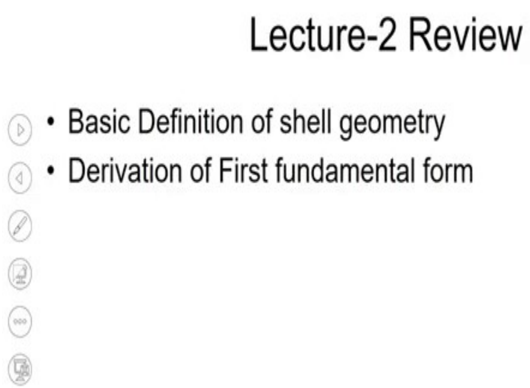

Dear learners welcome to the 3rd lecture of week one. In the second lecture, we have done the Basic Definitions of shell geometry and we have derived the first fundamental form of surfaces. Now, I will quickly review that how to derive the first fundamental form of surfaces.

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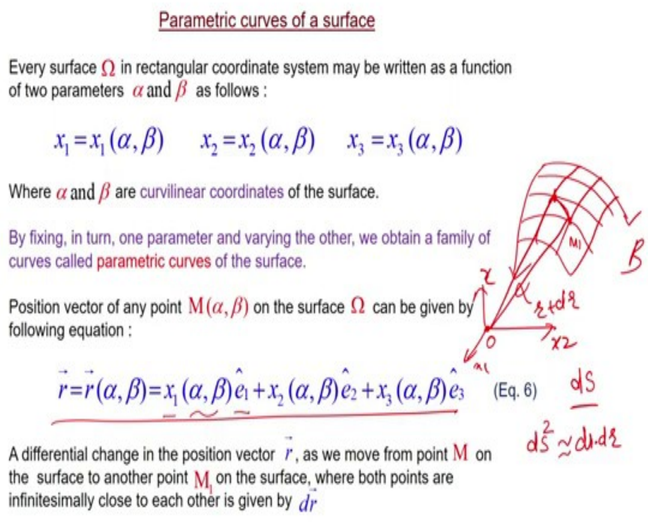

Let us say α and β are two curvilinear coordinates. We will assume a surface and divide it into grids, these lines are parallel to α coordinates and β coordinates. And from a reference coordinate system 'o' where x_1, x_2, and x_3 will get let us say point M and the position vector.

So, the position vector r can be written as

$$r = r(\alpha,\beta) = x_1(\alpha,\beta)\hat{e}_1 + x_2(\alpha,\beta)\hat{e}_2 + x_3(\alpha,\beta)\hat{e}_3 \ \hat{e}_3 .$$

We write a position vector like this, if this point moves to let us say M_1'.

So, change in curvature ds; curve length we are interested to know. So, it will change in r + dr, we can say that $(ds)^2 \simeq dr.dr$. So, first, we have to find out the dr. Now, r is a function of α and β.

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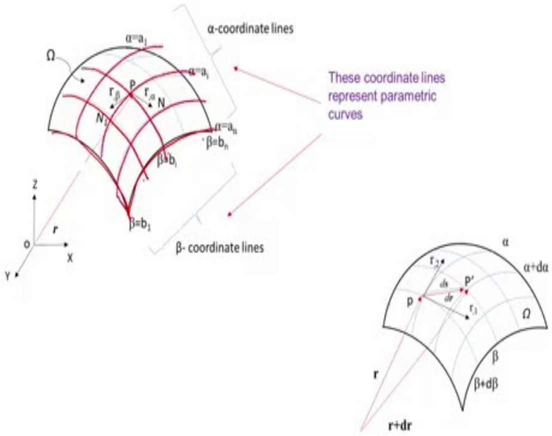

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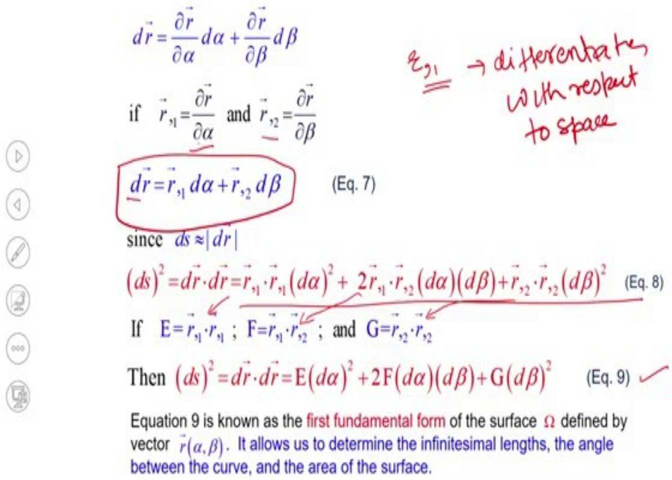

dr can be find by

$$d\vec{r} = \frac{\partial r}{\partial \alpha}d\alpha + \frac{\partial r}{\partial \beta}d\beta \quad d\beta$$

Where $\vec{r}_{,1} = \frac{\partial r}{\partial \alpha}$ and $\vec{r}_{,2} = \frac{\partial r}{\partial \beta}$

So, comma is representing the differentiation with respect to space coordinates.

So, ultimately the dr can be represented in a very compact form

$$dr = r_{,1}\, d\alpha + r_{,2}\, d\beta$$

As I have said that $(ds)^2 \simeq dr.dr$. So, dr is known. So, multiplying with that taking dot product gives you equation number (8).

$$(ds)^2 = dr.dr = r_{,1}.r_{,1}(d\alpha)^2 + 2r_{,1}.r_{,2}(d\alpha)(d\beta) + r_{,2}.r_{,2}(d\beta)^2$$

Now, again we assign $r_{,1}.\, r_{,1}$ as E and mixed derivative product of that i.e. $r_{,1}.r_{,2}$ is assigned as F and $r_{,2}.r_{,2}$ is assigned as G. Then,

$$(ds)^2 = dr.dr = E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2$$

This equation (9) is known as the first fundamental of the surface where E, F, and G are the distortion parameters or deformation parameters.

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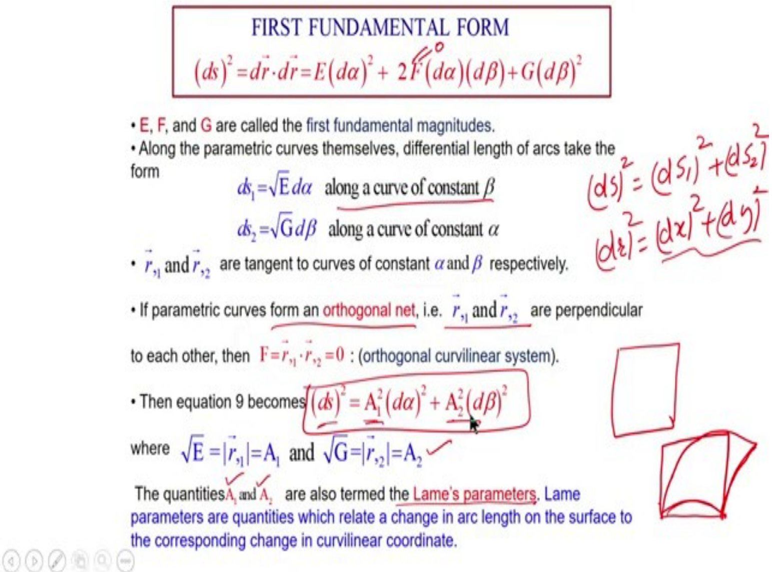

If we say that the curve length along a curve is α and constant β; go to back equation, in this figure, you can see that these are the line where β is going from b_1 to b_2 to b_3 and this line where β is constant. Over these lines β is constant and over these lines α is constant.

So, for that case, ds_1 the curve length along the constant curve of β can be written as

$ds_1 = \sqrt{E}d\alpha$ and $ds_2 = \sqrt{G}d\beta$.

We can write in terms of ds square $(ds)^2$ something like $(ds)^2 = (ds_1)^2 + (ds_2)^2$ it is same like in your rectangular coordinate system; where, $(dr)^2 = (dx)^2 + (dy)^2$ similarly like that. So, if parametric curves form an orthogonal net means they are perpendicular to each other and their tangents $r_{,1}$ and $r_{,2}$ are perpendicular then, we can say that this F will vanish. It will go to 0.

Then again $(ds)^2 = E(d\alpha)^2 + G(d\beta)^2$.

So, making this type of form $\sqrt{E} = A_1$ and $\sqrt{G} = A_2$. So, we can say

$(ds)^2 = A_1^2(d\alpha)^2 + A_2^2(d\beta)^2$

So, this is the most important and very useful form in the theory of surfaces. The quantities A_1 and A_2 are known as lame's parameters. For defining any cell surface, we must know the lame's parameters A_1 and A_2, these are nothing, but distortions parameters change in that.

If we take this perfect rectangle and if we say that this change is denoted by the constants A_1 and A_2.

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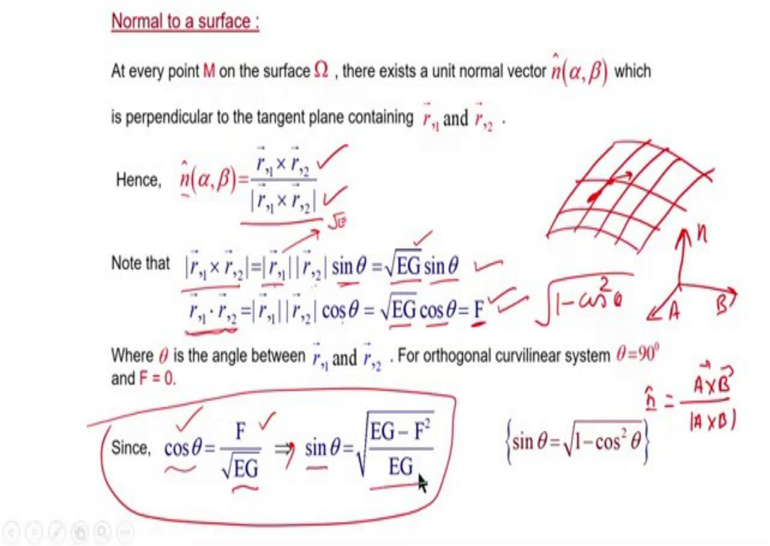

Now, we are interested to find the normal to a surface. We have said that over this surface at a particular point, the tangents are defined. So, they lie in a plane that is known as a tangent plane. Now we are interested to find out a unit normal like in your class 12 level or undergraduate level if 2 vectors are working if there is a normal out of the plane.

So, you can say that $\hat{n} = \frac{A x B}{|A x B|}$

Same way here $r_{,1}$ and $r_{,2}$ lies in one plane and $\hat{n}$ is perpendicular to that which is known as the surface normal or a unit normal vector that can be found out by

$$\hat{n}(\alpha,\beta) = \frac{r_{,1} x r_{,2}}{|r_{,1} x r_{,2}|}$$

So, the cross product of 2 vectors can find out by $r_{,1}$ and $r_{,2}$ magnitude and $\sin\theta$ in between that.

So, $r_{,1} = \sqrt{E}$ and $r_{,2} = \sqrt{G}$.

So, we can say that

$$|r_{,1}||r_{,2}|\sin\theta = \sqrt{EG}\sin\theta$$

$$|r_{,1}||r_{,2}|\cos\theta = \sqrt{EG}\cos\theta = F$$

As we have said that for an orthogonal curvilinear system if θ is 0. So, this product will vanish. So, we can find $\cos\theta = \frac{F}{\sqrt{EG}}$ and then $\sin\theta$ can find by $\sin\theta = \sqrt{1-\cos^2\theta}$. So, using these relations $\sin\theta = \sqrt{\frac{EG-F^2}{EG}}$. We can find the angles between, using this formula if F is there, if F is not there then it will be just 1 because θ is 90˚.

Sometimes we use these relations because we are going to develop a theory of surfaces for the general curvilinear system. And later on, we will specialize that it should be the only orthogonal system, but in the theory of surfaces, we consider that there is no orthogonal curvilinear system. So, we assume that F is there.

Then, $\cos\theta = \frac{F}{\sqrt{EG}}$ and $\sin\theta = \sqrt{\frac{EG-F^2}{EG}}$.

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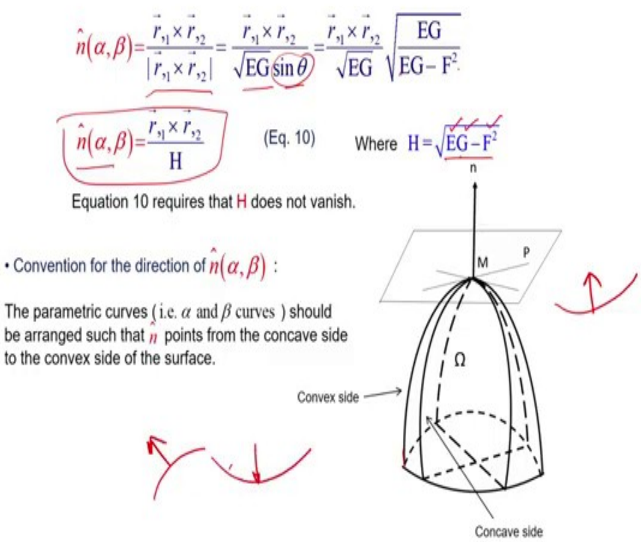

So, again using this concept, $\left|r_{,1} \times r_{,2}\right|$ can be written as $\sqrt{EG}\sin\theta$ and again

$$\sin\theta = \sqrt{\frac{EG - F^2}{EG}}.$$

So, $$\hat{n}(\alpha, \beta) = \frac{r_{,1} \times r_{,2}}{H}.$$

Where, $H = \sqrt{EG - F^2}$.

Calculating the EG and F are easy as compared to calculating the magnitude of this. So, the parametric curve should be arranged such that $\hat{n}$ points from the concave side to the convex side of the surface.

So, if the surfaces are like this, it will go outside. If this is inside then it will go inside. So, depending upon the surface basically outer to the surface normal of surfaces are defined like this.

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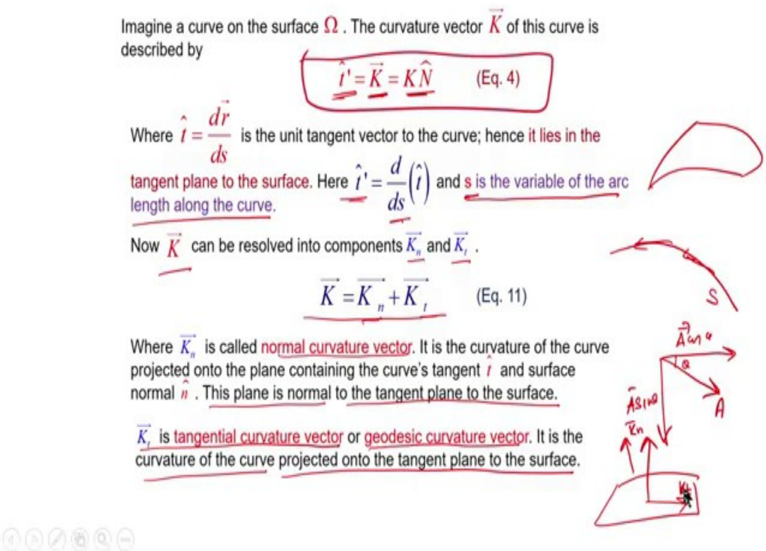

Now, imagine a curve on a surface omega (Ω) the curvature vector K of this curve can be described by $\hat{t}' = K = K\hat{N}\ K\hat{N}$. In the previous lecture we have derived this relation, where

$\hat{t}$ is nothing, but $\frac{dr}{ds}$, it is the unit tangent vector to the curve; hence it lies in the tangent plane of the surface and $\hat{t}' = \frac{d}{ds}\left(\hat{t}\right)$.

So, in most cases, we are using this parameter as ‘s’ is a variable arc length along the curve. It is a variable and it is varying along the arc length of the curve and this K vector can be resolved into two components. Let us say a vector A. It can be resolved in two components along the x-axis and along the y-axis if it makes some angle θ with the x-axis. we can say A cosθ and A sinθ.

In the same way, this K curvature vector because it is on a surface can be resolved into two components K_n and K_t ($K = K_n + K_t$), where K_n is the normal curvature vector and K_t is the tangential curvature vector. So, I will say that it is the curvature of the curve projected onto the plane containing the curves tangent $\hat{t}$ and surface normal $\hat{n}$. So, K_n lies on the plane where the tangent vector and surface normal are there. This plane is normal to the tangent plane to the surface. So, in that direction curvature K_n is defined.

Then, the tangential curvature: sometimes we called it geodesic curvature; it is the curvature of the curve projected on the tangent plane of the surface. So, K_n is outward normal along the normal vector of the surface and K_t is along the tangent of the surface.

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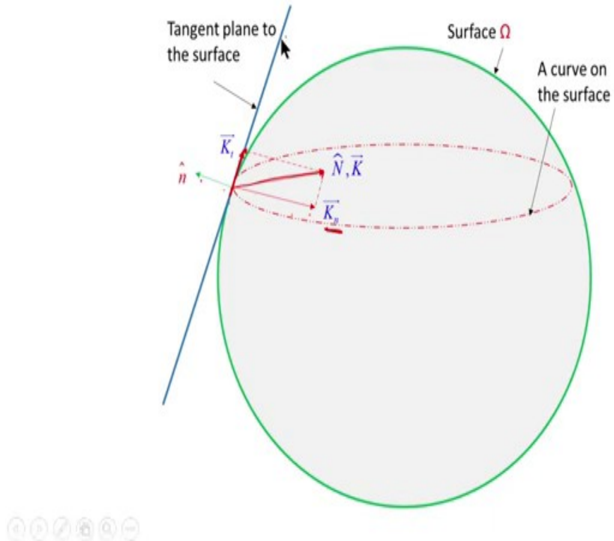

Here is a sphere or spherical surface. So, we can say that this is a curve on that surface. So, this is our vector K and this is our surface normal $\hat{n}$ and this is our tangent plane $\hat{t}$. So, that normal curvature vector will lie in the normal plane and K_t will lie on the tangent plane.

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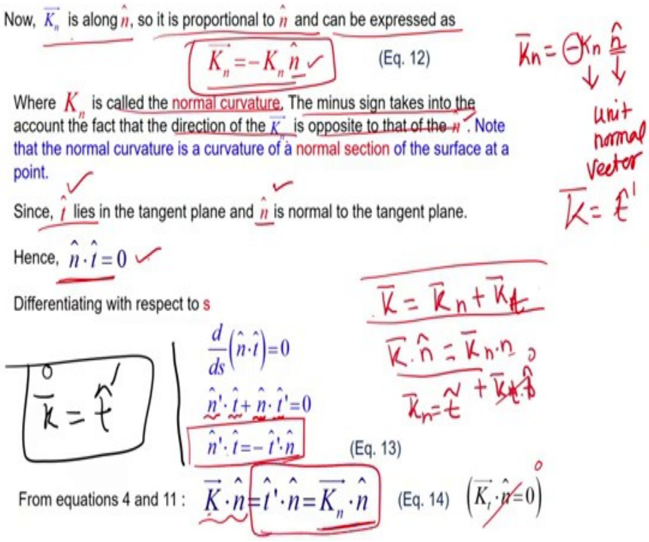

So, K_n is along $\hat{n}$ that is the most important part and it is proportional to $\hat{n}$ and can be expressed as $K_n = -K_n \hat{n}$. So, if you see that the direction of K is opposite to that of $\hat{n}$. So, it will be -K_n. K_n is called normal curvature and unit vector, already I have discussed that the minus sign indicates that direction of K_n is opposite to the $\hat{n}$. Normal curvature is a curvature of a normal section of the surface at a point.

So, this is a very important concept that a normal curvature can be written as minus -K_n. K_n is a normal curvature and $\hat{n}$ is the unit normal vector, where minus sign denotes that K_n is acting in the opposite direction of $\hat{n}$. Already we have said that it lies in the plane of $\hat{t}$ and $\hat{n}$ It lies in the plane of tangent and normal section. So, we can say that $\hat{n}.\hat{t} = 0$ because they are perpendicular to each other normal and tangent plane.

If we take the differentiation of this equation with respect to s

$$\frac{d}{ds}\left(\hat{n}.\hat{t}\right) = 0$$

$$\hat{n}'.\hat{t} + \hat{n}.\hat{t}' = 0$$

This gives a very important relation that differentiation of $\hat{n}'.\hat{t} = -\hat{t}'.\hat{n}$.

We are going to use this equation later on.

First of all, we will say that our basic equation of curvature vector can be divided into two components; normal curvature and tangential curvature. If we multiply this equation with the normal dot $\hat{n}$, then it will be $K_n.\hat{n} + K_t.\hat{n}$ because K_t lies in the tangential plane.

$$K.\hat{n} = \hat{t}'.\hat{n} = K_n.\hat{n} \quad , \quad (K_t.\hat{n} = 0)$$

So, this is our first equation and from there we can say that K is $\hat{t}'$. If you remember that in the previous relation, I said that it is $\hat{t}'$. So, $\hat{t}'.\hat{n} = K_n.\hat{n}$.

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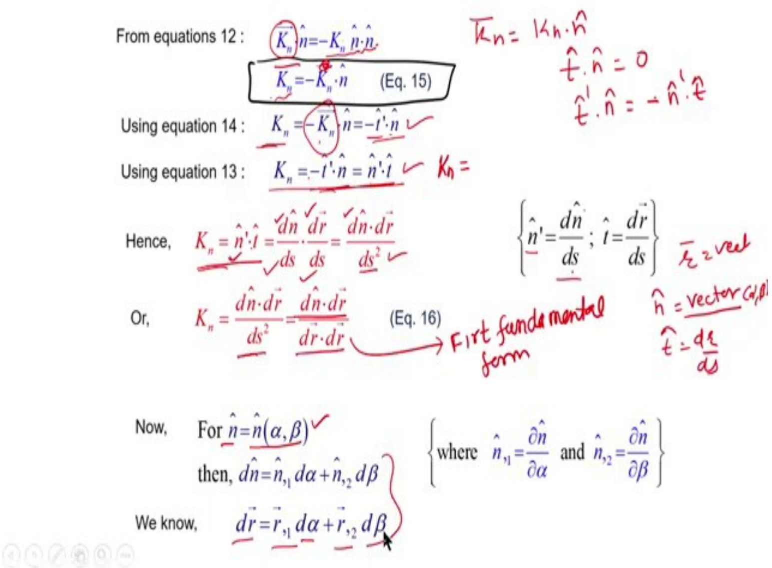

So, substituting $K_n.\hat{n}$,

$$K_n.\hat{n} = -K_n.\hat{n}.\hat{n}$$

So, $\hat{n}.\hat{n}$ becomes 1. So, K_n is just a reverse of it $K_n.\hat{n}$ substituting there.

Now $K_n = -K_n.\hat{n} = -\hat{t}'.\hat{n}$

We know that $K_n = \hat{t}'$.

So, we have obtained that $\hat{t}.\hat{n}$ = 0, from there we have obtained that $\hat{t}'.\hat{n} = -\hat{n}'.\hat{t}$. So, we can use these relations. Here (-) and (-) will get canceled out. Ultimately, we are saying that curvature can be represented as $\hat{n}'.\hat{t}$.

Now, what is $\hat{n}'$? $\hat{n}$ is a vector, like vector $r = r(\alpha, \beta)$. $\hat{n}' = \frac{d\hat{n}}{ds}$ and $\hat{t} = \frac{dr}{ds}$.

Hence, $K_n = \hat{n}'.\hat{t} = \frac{d\hat{n}}{ds}.\frac{dr}{ds} = \frac{d\hat{n}.dr}{(ds)^2}$ or $K_n = \frac{d\hat{n}}{ds}.\frac{dr}{ds} = \frac{d\hat{n}.dr}{(ds)^2}$

When we put it together, the bottom one gives you ds square $(ds)^2$ which is equal to $dr.dr$. So, ultimately it is the first fundamental form of the equation at the denominator expression for that.

The above expression $d\hat{n}.dr$ is like $r = r(\alpha, \beta)$, $\hat{n} = \hat{n}(\alpha, \beta)$ also. So, change in normal vector will be $d\hat{n} = \hat{n}_{,1}\, d\alpha + \hat{n}_{,2}\, d\beta$.

Then we already know that $dr = r_{,1}\, d\alpha + r_{,2}\, d\beta$. If we multiply these two numerators it will give you this form.

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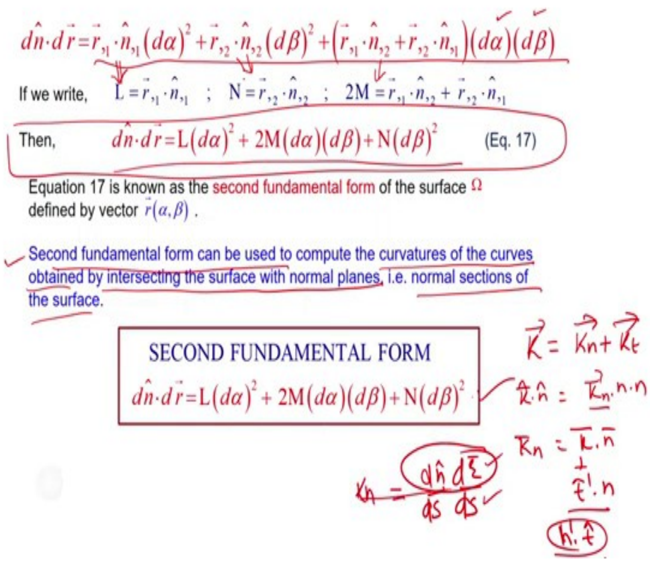

$$d\hat{n}.dr = r_{,1}.\hat{n}_{,1}(d\alpha)^2 + r_{,2}.\hat{n}_{,2}(d\beta)^2 + (r_{,1}.\hat{n}_{,2} + r_{,2}.\hat{n}_{,1})(d\alpha)(d\beta)$$

If we say $r_{,1}.\hat{n}_{,1} = L$, $r_{,2}.\hat{n}_{,2} = N$ and $r_{,1}.\hat{n}_{,2} + r_{,2}.\hat{n}_{,1} = 2M$.

Then, $d\hat{n}.dr = L(d\alpha)^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2$

This equation (17) is known as the second fundamental form of surfaces.

So, we have derived that first fundamental form of surfaces and second fundamental form of surfaces. The second fundamental form of surfaces can be used to compute the curvature of the curves obtained by intersecting the surfaces with the normal plane or the normal sections of the surfaces. How to derive the second fundamental form?

We have to start with the K vector. $K = K_n + K_t$ and then multiply dot $\hat{n}$; then it will give $K_n \hat{n}.\hat{n}$ and from there we can prove that $K_n = K.\hat{n}$ and $K_n = \hat{t}'.\hat{n}$ and $\hat{t}'.\hat{n} = \hat{n}'.\hat{t}$ and then from here $\hat{n}' = \frac{d\hat{n}}{ds}$ and $\hat{t} = \frac{dr}{ds}$.

So, the above equation is known as the second fundamental form of surfaces and the denominator is the first fundamental of surfaces. So, the normal curvature can be represented like this.

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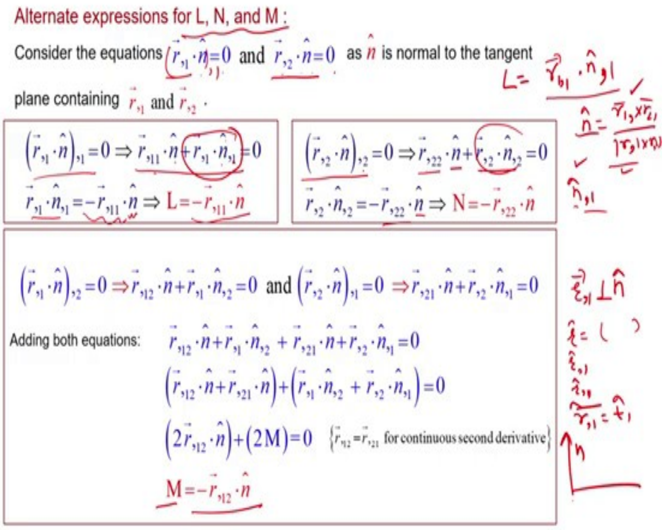

Now, we want to know the alternate expression. Why we are interested in having the alternate expression of L, N and M? The reason behind that is we are considering L= $r_{,1}.\hat{n}_{,1}$.

And you know that $\hat{n}$ expression is already complex. $r_{,1}.r_{,2}$ and its magnitude; if you know the position vector in an explicit form then evaluating $\hat{n}$ is complex and further derivative of it with respect to α and β becomes more complex. To get rid of that can we express it in terms of the normal vector.

We will say, $r_{,1}$ is perpendicular to $\hat{n}$; which is true because $r_{,1}$ is tangent vector lies in a tangent plane and $\hat{n}$ is a surface normal and they are perpendicular to each other. $r_{,1}.\hat{n}=0$ and similarly $r_{,2}.\hat{n}=0$. So, if you take that $r_{,1}=0$; then you can represent and take derivative with respect to 1

$$\left(r_{,1}.\hat{n}\right)_{,1}=0 \Rightarrow r_{,11}.\hat{n}+r_{,1}.\hat{n}_{,1}=0$$

$r_{,1} \cdot \hat{n}_{,1} = -r_{,11} \cdot \hat{n}$

So, this is the expression of L.

So, $L = -r_{,11} \cdot \hat{n}$

This is the simpler form because the position vector is always simple. So, the derivative taking in first and two is easy 2 times, but in a normal section or a normal vector, it is a difficult one.

You have to first find out the cross product then the magnitude of that and expression and then after that, you will take the derivative of that. Similarly, $r_{,2}$ and $\hat{n}$; if you differentiate it with 2.

Then, $\left(r_{,2} \cdot \hat{n} \right)_{,2} = 0 \Rightarrow r_{,22} \cdot \hat{n} + r_{,2} \cdot \hat{n}_{,2} = 0$

$r_{,2} \cdot \hat{n}_{,2} = -r_{,22} \cdot \hat{n}$

It is the expression for N.

$N = -r_{,22} \cdot \hat{n}$

If you take the derivative with respect to 2, this will give you the expression for M.

$M = -r_{,12} \cdot \hat{n}$

So, up to here, I have derived the first fundamental form of surfaces, the second fundamental form of surfaces, and alternate expression for L, M, and N because these are required for deriving the curvatures.

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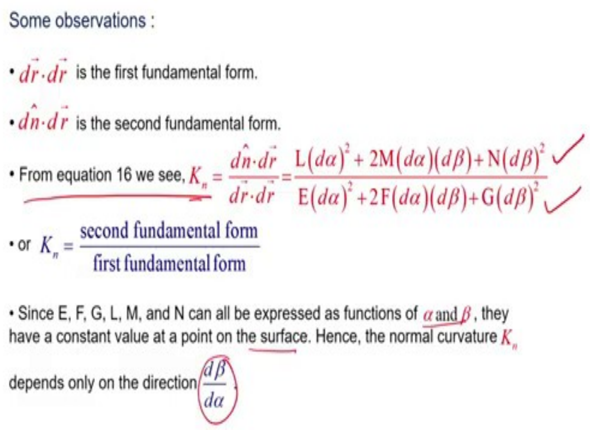

If I write the form of K_n;

$$K_n = \frac{d\hat{n}.dr}{dr.dr}.$$

So, above is the second fundamental bottom is the first fundamental form. Now E, F, G, L, M, and N, all can be expressed in terms of functions of α and β

$$K_n = \frac{d\hat{n}.dr}{dr.dr} = \frac{L(d\alpha)^2 + 2M(d\alpha)(d\beta) + N(d\beta)^2}{E(d\alpha)^2 + 2F(d\alpha)(d\beta) + G(d\beta)^2}.$$

And they will have a constant value at a point on the surface.

So, we aim to find out the maximum and minimum curvature because this is the definition of normal curvature and we are interested to find out the maximum and minimum normal curvature. And these are the function of α and β. So, first, we will take out common and make it $\frac{d\beta}{d\alpha}$.

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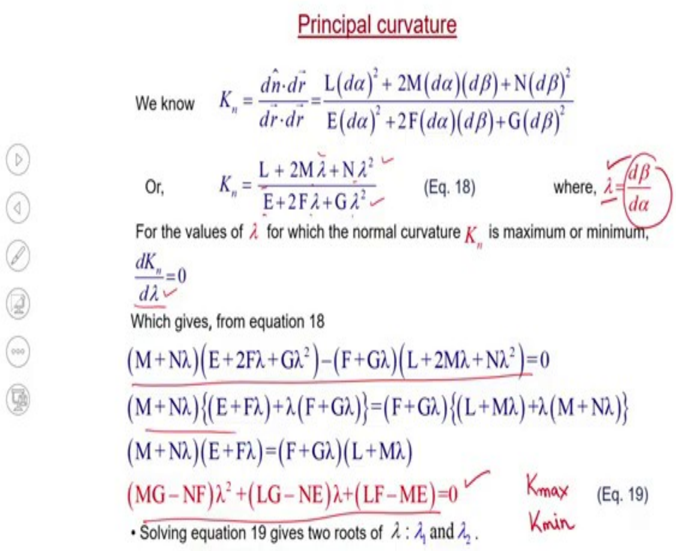

And put it as lambda (λ)

$$K_n = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$$

So, to define a parameter lambda $\lambda = \frac{d\beta}{d\alpha}$

Now we say that if we take the derivative of this K_n with respect to λ

$$\frac{dK_n}{d\lambda} = 0$$

This equation will give you two roots and the first root will be corresponding to maximum and the second root will be corresponding to the minimum.

So, we have just divided with respect to $d\alpha$ and from here $K_n = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$ and take derivative with respect to λ gives you this equation.

$$(M+N\lambda)(E+2F\lambda+G\lambda^2)-(F+G\lambda)(L+2M\lambda+N\lambda^2)=0$$

$$(MG-NF)\lambda^2+(LG-NE)\lambda+(LF-ME)=0$$

And, ultimately this form equates to 0, if we solve this quadratic equation, we will get two roots one corresponding to maximum curvature and the second corresponding to the minimum one.

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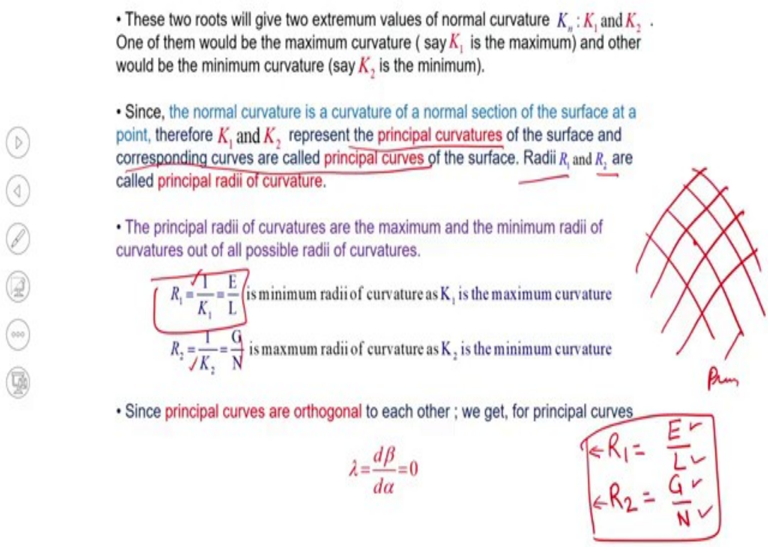

So, these two roots will give extremum values of normal curvature K_n. So, they will be denoted as K_1 and K_2; and they are known as principal curvatures and corresponding curves will be known as principal curves. So, there are several lines on that surface, they are all curves. So, the corresponding curves to the principal curvature are known as principal curves.

And radii R_1 and R_2 can be obtained by inversing the curvature. The important formula for an orthogonal system; if we say that our curvilinear parameters are orthogonal to each other then $R_1 = \frac{E}{L} = \frac{1}{K_1}$ and $R_2 = \frac{G}{N} = \frac{1}{K_2}$. We use this set of equations that $R_1 = \frac{E}{L}$ and $R_2 = \frac{G}{N}$

So, evaluating E, L, G, and N for a surface is easy. So, one can find out what is the maximum radius and what is the minimum radius. With this our topic of the first fundamental form of surfaces and the second fundamental of surfaces ends here, after that I will explain various theorems like the theorem of Weingarten, Rodriguez theorem, Gauss theorem, etc. These are also required for developing the shell theory.

Thank you.