Theory of Composite Shells

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Week – 03

Lecture – 01

Derivation of Governing equations

Dear learners welcome to the 3rd week, lecture 01. In this lecture, I am going to derive the basic Governing equations for a shell

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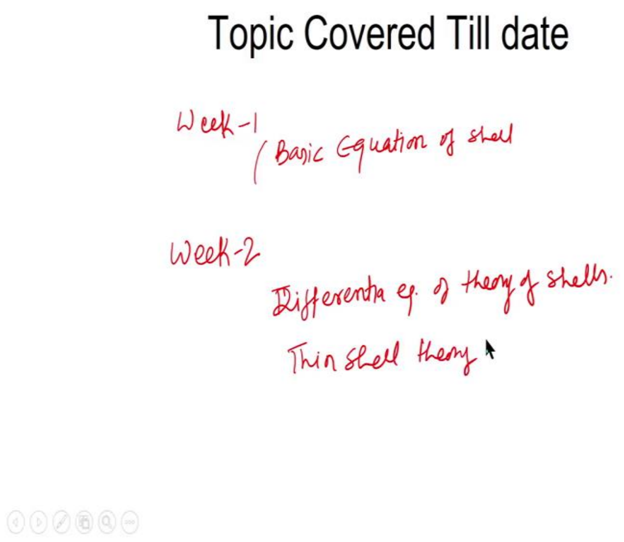

In the first week, we have covered the preliminary basic equations of shells. In the second week, we have derived the theory of differential equations of the theory of shells and started the basic formulation of thin shell theory. In the 3rd week, I am going to continue the development of the theory of shell governing equations.

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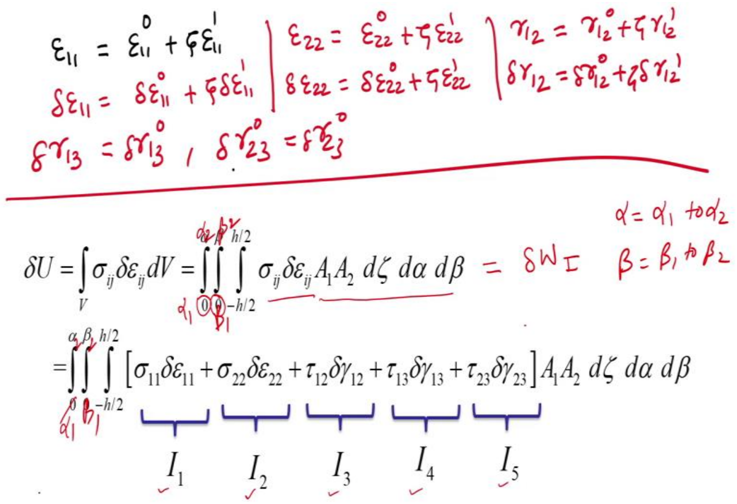

We were on the strain energy or elastic strain energy of a shell that can be written as ∂U or sometimes it is known as ∂W_I internal work done. For a 3-dimensional case:

$$\partial U = \int_v \sigma_{ij} \partial \varepsilon_{ij} dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{ij} \partial \varepsilon_{ij} A_1 A_2 (d\alpha)(d\beta)(d\varsigma)$$

Here, dV is the volume of the shell. we can write that volume of shell is $dV = A_1 A_2 (d\alpha)(d\beta)(d\varsigma)$.

Limits are 0 to α or 0 to β or sometimes we may say that α goes from α_1 to α_2 and β goes from β_1 to β_2. It doesn't need to be 0.

If we open this explicitly, ∂U will be:

$$\int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[\underbrace{\sigma_{11} \partial \varepsilon_{11}}_{I_1} + \underbrace{\sigma_{22} \partial \varepsilon_{22}}_{I_2} + \underbrace{\tau_{12} \partial \gamma_{12}}_{I_3} + \underbrace{\tau_{13} \partial \gamma_{13}}_{I_4} + \underbrace{\tau_{23} \partial \gamma_{23}}_{I_5} \right] A_1 A_2 (d\alpha)(d\beta)(d\varsigma)$$

It will be a very huge integral, but we will divide it into sub integrals like I_1, I_2, I_3, I_4 and I_5, and extract the terms in the integrations.

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$$\delta U = \int_V \sigma_{ij}\delta\varepsilon_{ij}dV = \int_0^{\alpha}\int_0^{\beta}\int_{-h/2}^{h/2} \sigma_{ij}\delta\varepsilon_{ij}A_1A_2\, d\zeta\, d\alpha\, d\beta$$ ← Variation of Strain energy term

$$= \int_0^{\alpha}\int_0^{\beta}\int_{-h/2}^{h/2} \left[\underbrace{\sigma_{11}\delta\varepsilon_{11}}_{I_1} + \underbrace{\sigma_{22}\delta\varepsilon_{22}}_{I_2} + \underbrace{\tau_{12}\delta\gamma_{12}}_{I_3} + \underbrace{\tau_{13}\delta\gamma_{13}}_{I_4} + \underbrace{\tau_{23}\delta\gamma_{23}}_{I_5}\right] A_1A_2\, d\zeta\, d\alpha\, d\beta$$

Calculation of I_1

$$I_1 = \int_0^{\alpha}\int_0^{\beta}\int_{-h/2}^{h/2} \left[\sigma_{11}\delta\varepsilon_{11}\right] A_1A_2\, d\zeta\, d\alpha\, d\beta$$

For the case of calculation:

$$I_1 = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\int_{\frac{-h}{2}}^{\frac{h}{2}} (\sigma_{11}\partial\varepsilon_{11}) A_1A_2 (d\alpha)(d\beta)(d\varsigma)$$

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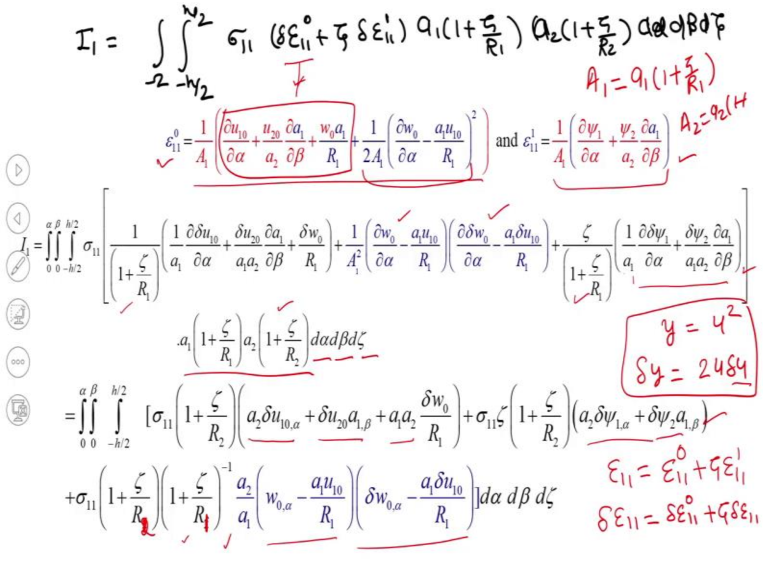

$$\varepsilon_{11} = \varepsilon_{11}^0 + \varsigma\varepsilon_{11}^1 .$$

If you take the first variation in this, it will be:

$$\partial\varepsilon_{11}^0 + \varsigma\, \partial\varepsilon_{11}^1 .$$

In the expression of ε_{11}^{0}:

$$\varepsilon_{11}^{0} = \frac{1}{A_1}\left(\frac{\partial u_{10}}{\partial \alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1} + \frac{1}{2A_1}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)^2\right); \varepsilon_{11}^{1} = \frac{1}{A_1}\left(\frac{\partial \psi_1}{\partial \alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right)$$

Here, $\frac{\partial u_{10}}{\partial \alpha} + \frac{u_{20}}{a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1}$ terms are related to linear, $\frac{1}{2A_1}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)^2$ these are the non-linear terms, and $\frac{1}{A_1}\left(\frac{\partial \psi_1}{\partial \alpha} + \frac{\psi_2}{a_2}\frac{\partial a_1}{\partial \beta}\right)$terms are related to the curvature.

Here, in ε_{11}^{0} we take 1 by A_1 common and in ε_{11}^{1} also 1 by A_1 will be common. Substitute the expression of $\partial \varepsilon_{11}$ by taking ∂:

$$I_1 = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left[\begin{array}{l} \frac{1}{\left(1+\frac{\varsigma}{R_1}\right)}\frac{1}{a_1}\frac{\partial u_{10}}{\partial \alpha} + \frac{\partial u_{20}}{a_1 a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1} \\ + \frac{1}{A_1^2}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial \partial w_0}{\partial \alpha} - \frac{a_1 \partial u_{10}}{R_1}\right) \\ + \frac{\varsigma}{\left(1+\frac{\varsigma}{R_1}\right)}\left(\frac{1}{a_1}\frac{\partial \partial \psi_1}{\partial \alpha} + \frac{\partial \psi_2}{a_1 a_2}\frac{\partial a_1}{\partial \beta}\right) \end{array}\right] a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)(d\alpha)(d\beta)(d\varsigma)$$

The linear term will be: $\frac{1}{\left(1+\frac{\varsigma}{R_1}\right)}\frac{1}{a_1}\frac{\partial u_{10}}{\partial \alpha} + \frac{\partial u_{20}}{a_1 a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0 a_1}{R_1}$

Similarly, the nonlinear term will be:

$$\frac{1}{A_1^2}\left(\frac{\partial w_0}{\partial \alpha} - \frac{a_1 u_{10}}{R_1}\right)\left(\frac{\partial \partial w_0}{\partial \alpha} - \frac{a_1 \partial u_{10}}{R_1}\right)$$

For example, y $= u^2$ and if you want to take the first variation of y, then $\partial y = 2u\partial u$

Similarly, twice will come up, it will get canceled and nonlinear terms will be expressed like that. Same way the curvature term will be:

$$\frac{\varsigma}{\left(1+\frac{\varsigma}{R_1}\right)}\left(\frac{1}{a_1}\frac{\partial\partial\psi_1}{\partial\alpha}+\frac{\partial\psi_2}{a_1a_2}\frac{\partial a_1}{\partial\beta}\right)$$

Here a_1 is kept inside because we are going to use this in the definition itself.

The dV will be: $a_1\left(1+\frac{\varsigma}{R_1}\right)a_2\left(1+\frac{\varsigma}{R_2}\right)(d\alpha)(d\beta)(d\varsigma)$

If you take it inside, then $\left(1+\frac{\varsigma}{R_1}\right)$ will be canceled, inside remains a_1, a_2 and $\left(1+\frac{\varsigma}{R_2}\right)$.

σ_{11} and $\left(1+\frac{\varsigma}{R_2}\right)$ will be taken common and if you put a_1, a_2 inside, then a_1 get canceled. Finally, the term will look like this:

$$I_1=\int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\int_{\frac{-h}{2}}^{\frac{h}{2}}\left[\begin{array}{l}\sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(a_2\partial u_{10},_{\alpha}+\partial u_{20}a_1,_{\beta}+a_1a_2\frac{\partial w_0}{R_1}\right)\\+\sigma_{11}\varsigma\left(1+\frac{\varsigma}{R_2}\right)\left(a_2\partial\psi_1,_{\alpha}+\partial\psi_2a_1,_{\beta}\right)\\+\sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)\frac{a_2}{a_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\partial w_0,_{\alpha}-\frac{a_1\partial u_{10}}{R_1}\right)\end{array}\right](d\alpha)(d\beta)(d\varsigma)$$

The linear term will be:

$$\sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(a_2\partial u_{10},_{\alpha}+\partial u_{20}a_1,_{\beta}+a_1a_2\frac{\partial w_0}{R_1}\right).$$

Same way, the non-linear part will be:

$$\sigma_{11}\varsigma\left(1+\frac{\varsigma}{R_2}\right)\left(a_2\partial\psi_1,_{\alpha}+\partial\psi_2a_1,_{\beta}\right).$$

In the non-linear part, there is A_1^2, so, A_1 will get canceled, and another A_1 remains at the bottom. The curvature term will be:

$$\sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)\frac{a_2}{a_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\partial w_0,_{\alpha}-\frac{a_1\partial u_{10}}{R_1}\right).$$

Now, we can use the definition of stress resultant.

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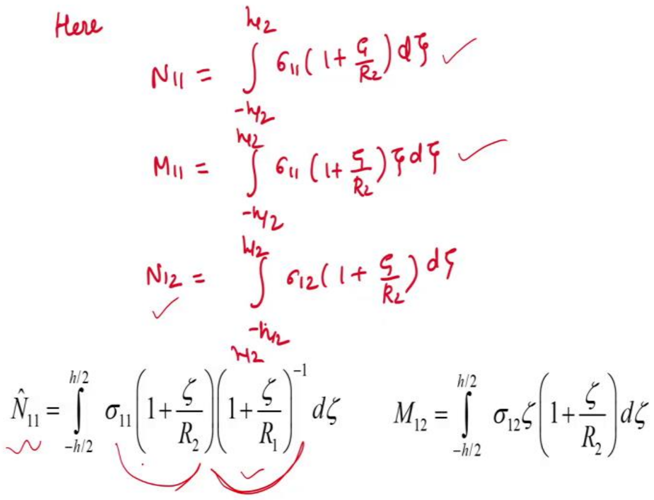

As per the definition of stress resultant:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right)(d\varsigma) \qquad M_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right)\varsigma(d\varsigma) \qquad N_{12} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{12} \left(1 + \frac{\varsigma}{R_2}\right)(d\varsigma)$$

Right now, we are not using the definition of N_{12}, but we are using the definition of N_{11}, where N_{11} is defined as another resultant:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11} \left(1 + \frac{\varsigma}{R_2}\right)\left(1 + \frac{\varsigma}{R_1}\right)^{-1} (d\varsigma)$$

When we derive a non-linear one, then we have to consider this new definition N_{11}. Sometimes in some books, it may be star, tilde, delta means we have to give a separate symbol for this definition.

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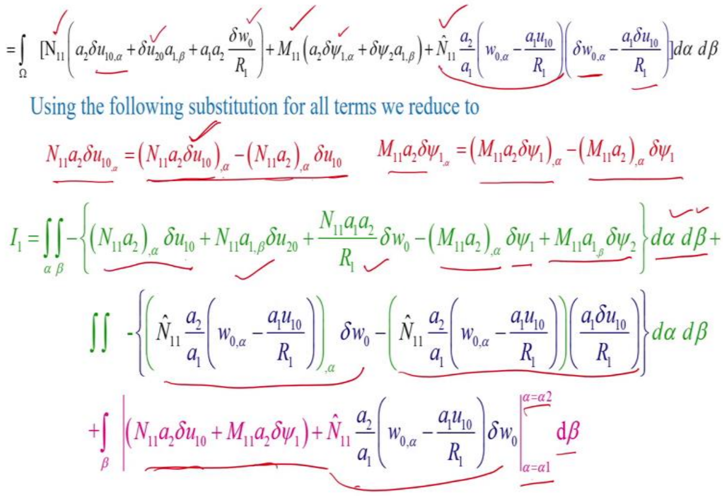

If we use these definitions, they can be written like this N_{11}, M_{11}, N_{11}. Now, in term $\partial u_{10},_{\alpha}$, there is a derivative. We want to reduce this derivative; we want to get rid of this derivative. The standard approach is that $N_{11}a_2\partial u_{10}$ can be written as:

$$N_{11}a_2\partial u_{10} = \left(N_{11}a_2\partial u_{10}\right),_{\alpha} - \left(N_{11}a_2\right),_{\alpha}\partial u_{10}$$

We have to apply the same procedure and $M_{11}a_2\partial\psi_1$ will be:

$$M_{11}a_2\partial\psi_1 = \left(M_{11}a_2\partial\psi_1\right),_{\alpha} - \left(M_{11}a_2\right),_{\alpha}\partial\psi_1 .$$

Same way, we will treat this N_{11} as one term, and here $\partial w_0,_{\alpha}$ is a derivative comma α so, it will give two terms and $\frac{a_1\partial u_{10}}{R_1}$ will give you one term.

From this, we will get:

$$I_1 = \int_{\alpha}\int_{\beta} -\left\{\left(N_{11}a_2\right),_{\alpha}\partial u_{10} + N_{11}a_1,_{\beta}\partial u_{20} + \frac{N_{11}a_1a_2}{R_1}\partial w_0 - \left(M_{11}a_2\right),_{\alpha}\partial\psi_1 + M_{11}a_1,_{\beta}\partial\psi_2\right\}(d\alpha)(d\beta) +$$

$$\iint -\left\{\left(N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha} - \frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}\partial w_0 - \left(N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha} - \frac{a_1u_{10}}{R_1}\right)\right)\left(\frac{a_1\partial u_{10}}{R_1}\right)\right\}(d\alpha)(d\beta) +$$

$$\int_{\beta}\left|\left(N_{11}a_2\partial u_{10} + M_{11}a_2\partial\psi_1\right) + \widehat{N_{11}\frac{a_2}{a_1}}\left(w_0,_{\alpha} - \frac{a_1u_{10}}{R_1}\right)\partial w_0\right|_{\beta=\beta_2}^{\alpha=\alpha_2}(d\beta)$$

I have done it purposefully separately so that we can, later on, take it out of the equation.

$$.\left(N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}\partial w_0$$ will be one contribution and

$$-\left(N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)\left(\frac{a_1\partial u_{10}}{R_1}\right)$$ will be the second contribution and the very first term

$N_{11}a_2\partial u_{10}$ will go to the boundary along α because here α is derivative. We can integrate along α, α will be limits, α is equal to α_1 to α_2.

One from linear $N_{11}a_2\partial u_{10}$, one from the moment $M_{11}a_2\partial\psi_1$, and one from non-linear $N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\partial w_0$, we will have three terms which will be on the boundary and integration on the $d\beta$. These are generally colored pink so that we can categorize them as boundary terms. For I_2, I_3, I_4 and I_5 we will proceed with the same procedure.

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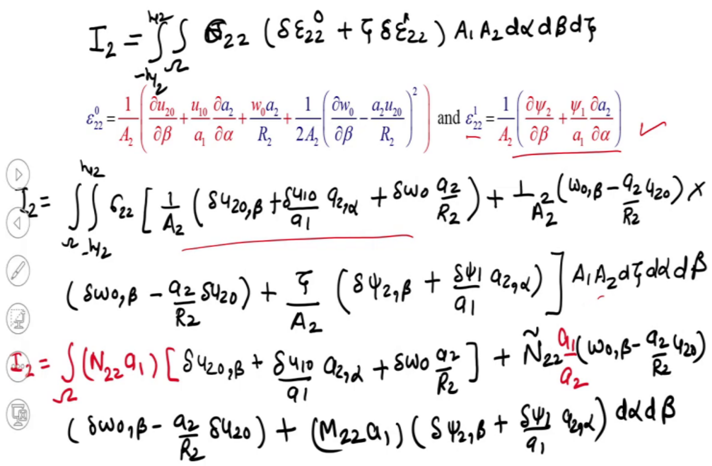

$$I_2=\int_{\Omega}\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{22}\left(\partial\varepsilon_{22}^{0}+\varsigma\partial\varepsilon_{22}^{1}\right)A_1A_2\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right).$$

First of all, we have to write the explicit expression of $\partial\varepsilon_{22}^{0}$ and $\partial\varepsilon_{22}^{1}$.

$$\varepsilon_{22}^{0}=\frac{1}{A_2}\left(\frac{\partial u_{20}}{\partial\beta}+\frac{u_{10}}{a_1}\frac{\partial a_2}{\partial\alpha}+\frac{w_0a_2}{R_2}+\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\beta}-\frac{a_2u_{20}}{R_2}\right)^2\right);\quad \varepsilon_{22}^{1}=\frac{1}{A_2}\left(\frac{\partial\psi_2}{\partial\beta}+\frac{\psi_1}{a_1}\frac{\partial a_2}{\partial\alpha}\right)$$

If we put the variation, here you take $\frac{1}{A_2}$ common, from there, A_2 will get canceled A_1 will remain.

$$I_2=\int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_{22}\left[\begin{array}{l}\frac{1}{A_2}\left(\partial u_{20},_{\beta}+\frac{\partial u_{10}}{a_1}a_2,_{\alpha}+\partial w_0\frac{a_2}{R_2}\right)+\frac{1}{A_2{}^2}\left(w_0,_{\beta}-\frac{a_2}{R_2}u_{20}\right)\\ \left(\partial w_0,_{\beta}-\frac{a_2}{R_2}\partial u_{20}\right)+\frac{\varsigma}{A_2}\left(\partial\psi_2,_{\beta}+\frac{\partial\psi_1}{a_1}a_2,_{\alpha}\right)\end{array}\right](d\alpha)(d\beta)(d\varsigma)$$

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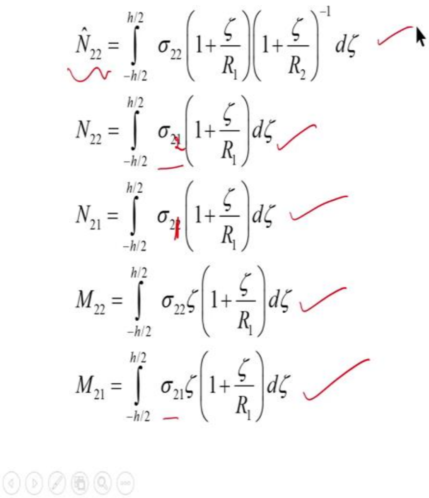

The definition of N_{22} is:

$$N_{22}=\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_{22}\left(1+\frac{\varsigma}{R_1}\right)(d\varsigma)$$

The definition of N_{21} is:

$$N_{21}=\int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_{21}\left(1+\frac{\varsigma}{R_1}\right)(d\varsigma),$$

$$M_{22} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22}\varsigma\left(1+\frac{\varsigma}{R_1}\right)(d\varsigma)$$

$$M_{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{21}\varsigma\left(1+\frac{\varsigma}{R_1}\right)(d\varsigma)$$

Due to the non-linear term, we have to define a new variable which is N_{22} :

$$N_{22} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{22}\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)^{-1}(d\varsigma) \; .$$

If we use these definitions, the expression for I_2 will become:

$$I_2 = \int_{\Omega}(N_{22}a_1)\left(\partial u_{20},_{\beta} + \frac{\partial u_{10}}{a_1}a_2,_{\alpha} + \partial w_0\frac{a_2}{R_2}\right) + N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta} - \frac{a_2}{R_2}u_{20}\right)\left(\partial w_0,_{\beta} - \frac{a_2}{R_2}\partial u_{20}\right)$$
$$+(M_{22}a_1)\left(\partial\psi_2,_{\beta} + \frac{\partial\psi_1}{a_1}a_2,_{\alpha}\right)(d\alpha)(d\beta)$$

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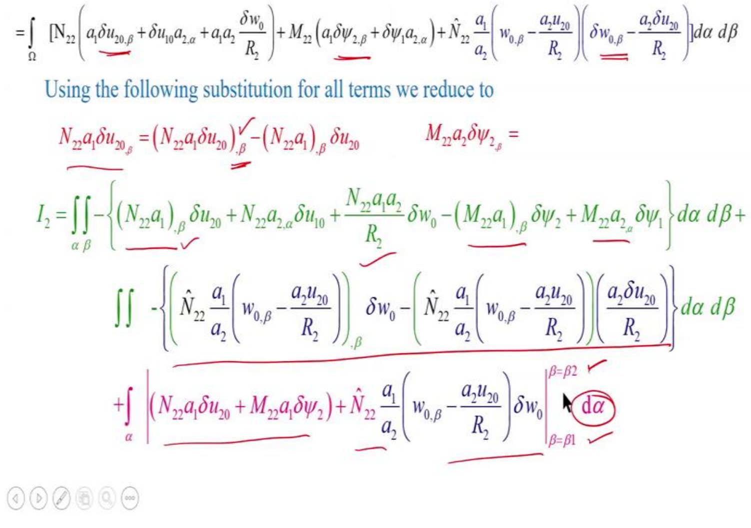

If we substitute here and following the same procedure as we have done for I_1 case to get rid of derivative β terms, ultimately, it will give you this expression:

$$I_2 = \int_{\alpha}\int_{\beta} -\left\{ \left(N_{22}a_1\right),_{\beta}\partial u_{20} + N_{22}a_2,_{\alpha}\partial u_{10} + \frac{N_{22}a_1a_2}{R_2}\partial w_0 - \left(M_{22}a_1\right),_{\beta}\partial\psi_2 + M_{22}a_2,_{\alpha}\partial\psi_1 \right\}(d\alpha)(d\beta)+$$

$$\int\int -\left\{ \left(N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta} - \frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}\partial w_0 - \left(N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta} - \frac{a_2u_{20}}{R_2}\right)\right)\left(\frac{a_2\partial u_{20}}{R_2}\right)\right\}(d\alpha)(d\beta)+$$

$$\int_{\alpha}\left|\left(N_{22}a_1\partial u_{20} + M_{22}a_1\partial\psi_2\right) + N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta} - \frac{a_2u_{20}}{R_2}\right)\partial w_0\right|_{\beta=\beta_1}^{\beta=\beta_2}(d\alpha)$$

Here, the first term is the linear contribution, second term is the non- linear contribution and the last term is the boundary term, but this time, it will be on the boundary of β , β_1 to β_2 and integration over $d\alpha$.

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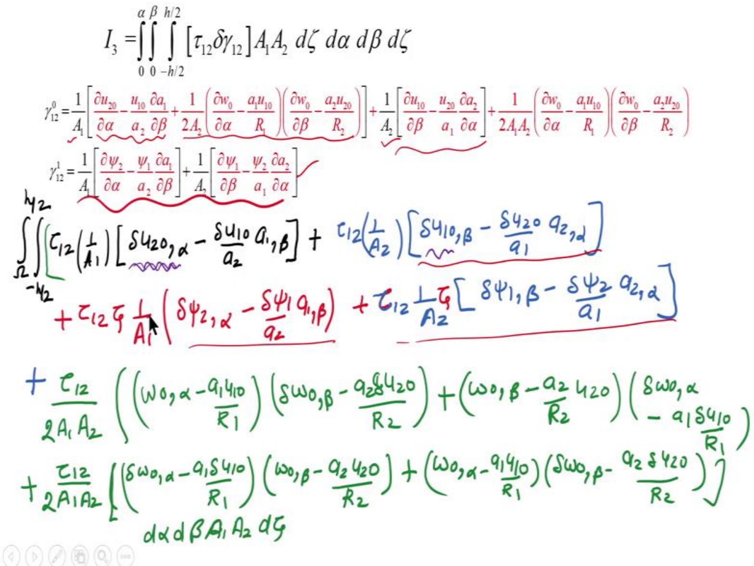

Now, the integration of I_3:

$$I_3 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}}\left(\tau_{12}\partial\gamma_{12}\right)A_1A_2(d\alpha)(d\beta)(d\varsigma)$$

γ_{12}^0 expression is very big and it has two parts: one part in which $\frac{1}{A_1}$ is common in another part $\frac{1}{A_2}$ is common.

$$\gamma_{12}^{0}=\frac{1}{A_1}\left(\frac{\partial u_{20}}{\partial\alpha}+\frac{u_{10}}{a_2}\frac{\partial a_1}{\partial\beta}+\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta}-\frac{a_2u_{20}}{R_2}\right)\right)$$

$$+\frac{1}{A_2}\left(\frac{\partial u_{10}}{\partial\beta}+\frac{u_{20}}{a_1}\frac{\partial a_2}{\partial\alpha}\right)+\frac{1}{2A_1A_2}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta}-\frac{a_2u_{20}}{R_2}\right)$$

$\frac{\partial u_{20}}{\partial\alpha}+\frac{u_{10}}{a_2}\frac{\partial a_1}{\partial\beta}$ is corresponding to the linear part

$\frac{1}{2A_2}\left(\frac{\partial w_0}{\partial\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\frac{\partial w_0}{\partial\beta}-\frac{a_2u_{20}}{R_2}\right)$ is corresponding to the non-linear part.

And $\gamma_{12}^{1}=\frac{1}{A_1}\left(\frac{\partial\psi_2}{\partial\alpha}+\frac{\psi_1}{a_2}\frac{\partial a_1}{\partial\beta}\right)+\frac{1}{A_2}\left(\frac{\partial\psi_1}{\partial\beta}+\frac{\psi_2}{a_1}\frac{\partial a_2}{\partial\alpha}\right)$

Here, 1 by A_1 having some coefficients and 1 by A_2 having some coefficients. From here, I_3 will be:

$$I_3=\int_{\Omega}\int_{\frac{-h}{2}}^{\frac{h}{2}}\begin{bmatrix}\tau_{12}\frac{1}{A_1}\left(\partial u_{20},_{\alpha}-\frac{\partial u_{10}}{a_2}a_1,_{\beta}\right)+\tau_{12}\frac{1}{A_2}\left(\partial u_{10},_{\beta}-\frac{\partial u_{20}}{a_1}a_2,_{\alpha}\right)\\+\tau_{12}\varsigma\frac{1}{A_1}\left(\partial\psi_2,_{\alpha}-\frac{\partial\psi_1}{a_2}a_1,_{\beta}\right)+\tau_{12}\frac{1}{A_2}\varsigma\left(\partial\psi_1,_{\beta}-\frac{\partial\psi_2}{a_1}a_2,_{\alpha}\right)\\+\frac{\tau_{12}}{2A_1A_2}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\partial w_0,_{\beta}-\frac{a_2\partial u_{20}}{R_2}\right)+\left(w_0,_{\beta}-\frac{a_2}{R_2}u_{20}\right)\left(\partial w_0,_{\alpha}-\frac{a_1\partial u_{10}}{R_1}\right)\\+\frac{\tau_{12}}{2A_1A_2}\left(\partial w_0,_{\alpha}-\frac{a_1\partial u_{10}}{R_1}\right)\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)+\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\left(\partial w_0,_{\beta}-\frac{a_2\partial u_{20}}{R_2}\right)\end{bmatrix}$$

$$A_1A_2(d\alpha)(d\beta)(d\varsigma)$$

At the last the green one is the non-linear term.

In the first term, A_1 will get canceled, A_2 will be above. In the second term, A_2 will be canceled and A_1 will be above. We will get the two contributions.

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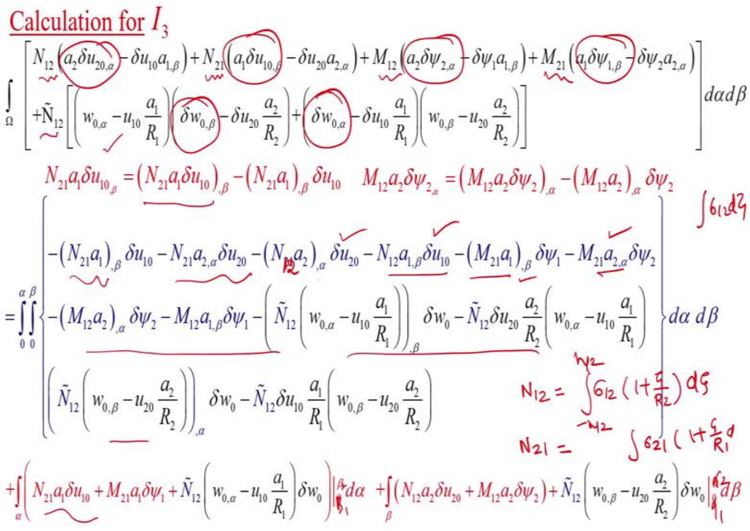

We already know the definition of N_{12} is:

$$N_{12} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{12}\left(1+\frac{\varsigma}{R_2}\right)(d\varsigma).$$

The definition of N_{21} is:

$$N_{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{21}\left(1+\frac{\varsigma}{R_1}\right)(d\varsigma).$$

$$I_3 = \int_{\Omega} \left[\begin{array}{l} N_{12}\left(a_2\partial u_{20,\alpha} - \partial u_{10} a_1,_{\beta}\right) + N_{21}\left(a_1\partial u_{10,\beta} - \partial u_{20} a_2,_{\alpha}\right) \\ +M_{12}\left(a_2\partial \psi_{2,\alpha} - \partial \psi_1 a_1,_{\beta}\right) + M_{21}\left(a_1\partial \psi_{1,\beta} + \partial \psi_2 a_{2,\alpha}\right) \\ +N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\left(\partial w_{0,\beta} - \partial u_{20}\frac{a_2}{R_2}\right) + \left(\partial w_{0,\alpha} - \partial u_{10}\frac{a_1}{R_1}\right)\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right) \end{array} \right] (d\alpha)(d\beta)$$

The first term will give a contribution of N_{12}, the second term will give a contribution of N_{21} and third & fourth terms are ς M_{12} and ς M_{21}. And here is a new contribution N.

If you go back in this combination A_1 A_2 will get canceled. Ultimately, we have a new definition of N_{12} is:

$$\tilde{N}_{12} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{12}\left(d\varsigma\right)$$

$N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\left(\partial w_{0,\beta} - \partial u_{20}\frac{a_2}{R_2}\right) + \left(\partial w_{0,\alpha} - \partial u_{10}\frac{a_1}{R_1}\right)\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)$ This is the non-linear contribution. The terms like $a_2\partial u_{20,\alpha}$, $a_1\partial u_{10,\beta}$, $a_2\partial\psi_{2,\alpha}$, $a_1\partial\psi_{1,\beta}$, $\partial w_{0,\beta}$, and $\partial w_{0,\alpha}$ are having derivative along α and β.

We have to convert it into a primary form and proceed further. It will become:

$$\int_0^{\alpha}\int_0^{\beta}\left[\begin{array}{l} \left(N_{21}a_1\right)_{,\beta}\partial u_{10} - N_{21}a_{2,\alpha}\partial u_{20} - \left(N_{12}a_2\right)_{,\alpha}\partial u_{20} - N_{12}a_{1,\beta}\partial u_{10} - \left(M_{21}a_1\right)_{,\beta}\partial\psi_1 - M_{21}a_{2.\alpha}\partial\psi_2 \\ -\left(M_{12}a_2\right)_{,\alpha}\partial\psi_2 - M_{12}a_{1,\beta}\partial\psi_1 - N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\partial w_0 - N_{12}\partial u_{20}\frac{a_2}{R_2}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right) \\ N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\partial w_0 - N_{12}\partial u_{10}\frac{a_1}{R_1}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right) \end{array}\right]\left(d\alpha\right)\left(d\beta\right)$$

In this way, you will get all the terms that will be in the area and the rest of the terms like the whole derivative with respect to β and whole derivative with respect to α going to the boundary.

There will be some terms which will be on the boundary of β: β_1 to β_2

$$\int_{\beta}\left(N_{12}a_2\partial u_{20} + M_{12}a_2\partial\psi_2 + N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\partial w_0\right)\Bigg|_{\alpha_1}^{\alpha_2} d\beta$$

And there will be some terms which will be on the boundary α: α_1 to α_2

$$\int_{\alpha}\left(N_{21}a_1\partial u_{10} + M_{21}a_1\partial\psi_1 + N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\partial w_0\right)\Bigg|_{\beta_1}^{\beta_2} d\alpha \,.$$

Whenever we have a whole derivative with respect to β, that will go to the β boundary. When we have a whole derivative with respect to α, then it will go to α boundary.

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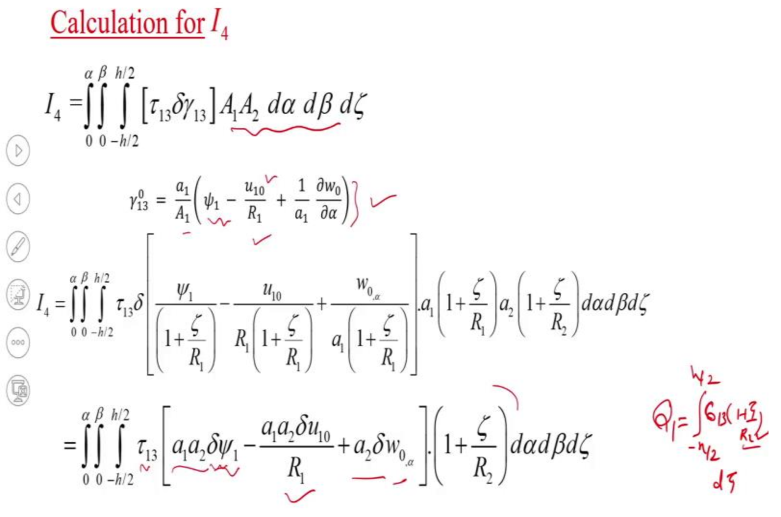

Now, come to the I_4 expression:

$$I_4 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left(\tau_{13}\partial\gamma_{13}\right) A_1 A_2 \left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

Here γ_{12}^{0} will be:

$$\gamma_{12}^{0} = \frac{a_1}{A_1}\left(\psi_1 - \frac{u_{10}}{R_1} + \frac{1}{a_1}\frac{\partial w_0}{\partial \alpha}\right)$$

If you substitute it in the equation I_4 will be:

$$\int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{13}\partial\left(\frac{\psi_1}{\left(1+\frac{\varsigma}{R_1}\right)} - \frac{u_{10}}{R_1\left(1+\frac{\varsigma}{R_1}\right)} + \frac{w_{0,\alpha}}{a_1\left(1+\frac{\varsigma}{R_1}\right)}\right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right).$$

$$I_4 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{13}\partial\left(a_1 a_2\partial\psi_1 - \frac{a_1 a_2 \partial u_{10}}{R_1} + a_2\partial w_{0,\alpha}\right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

The definition of Q_1:

$$Q_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{13}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$$

It will be replaced by Q_1 and the whole integral becomes in plane integral α and β.

$$I_4 = \int_0^{\alpha}\int_0^{\beta}\left[Q_1\left(a_1a_2\partial\psi_1\right) - Q_1\frac{a_1a_2\partial u_{10}}{R_1} + Q_1a_2\partial w_{0,\alpha}\right](d\alpha)(d\beta)$$

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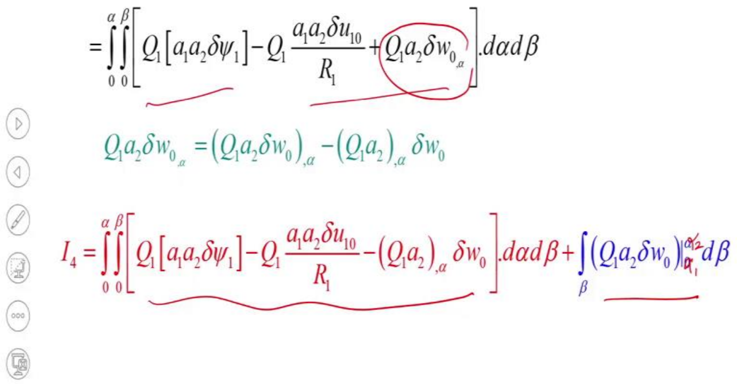

And finally, the terms will look like this

$$I_4 = \int_0^{\alpha}\int_0^{\beta}\left[Q_1\left(a_1a_2\partial\psi_1\right) - Q_1\frac{a_1a_2\partial u_{10}}{R_1} + Q_1a_2\partial w_{0,\alpha}\right](d\alpha)(d\beta).$$

This term $Q_1a_2\partial w_{0,\alpha}$ is having derivative with respect to α. Using the same concept, they become:

$$I_4 = \int_0^{\alpha}\int_0^{\beta}\left[Q_1\left(a_1a_2\partial\psi_1\right) - Q_1\frac{a_1a_2\partial u_{10}}{R_1} + \left(Q_1a_2\right)_{,\alpha}\partial w_0\right](d\alpha)(d\beta) \text{ on the area}$$

and this term $\int_{\beta}\left(Q_1a_2\partial w_0\right)\Big|_{\alpha_1}^{\alpha_2}d\beta$ will be on the boundary.

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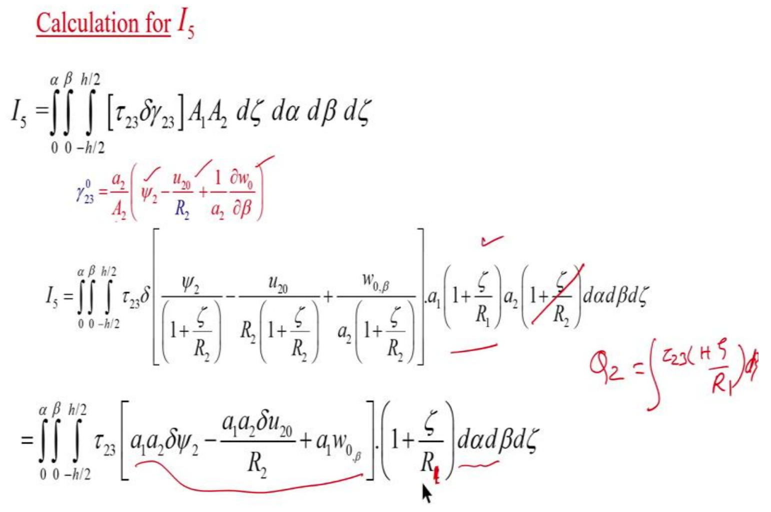

Similarly, I_5 will be:

$$I_5 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left(\tau_{23}\partial\gamma_{23}\right) A_1 A_2 \left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

$$\gamma_{23}^{0} = \frac{a_2}{A_2}\left(\psi_2 - \frac{u_{20}}{R_2} + \frac{1}{a_2}\frac{\partial w_0}{\partial \beta}\right)$$

If we substitute the value taking del:

$$I_5 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{23}\partial\left(\frac{\psi_2}{\left(1+\frac{\varsigma}{R_2}\right)} - \frac{u_{20}}{R_2\left(1+\frac{\varsigma}{R_2}\right)} + \frac{w_{0,\beta}}{a_2\left(1+\frac{\varsigma}{R_2}\right)}\right) a_1\left(1+\frac{\varsigma}{R_1}\right) a_2\left(1+\frac{\varsigma}{R_2}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right)$$

Then, the terms will look like this:

$$I_5 = \int_0^{\alpha}\int_0^{\beta}\int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{13}\partial\left(a_1 a_2\partial\psi_2 - \frac{a_1 a_2\partial u_{20}}{R_2} + a_1\partial w_{0,\beta}\right)\left(1+\frac{\varsigma}{R_1}\right)\left(d\alpha\right)\left(d\beta\right)\left(d\varsigma\right).$$

The definition of Q_2 is:

$$Q_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{23}\left(1+\frac{\varsigma}{R_1}\right)d\varsigma$$

After replacing with Q_2 the term will be:

$$I_5 = \int_0^{\alpha}\int_0^{\beta}\left[Q_2\left(a_1a_2\partial\psi_2\right) - Q_2\frac{a_1a_2\partial u_{20}}{R_1} + Q_2a_1\partial w_{0,\beta}\right](d\alpha)(d\beta).$$

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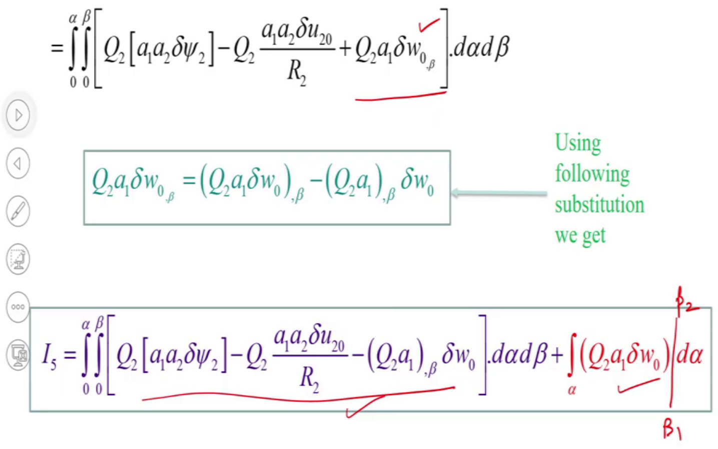

There is a derivative with respect to β, we have to get rid of that and then, I_5 will be:

$$\int_0^{\alpha}\int_0^{\beta}\left[Q_2\left(a_1a_2\partial\psi_2\right) - Q_2\frac{a_1a_2\partial u_{20}}{R_2} + \left(Q_2a_1\right)_{,\alpha}\partial w_0\right](d\alpha)(d\beta) \text{ on the area.}$$

And only this $\int_{\alpha}\left(Q_2a_1\partial w_0\right)\Big|_{\beta_1}^{\beta_2} d\alpha$ term is on the boundary. The boundary will go to β; β_1 to β_2.

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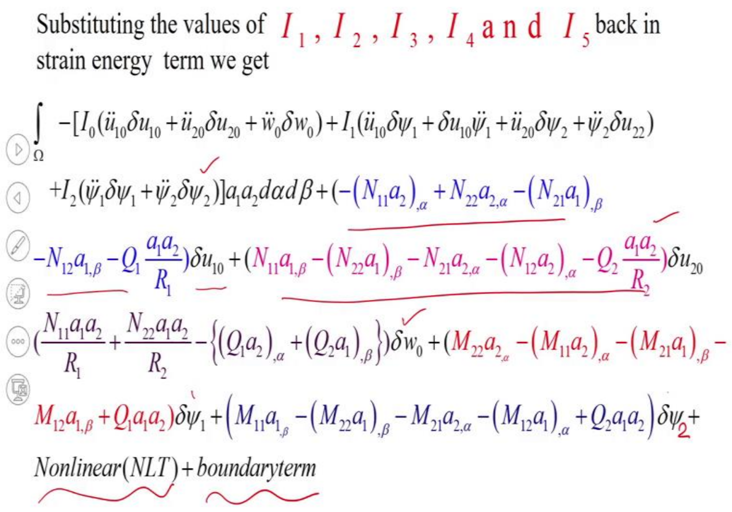

Now, we have to club all the terms, like the terms corresponding to the kinetic energy and the terms corresponding to the potential energy.

$$
\begin{aligned}
\int_\Omega & \begin{bmatrix} -I_0\left(\ddot{u}_{10}\partial u_{10}+\ddot{u}_{20}\partial u_{20}+\ddot{w}_0\partial w_0\right)+I_1\left(\ddot{u}_{10}\partial\psi_1+\partial u_{10}\ddot{\psi}_1+\ddot{u}_{20}\partial\psi_2+\ddot{\psi}_2\partial u_{20}\right) \\ +I_2\left(\ddot{\psi}_1\partial\psi_1+\ddot{\psi}_2\partial\psi_2\right) \end{bmatrix} a_1a_2(d\alpha)(d\beta)+ \\
& \left[-\left(N_{11}a_2\right)_{,\alpha}+N_{22}a_{2,\alpha}-\left(N_{21}a_1\right)_{,\beta}-N_{12}a_{1,\beta}-Q_1\frac{a_1a_2}{R_1}\right]\partial u_{10}+ \\
& \left[N_{11}a_{1,\beta}-\left(N_{22}a_1\right)_{,\beta}-N_{21}a_{2,\alpha}-\left(N_{12}a_2\right)_{,\alpha}-Q_2\frac{a_1a_2}{R_1}\right]\partial u_{20}+ \\
& \left[\frac{N_{11}a_1a_2}{R_1}+\frac{N_{22}a_1a_2}{R_2}-\left[\left(Q_1a_2\right)_{,\alpha}+\left(Q_2a_1\right)_{,\beta}\right]\right]\partial w_0+ \\
& \left(M_{22}a_{2,\alpha}-\left(M_{11}a_2\right)_{,\alpha}-\left(M_{21}a_1\right)_{,\beta}-M_{12}a_{1,\beta}+Q_1a_1a_2\right)\partial\psi_1+ \\
& \left[M_{11}a_{1,\beta}-\left(M_{22}a_1\right)_{,\beta}-M_{21}a_{2,\alpha}-\left(M_{12}a_1\right)_{,\alpha}+Q_2a_1a_2\right]\partial\psi_2+\textit{nonlinear term}+\textit{boundary term}
\end{aligned}
$$

In this slide, I have only put the terms kinetic energy plus the linear contribution of the strain energy, there will be some non-linear contribution and boundary terms also.

It is a very big equation and the most important part is that here I have arranged the coefficient in terms of ∂u_{10}, ∂u_{20}, ∂w_0, $\partial\psi_1$ and $\partial\psi_2$. If we add all I_1, I_2, I_3, I_4 and I_5 and clubbing the coefficient of ∂u_{10}, ∂u_{20}, ∂w_0, $\partial\psi_1$ and $\partial\psi_2$ together will help us to frame up the governing equation easily.

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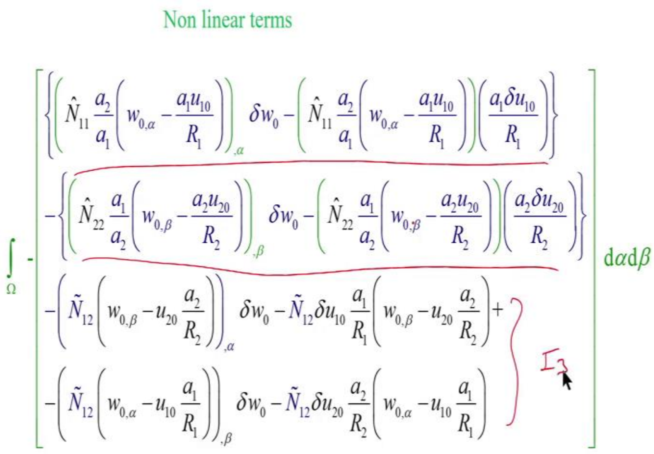

$$\int_{\Omega}\left[\begin{array}{l}\left\{\left(N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}\partial w_0-\left(N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)\left(\frac{a_1\partial u_{10}}{R_1}\right)\right\}\\ -\left\{\left(N_{22}\frac{a_1}{a_2}\left(w_{0},_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}\partial w_0-\left(N_{22}\frac{a_1}{a_2}\left(w_{0},_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)\left(\frac{a_2\partial u_{20}}{R_2}\right)\right\}\\ -\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}\partial w_0-\widetilde{N_{12}\partial u_{10}}\frac{a_1}{R_1}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)+\\ -\left(\widetilde{N_{12}}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\partial w_0-\widetilde{N_{12}\partial u_{20}}\frac{a_2}{R_2}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\end{array}\right](d\alpha)(d\beta)$$

These are the non-linear terms. The first contribution is due to I_1, the second contribution from I_2 and third and fourth contribution from I_3.

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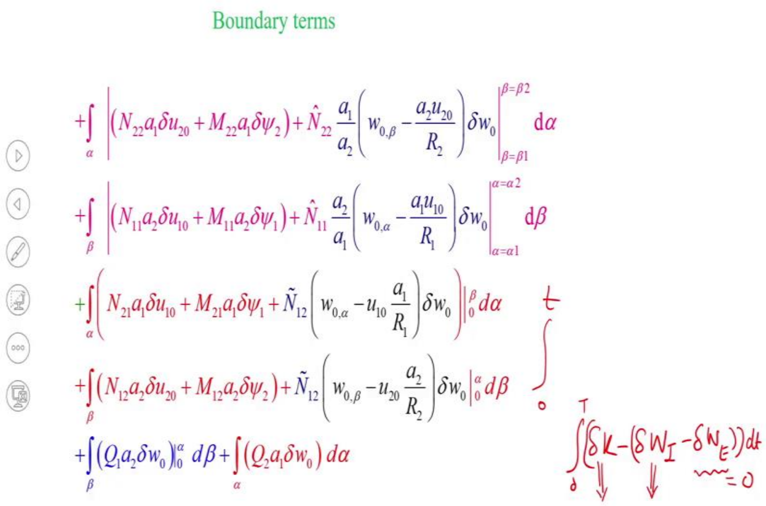

$$+\int_{\alpha}\left|\left(N_{22}a_1\partial u_{20}+M_{22}a_1\partial\psi_2\right)+N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)\partial w_0\right|_{\beta=\beta_1}^{\beta=\beta_2}d\alpha$$

$$+\int_{\beta}\left|\left(N_{11}a_2\partial u_{10}+M_{11}a_2\partial\psi_1\right)+N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\partial w_0\right|_{\beta=\beta_2}^{\alpha=\alpha_2}d\beta$$

$$+\int_{\alpha}\left(N_{21}a_1\partial u_{10}+M_{21}a_1\partial\psi_1+N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\partial w_0\right)\Bigg|_{\beta_1}^{\beta_2}d\alpha$$

$$+\int_{\beta}\left(N_{12}a_2\partial u_{20}+M_{12}a_2\partial\psi_2+N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\partial w_0\right)\Bigg|_{\alpha_1}^{\alpha_2}d\beta$$

$$+\int_{\beta}\left(Q_2a_1\partial w_0\right)\Big|_{\alpha_1}^{\alpha_2}d\beta+\int_{\alpha}\left(Q_2a_1\partial w_0\right)d\alpha$$

And these are the boundary terms. The first contribution is from I_2, second from I_1, third and fourth from I_3, and the last contribution is from I_4 and I_5. These are the boundary terms. Here, the pink and red ones are the linear contributions, the black and blue ones are the non-linear contribution. Till now, we have clubbed the coefficients, boundary terms, non-linear terms, and linear terms separately.

Now, the External work done is left. We have to add it because in Hamilton's principle, $\int_0^T \partial K-\left(\partial W_I-\partial W_E\right)dt=0$. We have identified the terms, the contribution of kinetic energy, and contribution due to the internal energy and put it in a simplified form arranged in $\partial u, \partial w, \partial \psi_1, \partial \psi_0$ coefficients. Now, we have to evaluate external work done.

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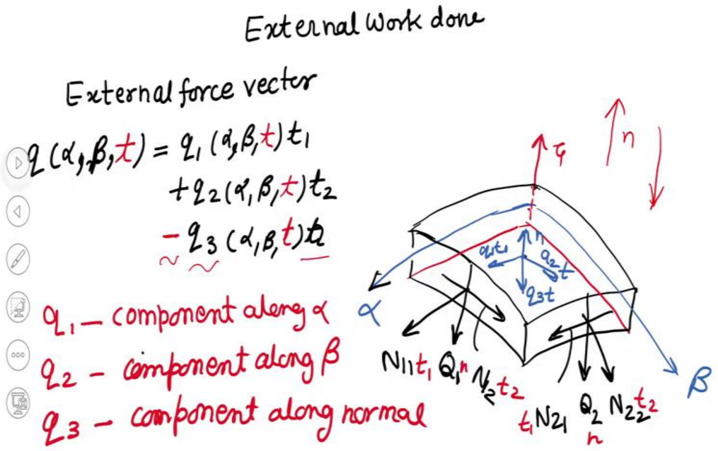

In the external force vector: let us say a force vector q is acting on a surface. It is acting on a reference surface q because the shell is thick, so, we assume that though sometimes it is acting on the top of the shell, the external force vector is taken as at the reference surface.

$$q(\alpha, \beta, t) = q_1(\alpha, \beta, t)t_1 + q_2(\alpha, \beta, t)t_2 - q_3(\alpha, \beta, t)t_3$$

q_1 component along the α direction, q_2 component along β direction, and q_3 component along normal (n). Generally, the surface normal (n) is in the upward direction, but when you apply pressure, it will be in the opposite direction that is why it is taken as q_3 and normal is in the downward direction.

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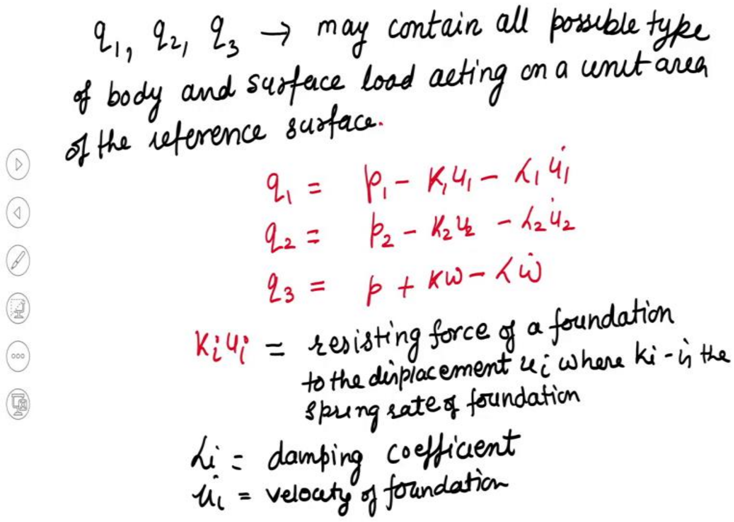

Now, these q_1, q_2, and q_3 may contain all possible types of body or surface load acting on a unit area of the reference surface. It is not just a pressure that you say that only mechanical pressure is not applied, let us say the shell is resting on an elastic foundation, the stiffness of the foundation resisting force can be modeled as $k_i u_i$, time, damping coefficients λ_i, and velocity of the foundation μ_i.

q_1 can be defined as:

$$q_1 = p_1 - k_1 u_1 - \lambda_1 \dot{u}_1$$

Where, p_1 is purely mechanical pressure, $k_1 u_1$ is the stiff foundation or stiffness of a foundation resisting, and $\lambda_1 \dot{u}_1$ the damping coefficient and the velocity of the foundation. Similarly, we can find q_2 and q_3:

$$q_2 = p_2 - k_2 u_2 - \lambda_2 \dot{u}_2$$
$$q_3 = p_3 - k_3 u_3 - \lambda_3 \dot{u}_3$$

We can say that we are going to analyze a shell resting on an elastic foundation and some damping coefficient is also considered or pure mechanical loading. In this way you can take consideration in this q_1, q_2, and q_3.

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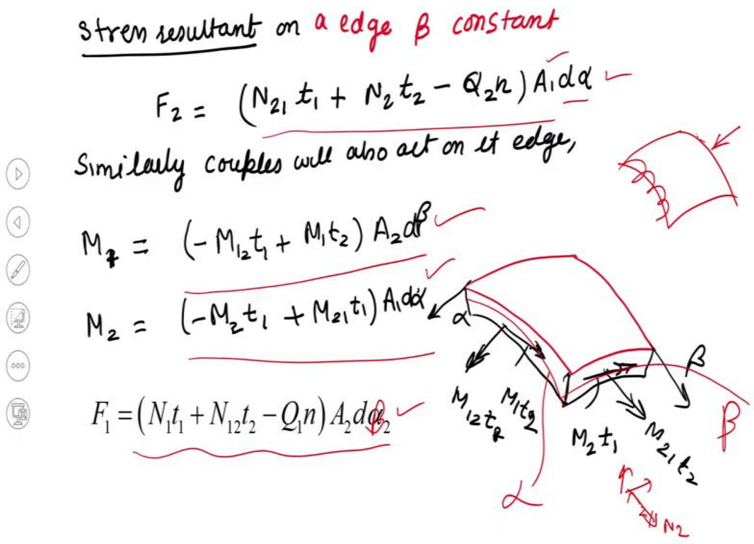

Now, we are coming to the edges. Let us say there will be stress resultants acting. If we talk about the in-plane resultants or those force components, over this edge β is constant and over this edge α is constant. Stress resultant on an edge where β is constant: there will be force resultant N_{21}, N_{22} and shear force Q_2 will be acting. The total force will be:

$$F_2 = \left(N_{21}t_1 + N_2 t_1 + Q_2 n\right) A_1 d\alpha \ .$$

Similarly, the couples M_1 and M_2 may also act on the edges.

$$M_1 = \left(-M_{12}t_1 + M_1 t_2\right) A_2 d\beta$$
$$M_2 = \left(-M_2 t_1 + M_{21}t_1\right) A_1 d\alpha$$

We can say that the resultant M_1 will be on the edge where α is constant and M_2 will be on the edge, where β is constant. Here, the force F_1 will be:

$$F_1 = \left(N_1 t_1 + N_{12}t_2 + Q_1 n\right) A_1 d\alpha_2 \, .$$

In this way, we can say that on the edges; there may be some force resultant and coupled resultants are acting.

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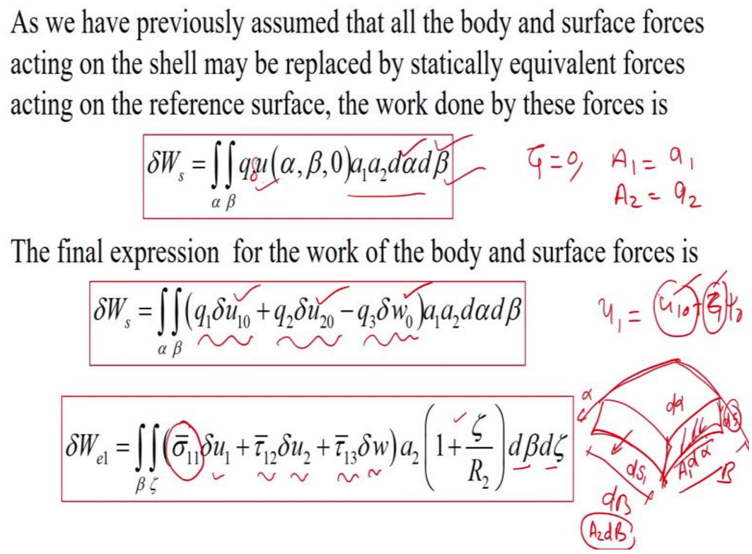
As we have previously assumed that all the body and surface forces acting on the shell may be replaced by statically equivalent forces acting on the reference surface, the work done by these forces is

$$\delta W_s = \int_\alpha \int_\beta q \delta u(\alpha, \beta, 0) a_1 a_2 d\alpha d\beta$$

The final expression for the work of the body and surface forces is

$$\delta W_s = \int_\alpha \int_\beta (q_1 \delta u_{10} + q_2 \delta u_{20} - q_3 \delta w_0) a_1 a_2 d\alpha d\beta$$

$$\delta W_{e1} = \int_\beta \int_\zeta (\bar{\sigma}_{11} \delta u_1 + \bar{\tau}_{12} \delta u_2 + \bar{\tau}_{13} \delta w) a_2 \left(1 + \frac{\zeta}{R_2}\right) d\beta d\zeta$$

Now, if you say that the outside forces are acting on these edges, they may also contribute to the external work done. Later on, you may think that a problem that in which a shell and some edge moment is applied over that edge or that some normal resultant force is acting on the edge, we may consider this kind of situation, for that purpose let us say some work is done.

And work done due to surface forces ∂W_s will be:

$$\partial W_s = \int_\alpha \int_\beta q \partial u(\alpha, \beta, 0) a_1 a_2 d\alpha d\beta$$

At the reference surface ς is 0, therefore, A_1 reduces to a_1 and A_2 reduces to a_2. ∂W_s the external work done due to the surface forces will be:

$$\partial W_s = \int_\alpha \int_\beta \left(q_1 \partial u_{10} + q_2 \partial u_{20} - q_3 \partial w_0\right) a_1 a_2 d\alpha d\beta .$$

Here, you can write u_1 instead of u_{10}, as u_1 is equal to $u_{10} + \varsigma\, \psi_1$, here ς is 0, only the contribution is due to u_{10}.

Now, on the edges the work done will be: let us say this is the β edge and this is the α edge, will be $d\beta$ and $d\varsigma$. ∂W_{e1} will be:

$$\partial W_{e1} = \int_\beta \int_\varsigma \left(\bar{\sigma}_{11} \partial u_1 + \bar{\tau}_{12} \partial u_2 + \bar{\tau}_{13} \partial w\right) a_2 \left(1 + \frac{\varsigma}{R_2}\right) d\beta d\varsigma .$$

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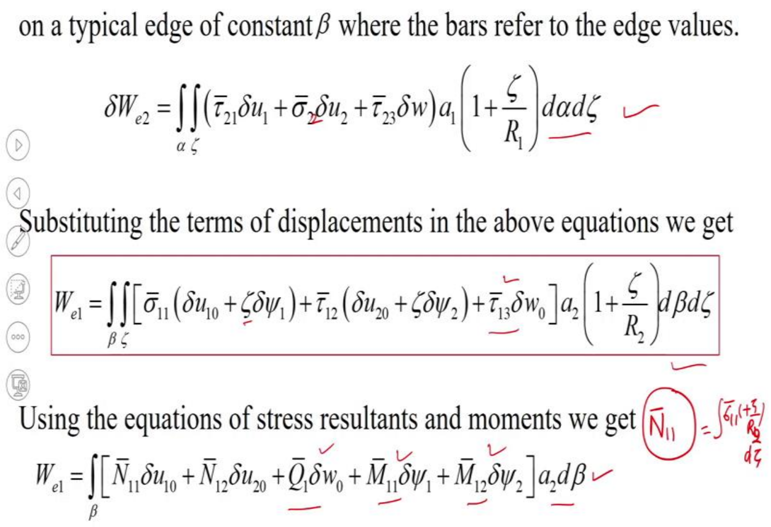

Using the definition, here ς is the integration. We can use those concepts of integration and before going to proceed further, let us define the work done over this edge where β is constant. ∂W_{e2} will be:

$$\partial W_{e2} = \int_{\beta}\int_{\varsigma} \left(\bar{\tau}_{21}\partial u_1 + \bar{\sigma}_{22}\partial u_2 + \bar{\tau}_{23}\partial w\right) a_1 \left(1+\frac{\varsigma}{R_1}\right) d\alpha d\varsigma .$$

If we substitute the value of u_1, u_2 and u_3 because we know $u_1 = u_{10} + \varsigma\,\psi_1$, and so on then, ∂W_{e1} will be:

$$\partial W_{e1} = \int_{\beta}\int_{\varsigma} \left(\bar{\sigma}_{11}\left(\partial u_{10} + \varsigma\partial\psi_1\right) + \bar{\tau}_{12}\left(\partial u_{20} + \varsigma\partial\psi_2\right) + \bar{\tau}_{13}\partial w_0\right) a_2 \left(1+\frac{\varsigma}{R_2}\right) d\beta d\varsigma$$

Using the concept of stress resultants and moments here we get $\bar{N}_{11}$ means the applied outside. It is slightly different, but definition wise same as the bar is R_2.

∂W_{e1} will be:

$$\partial W_{e1} = \int_{\beta}\left(\bar{N}_{11}\partial u_{10} + \bar{N}_{12}\partial u_{20} + \bar{Q}_1\partial w_0 + \bar{M}_{11}\partial\psi_1 + \bar{M}_{12}\partial\psi_2\right) a_2 d\beta .$$

This will be the contribution from edge 1.

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$$W_{e2}=\int_{\alpha}\int_{\zeta}\left[\bar{\sigma}_{22}\left(\delta u_{20}+\zeta\delta\psi_{2}\right)+\bar{\tau}_{21}\left(\delta u_{10}+\zeta\delta\psi_{1}\right)+\bar{\tau}_{23}\delta w_{0}\right]a_{1}\left(1+\frac{\zeta}{R_{1}}\right)d\alpha d\zeta$$

Using the equations of stress resultants and moments we get

$$W_{e2}=\int_{\alpha}\left[\bar{N}_{21}\delta u_{10}+\bar{N}_{22}\delta u_{20}+\bar{Q}_{2}\delta w+\bar{M}_{21}\delta\psi_{1}+\bar{M}_{22}\delta\psi_{2}\right]a_{1}d\alpha$$

Then, the contribution from edge 2 ∂W_{e2} will be:

$$\int_{\beta}\int_{\varsigma}\left(\bar{\sigma}_{22}\left(\partial u_{20}+\varsigma\partial\psi_{2}\right)+\bar{\tau}_{21}\left(\partial u_{10}+\varsigma\partial\psi_{1}\right)+\bar{\tau}_{23}\partial w_{0}\right)a_{1}\left(1+\frac{\varsigma}{R_{1}}\right)d\alpha d\varsigma$$

And if you use the concept of stress resultant and moments, it will become:

$$\partial W_{e2}=\int_{\beta}\left(\bar{N}_{21}\partial u_{10}+\bar{N}_{22}\partial u_{20}+\bar{Q}_{2}\partial w_{0}+\bar{M}_{21}\partial\psi_{1}+\bar{M}_{22}\partial\psi_{2}\right)a_{1}d\alpha\,.$$

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Total work done

$$\delta W_{s}+\delta W_{e1}+\delta W_{e2}=\int_{\alpha}\int_{\beta}\left(q_{1}\delta u_{10}+q_{2}\delta u_{20}-q_{3}\delta w_{0}\right)a_{1}a_{2}d\alpha d\beta$$

$$+\int_{\beta}\left[\bar{N}_{11}a_{2}\delta u_{10}+\bar{N}_{12}a_{2}\delta u_{20}+\bar{Q}_{1}a_{2}\delta w_{0}+\bar{M}_{11}a_{2}\delta\psi_{1}+\bar{M}_{12}a_{2}\delta\psi_{2}\right]d\beta$$

$$\int_{\alpha}\left[\bar{N}_{21}a_{1}\delta u_{10}+\bar{N}_{22}a_{1}\delta u_{20}+\bar{Q}_{2}a_{1}\delta w+\bar{M}_{21}a_{1}\delta\psi_{1}+\bar{M}_{22}a_{1}\delta\psi_{2}\right]d\alpha$$

$\alpha=\alpha_2$, $\beta_1=\beta_2$, $\alpha_2=\alpha_1$, $\beta=\beta_1$

Now, we have to club all the work done $\partial W_s + \partial W_{e1} + \partial W_{e2}$ and it will be:

$$\int_\alpha \int_\beta \left(q_1 \partial u_{10} + q_2 \partial u_{20} - q_3 \partial w_0 \right) a_1 a_2 d\alpha d\beta$$
$$+ \int_\beta \left(\bar{N}_{11} a_2 \partial u_{10} + \bar{N}_{12} a_2 \partial u_{20} + \bar{Q}_1 a_2 \partial w_0 + \bar{M}_{11} a_2 \partial \psi_1 + \bar{M}_{12} a_2 \partial \psi_2 \right) \Big|_{\alpha_1}^{\alpha_2} d\beta$$
$$+ \int_\beta \left(\bar{N}_{21} a_1 \partial u_{10} + \bar{N}_{22} a_1 \partial u_{20} + \bar{Q}_2 a_1 \partial w_0 + \bar{M}_{21} a_1 \partial \psi_1 + \bar{M}_{22} a_1 \partial \psi_2 \right) \Big|_{\beta_1}^{\beta_2} d\alpha$$

The first term will contribute to the area and the other terms ∂W_{e1} and ∂W_{e2} will contribute to the boundary terms.

$$\int_\beta \left(\bar{N}_{11} a_2 \partial u_{10} + \bar{N}_{12} a_2 \partial u_{20} + \bar{Q}_1 a_2 \partial w_0 + \bar{M}_{11} a_2 \partial \psi_1 + \bar{M}_{12} a_2 \partial \psi_2 \right) \Big|_{\alpha_1}^{\alpha_2} d\beta$$

Here α is equal to α_1 to α_2

$$\int_\beta \left(\bar{N}_{21} a_1 \partial u_{10} + \bar{N}_{22} a_1 \partial u_{20} + \bar{Q}_2 a_1 \partial w_0 + \bar{M}_{21} a_1 \partial \psi_1 + \bar{M}_{22} a_1 \partial \psi_2 \right) \Big|_{\beta_1}^{\beta_2} d\alpha$$

Here β will be β_1 to β_2.

We have obtained the total work done, strain energy, kinetic energy and we have clubbed them together. In the next lecture, I will be using the fundamental lemma of variational principle to develop the governing equations.

Thank you very much.