Theory of Composite Shells

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Week – 03

Lecture – 02

Shell governing equation

Dear learners welcome to week- 03, lecture- 02. In this lecture, I will cover the Shell Governing Equations that we already have obtained in lecture-01 of week- 03. Now, we will discuss those in more detail and the associated boundary conditions also.

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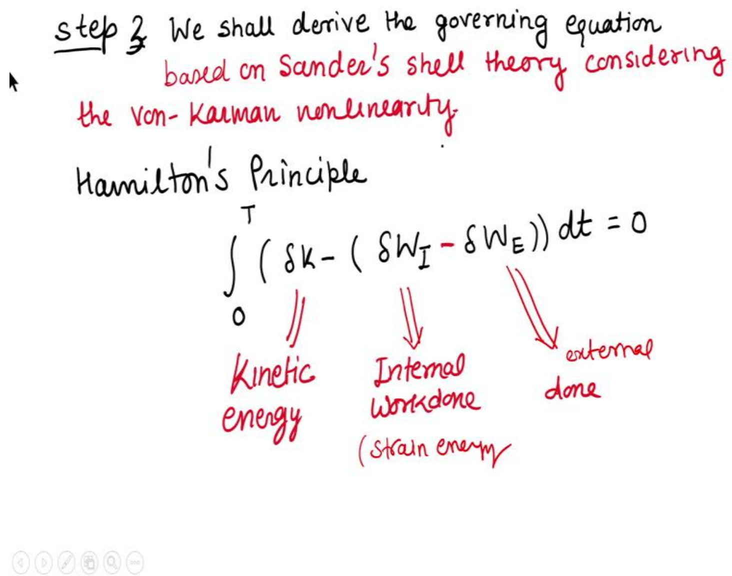

We shall derive the set of governing equations. In the last lecture, I have covered the kinetic energy, internal work done, and derive the relations of external work done. I clubbed all the terms together and equated it to 0.

As per the Hamilton principle:

$$\int_0^T \left(\partial K - \left(\partial W_I - \partial W_E\right)\right) dt = 0$$

Potential energy will contain two contributions, the first one is corresponding to the internal work done ∂W_I and the second one is corresponding to the external work done ∂W_E. ∂W_I is the strain energy of an elastic body.

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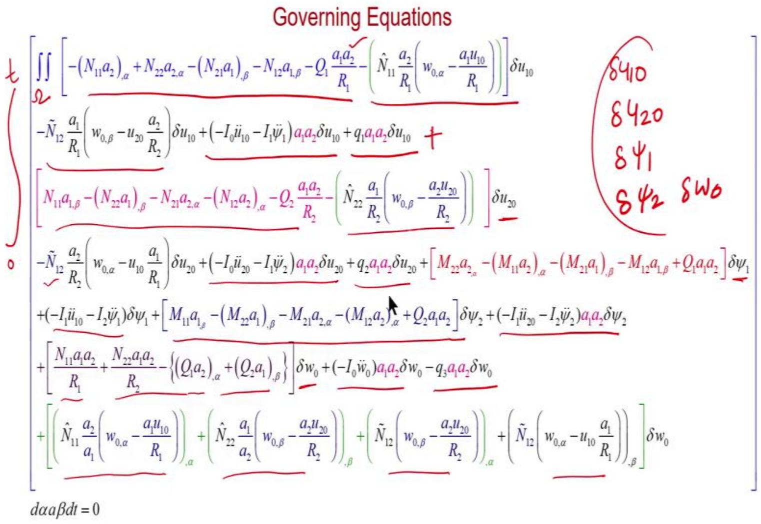

If we club all these equations; contribution of kinetic energy ∂K, the contribution of strain energy ∂W_I, and contribution of external work done ∂W_E.

$$\int_0^t \left[\begin{aligned} &\iint_\Omega \left[-\left(N_{11}a_2\right)_{,\alpha} + N_{22}a_{2,\alpha} - \left(N_{21}a_1\right)_{,\beta} - N_{12}a_{1,\beta} - Q_1\frac{a_1a_2}{R_1} - \left(N_{11}\frac{a_2}{R_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right)\right)\right]\partial u_{10} \\ &- N_{12}\frac{a_1}{R_1}\left(w_0,_\beta - \frac{a_2}{R_2}\right)\partial u_{10} + \left(-I_0\ddot{u}_{10} - I_1\ddot{\psi}_1\right)a_1a_2\partial u_{10} + q_1a_1a_2\partial u_{10} \\ &+\left[N_{11}a_{1,\beta} - \left(N_{22}a_1\right)_{,\beta} - N_{21}a_{2,\alpha} - \left(N_{12}a_2\right)_{,\alpha} - Q_2\frac{a_1a_2}{R_1} - \left(N_{22}\frac{a_1}{R_2}\left(w_0,_\beta - \frac{a_2u_{20}}{R_2}\right)\right)\right]\partial u_{20} \\ &- \widetilde{N_{12}\frac{a_2}{R_2}}\left(w_0,_\alpha - \frac{a_1}{R_1}\right)\partial u_{20} + \left(-I_0\ddot{u}_{20} - I_1\ddot{\psi}_2\right)a_1a_2\partial u_{20} + q_2a_1a_2\partial u_{20} \\ &+\left(M_{22}a_{2,\alpha} - \left(M_{11}a_2\right)_{,\alpha} - \left(M_{21}a_1\right)_{,\beta} - M_{12}a_{1,\beta} + Q_1a_1a_2\right)\partial\psi_1 \\ &+\left(-I_1\ddot{u}_{10} - I_2\ddot{\psi}_1\right)\partial\psi_1 + \left[M_{11}a_{1,\beta} - \left(M_{22}a_1\right)_{,\beta} - M_{21}a_{2,\alpha} - \left(M_{12}a_1\right)_{,\alpha} + Q_2a_1a_2\right]\partial\psi_2 \\ &+\left(-I_1\ddot{u}_{20} - I_2\ddot{\psi}_2\right)a_1a_2\partial\psi_2 + \left[\frac{N_{11}a_1a_2}{R_1} + \frac{N_{22}a_1a_2}{R_2} - \left[\left(Q_1a_2\right)_{,\alpha} + \left(Q_2a_1\right)_{,\beta}\right]\right]\partial w_0 \\ &+\left(-I_1\ddot{w}_0\right)a_1a_2\partial w_0 - q_3a_1a_2\partial w_0 + \left(\widetilde{N_{11}\frac{a_2}{a_1}}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha} + \left(\widetilde{N_{22}\frac{a_1}{a_2}}\left(w_0,_\beta - \frac{a_2u_{20}}{R_2}\right)\right)_{,\beta} \\ &+\left(\widetilde{N_{12}}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha} + \left(\widetilde{N_{12}}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\partial w_0 \end{aligned} \right] d\alpha\, d\beta\, dt = 0$$

Here in the first equation, $\left(-I_0\ddot{u}_{10} - I_1\ddot{\psi}_1\right)a_1a_2\partial u_{10}$ is the contribution of kinetic energy

$q_1a_1a_2\partial u_{10}$ is the contribution of external work done

$-\left(N_{11}a_2\right)_{,\alpha} + N_{22}a_{2,\alpha} - \left(N_{21}a_1\right)_{,\beta} - N_{12}a_{1,\beta} - Q_1\frac{a_1a_2}{R_1}$ is the contribution of internal work

done having linear contribution, and $N_{11}\frac{a_2}{R_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right)$ is the non-linear contribution.

Here you see that all are having ∂u_{10} coefficient.

We have clubbed ∂u_{10} coefficient, kinetic energy, external work done, and internal work done. 0 to t integration outside the whole expression and area integration is outside N and that is going to be 0 plus the contribution the coefficient of ∂u_{20}.

∂u_{20} coefficient will have:

$$N_{11}a_{1,\beta}-\left(N_{22}a_1\right)_{,\beta}-N_{21}a_{2,\alpha}-\left(N_{12}a_2\right)_{,\alpha}-Q_2\frac{a_1a_2}{R_1}$$ linear terms

$$N_{22}\frac{a_1}{R_2}\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)\ -N_{12}\frac{a_2}{R_2}\left(w_0,_{\alpha}-\frac{a_1}{R_1}\right)\partial u_{20}$$ non-linear terms

$\left(-I_0\ddot{u}_{20}-I_1\ddot{\psi}_2\right)a_1a_2\partial u_{20}$ kinetic energy and

$q_2a_1a_2\partial u_{20}$ external work done.

In $\partial\psi_1$ coefficient, there is:

$M_{22}a_{2,\alpha}-\left(M_{11}a_2\right)_{,\alpha}-\left(M_{21}a_1\right)_{,\beta}-M_{12}a_{1,\beta}+Q_1a_1a_2$ the linear term for internal work done

and $\left(-I_1\ddot{u}_{10}-I_2\ddot{\psi}_1\right)\partial\psi_1$ kinetic energy.

We do not have even the external work done in $\partial\psi_1$.

The coefficient of $\partial\psi_2$ has:

$M_{11}a_{1,\beta}-\left(M_{22}a_1\right)_{,\beta}-M_{21}a_{2,\alpha}-\left(M_{12}a_2\right)_{,\alpha}+Q_2a_1a_2$ internal work done

$\left(-I_1\ddot{u}_{20}-I_2\ddot{\psi}_2\right)a_1a_2\partial\psi_2$ kinetic energy contribution.

And in ∂w_0 coefficient we have:

$$\left[\frac{N_{11}a_1a_2}{R_1}+\frac{N_{22}a_1a_2}{R_2}-\left[\left(Q_1a_2\right)_{,\alpha}+\left(Q_2a_1\right)_{,\beta}\right]\right]\partial w_0$$ the linear term

$$\left(N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}+\left(N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}$$
$$+\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}+\left(N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\partial w_0$$ nonlinear term

$\left(-I_1\ddot{w}_0\right)a_1a_2\partial w_0$ kinetic energy

- $q_3a_1a_2\partial w_0$ external work done.

Now, we have clubbed all the equations at one place and integration from 0 to t, and these ∂u_{10}, ∂u_{20}, $\partial \psi_1$, $\partial \psi_2$, and ∂w_0 are the arbitrary variations. And, their coefficients are integrable over the range α_1 to α_2.

We can use the fundamental theorem of variational principle, which we call the fundamental lemma of a variational principle. If we use that these ∂u_{10}, ∂u_{20}, $\partial \psi_1$, $\partial \psi_2$, and ∂w_0 are arbitrary. So, these coefficients must vanish. This will help us to get ordinary differential equations.

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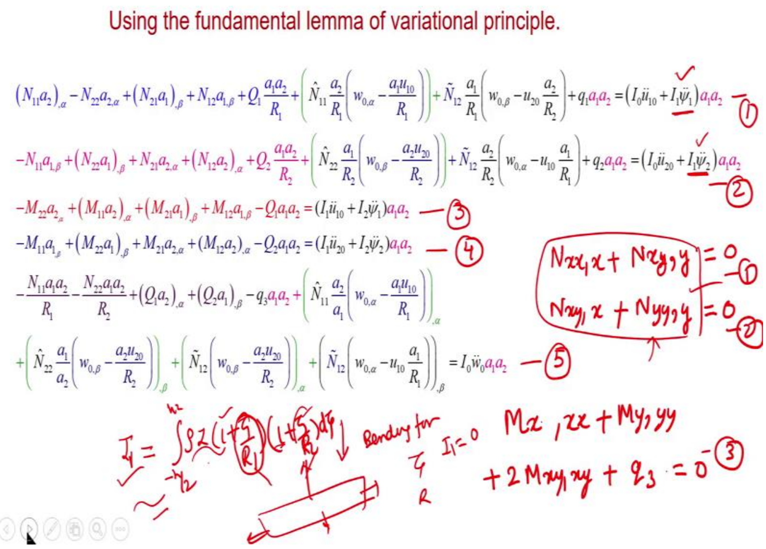

We will get 5 ordinary differential equations:

$$\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta} + Q_1\frac{a_1a_2}{R_1} + \left(N_{11}\frac{a_2}{R_1}\left(w_{0,\alpha} - \frac{a_1u_{10}}{R_1}\right)\right) + N_{12}\frac{a_1}{R_1}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1a_1a_2 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)a_1a_2 \quad \text{equation 1}$$

$$-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha} + Q_2\frac{a_1a_2}{R_1} + \left(N_{22}\frac{a_1}{R_2}\left(w_{0,\beta} - \frac{a_2u_{20}}{R_2}\right)\right) + N_{12}\frac{a_2}{R_2}\left(w_{0,\alpha} - u_{10}\frac{a_2}{R_2}\right) + q_2a_1a_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right)a_1a_2 \quad \text{equation 2}$$

$$-M_{22}a_{2,\alpha} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta} - Q_1a_1a_2 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right)a_1a_2 \quad \text{equation 3}$$

$$-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}-Q_2a_1a_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)a_1a_2 \quad \text{equation 4}$$

$$-\frac{N_{11}a_1a_2}{R_1}-\frac{N_{22}a_1a_2}{R_2}+\left(Q_1a_2\right)_{,\alpha}+\left(Q_2a_1\right)_{,\beta}-q_3a_1a_2+\left(N_{11}\frac{a_2}{a_1}\left(w_{0,\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}$$
$$+\left(N_{22}\frac{a_1}{a_2}\left(w_{0,\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}+\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}+\left(N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta} \quad \text{equation 5}$$
$$=I_0\ddot{w}_0a_1a_2$$

In some of the shell theories book the equations (3) and (4) are kept at the bottom and in some of the theories it is kept in between.

Here, we can see that all the equations are coupled, you see that these terms Q_1 and Q_2 are in equation (5), Q_1 is in equations (1) and (3), Q_2 is in equations (2) and (4). They are coupled through this shear and the non-linear term also equations are coupled together.

If you talk about a plate equation, for a rectangular plate:

$N_{xx,x}+N_{xy,y}=0$ equation (1)

For a static case:

$N_{xy,x}+N_{yy,y}=0$. equation (2)

If you talk about a classical plate:

$M_{x,xx}+M_{y,yy}+2M_{xy,xy}+q_3=0$. equation (3)

Here these in-plane equations are not coupled through equation (3), the moment equations are not here, or if you write in terms of FSDT here Q_1 & Q_2 do not come.

We can solve these two equations equation (1) and equation (2) independently and equation (3) independently, but for the case of the shell it is not true they are coupled through Q_1 & Q_2. So, we cannot solve those equations independently. Each equation affects the other, they are related to each other.

If a shell is thin, the bending effect or a bending force may cause extension in stretching, large stretching, significant stretching but, in the plate, it will not. This is the first observation and the second observation is that we can take consideration of non-linear terms means, without non-linear terms we can consider. Generally, for thin shell theories, non-linear terms are not considered but linear terms are considered.

These are the standard general equations. If a shell is symmetric, then I_1 the mass moment of inertia will be:

$$I_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho \varsigma \left(1 + \frac{\varsigma}{R_1}\right)\left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$$

If we talk about a thin shell, these terms are neglected. Whether if you talk about Sander's theory or love Kirchhoff of shells. Generally, they don't consider this effect, because the thickness is small and radius is very large as compared to one these are negligible.

This I_1 can be 0, but if you talk about a thick shell, then definitely it will have some contribution. In the other way, if you talk about a plate. For the case of a symmetrical plate, $I_1 = 0$ but for the case of a shell, $I_1 \neq 0$. That is why we have kept I_1 in the equations. Here, in the equation you see a_1 & a_2, in most of the books this a_1 & a_2, is taken common.

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$$\frac{1}{a_1 a_2}\left[(N_{11}a_2)_{,\alpha} - N_{22}a_{2,\alpha} + (N_{21}a_1)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + \left(\hat{N}_{11}\frac{1}{a_1 R_1}\left(w_{0,\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right)$$
$$+\tilde{N}_{12}\frac{1}{a_2 R_1}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1 = \left(I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1\right) \quad \text{①}$$

$$\frac{1}{a_1 a_2}\left[-N_{11}a_{1,\beta} + (N_{22}a_1)_{,\beta} + N_{21}a_{2,\alpha} + (N_{12}a_2)_{,\alpha}\right] + \frac{Q_2}{R_2} + \left(\hat{N}_{22}\frac{1}{a_2 R_2}\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right)\right)$$
$$+\tilde{N}_{12}\frac{1}{a_1 R_2}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right) + q_2 = \left(I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2\right) \quad \text{②}$$

$$\frac{1}{a_1 a_2}\left[-M_{22}a_{2,\alpha} + (M_{11}a_2)_{,\alpha} + (M_{21}a_1)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = (I_1 \ddot{u}_{10} + I_2 \ddot{\psi}_1) \quad \text{③}$$

$$\frac{1}{a_1 a_2}\left[-M_{11}a_{1,\beta} + (M_{22}a_1)_{,\beta} + M_{21}a_{2,\alpha} + (M_{12}a_2)_{,\alpha}\right] - Q_2 = (I_1 \ddot{u}_{20} + I_2 \ddot{\psi}_2) \quad \text{④}$$

$$\frac{1}{a_1 a_2}\left\{\left(\hat{N}_{11}\frac{a_2}{a_1}\left(w_{0,\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right)_{,\alpha} + \left(\hat{N}_{22}\frac{a_1}{a_2}\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right)\right)_{,\beta} + \left(\tilde{N}_{12}\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right)\right)_{,\alpha} + \left(\tilde{N}_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\right\}$$
$$+\left[-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right] + \frac{(Q_1 a_2)_{,\alpha}}{a_1 a_2} + \frac{(Q_2 a_1)_{,\beta}}{a_1 a_2} - q_3 = I_0 \ddot{w}_0 \quad \text{⑤}$$

If we take a_1 & a_2 common, then this governing equation will look like this

$$\frac{1}{a_1 a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + \left(N_{11}\frac{1}{a_1 R_1}\left(w_0,_{\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right)$$
$$+N_{12}\frac{1}{a_2 R_1}\left(w_0,_{\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)$$

The loading term and the dynamic term will not contain a_1 & a_2.

In the book of the theory of shells, you will find the final form of governing equations. The previous set of equations was the intermediate part just after the integration. This is the exact form that is represented in various textbook taking $\frac{1}{a_1 a_2}$. If we want to work with this, we can work on it. We have 5 governing equations.

$$\frac{1}{a_1 a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} +$$
$$\left(N_{11}\frac{1}{a_1 R_1}\left(w_0,_{\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right) + N_{12}\frac{1}{a_2 R_1}\left(w_0,_{\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right) \quad \text{equation (1)}$$

$$\frac{1}{a_1 a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha}\right] + \frac{Q_2}{R_2} +$$
$$\left(N_{22}\frac{1}{a_2 R_2}\left(w_0,_{\beta} - \frac{a_2 u_{20}}{R_2}\right)\right) + N_{12}\frac{1}{a_1 R_2}\left(w_0,_{\beta} - u_{10}\frac{a_1}{R_1}\right) + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right) \quad \text{equation (2)}$$

$$\frac{1}{a_1 a_2}\left[-M_{22}a_{2,\alpha} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right) \quad \text{equation (3)}$$

$$\frac{1}{a_1 a_2}\left[-M_{11}a_{1,\beta} + \left(M_{22}a_1\right)_{,\beta} + M_{21}a_{2,\alpha} + \left(M_{12}a_2\right)_{,\alpha}\right] - Q_2 = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right) \quad \text{equation (4)}$$

$$\frac{1}{a_1 a_2}\left[\begin{array}{l}\left(N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right)_{,\alpha} + \left(N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta} - \frac{a_2 u_{20}}{R_2}\right)\right)_{,\beta} + \left(N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha} \\ +\left(N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\end{array}\right]$$
$$+\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right) + \frac{\left(Q_1 a_2\right)_{,\alpha}}{a_1 a_2} + \frac{\left(Q_2 a_1\right)_{,\beta}}{a_1 a_2} = I_0\ddot{w}_0 \quad \text{equation (5)}$$

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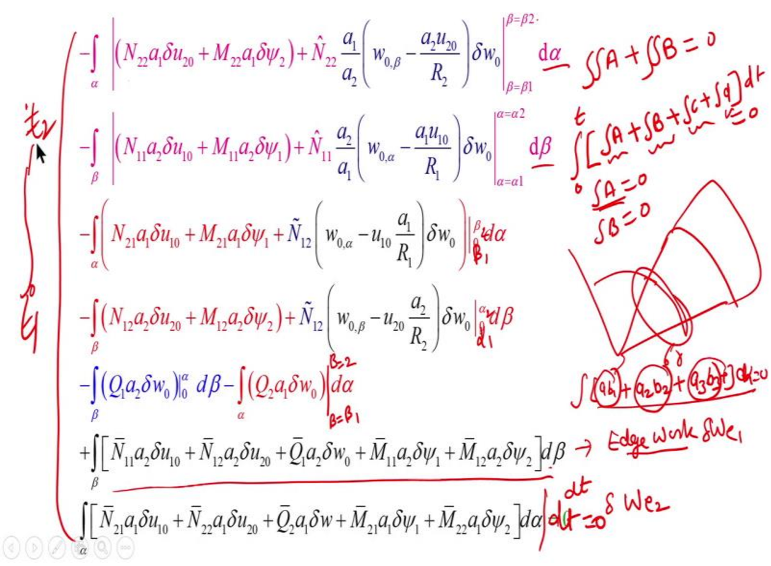

Now, we move to the boundary conditions. In lecture-01 of week-03:

$$-\int_{\alpha}\left|\left(N_{22}a_1\partial u_{20}+M_{22}a_1\partial\psi_2\right)+N_{22}\frac{a_1}{a_2}\left(w_{0},_{\beta}-\frac{a_2u_{20}}{R_2}\right)\partial w_0\right|_{\beta=\beta_1}^{\beta=\beta_2}d\alpha$$

It is the contribution due to I_1,

$$-\int_{\beta}\left|\left(N_{11}a_2\partial u_{10}+M_{11}a_2\partial\psi_1\right)+N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\partial w_0\right|_{\alpha=\alpha_1}^{\alpha=\alpha_2}d\beta$$

It is the contribution due to I_2

$$-\int_{\alpha}\left(N_{21}a_1\partial u_{10}+M_{21}a_1\partial\psi_1+N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\partial w_0\right)\Bigg|_{\beta_1}^{\beta_2}d\alpha$$

$$-\int_{\beta}\left(N_{12}a_2\partial u_{20}+M_{12}a_2\partial\psi_2+N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\partial w_0\right)\Bigg|_{\alpha_1}^{\alpha_2}d\beta$$

These are the contribution due to I_3

$$-\int_{\beta}\left(Q_2 a_1 \partial w_0\right)\Big|_{\alpha_1}^{\alpha_2} d\beta + \int_{\alpha}\left(Q_2 a_1 \partial w_0\right)\Big|_{\beta_1}^{\beta_2} d\alpha$$

This is the contribution due to I_4 and I_5.

Here α_1 to α_2 and β_1 to β_2 is more consistent compared to writing 0 to α or 0 to β. The reason behind that is θ can take any value, it is not necessary to start from 0, it can be anything.

You may say that 0 to 30° or 30° to 60° between that some component is there of volume element. And, these are the contribution of the edge work:

$$+\int_{\beta}\left(\bar{N}_{11} a_2 \partial u_{10} + \bar{N}_{12} a_2 \partial u_{20} + \bar{Q}_1 a_2 \partial w_0 + \bar{M}_{11} a_2 \partial \psi_1 + \bar{M}_{12} a_2 \partial \psi_2\right)\Big|_{\alpha_1}^{\alpha_2} d\beta \quad \text{due to } \partial W_{e1}$$

$$+\int_{\beta}\left(\bar{N}_{21} a_1 \partial u_{10} + \bar{N}_{22} a_1 \partial u_{20} + \bar{Q}_2 a_1 \partial w_0 + \bar{M}_{21} a_1 \partial \psi_1 + \bar{M}_{22} a_1 \partial \psi_2\right)\Big|_{\beta_1}^{\beta_2} d\alpha \quad \text{due to } \partial W_{e2}$$

We have put it together and equate it to 0. Ultimately, the area integral and the boundary integral are going to be 0:

$$\int\int A + \int\int B = 0$$

definitely the time will be there. Taking integration from 0 to t or t_2 to t_1. And at the end add dt. The whole equation will be equal to 0. Before proceeding further, let us say we have a term:

$$\int_0^t \left(\int A + \int B + \int C + \int D\right)\ dt = 0$$

We will say that all individual integration to satisfy this, $\int A$, $\int B$ need to be 0, that is the first thing, this integration $d\alpha$ is going to be 0, this integration $d\beta$ is going to be 0, and so on.

Inside this "A", we have a term:

$$\int\left(a_1 b_1 + a_2 b_2 + a_3 b_3\right) dt = 0$$

To satisfy this equation individually each term needs to be 0.

We will club together all the integration which are under $d\alpha$ and under $d\beta$ line. These terms will go to $d\beta$ and these terms will go to $d\alpha$. We will club all the terms under one head along β edge and α edge.

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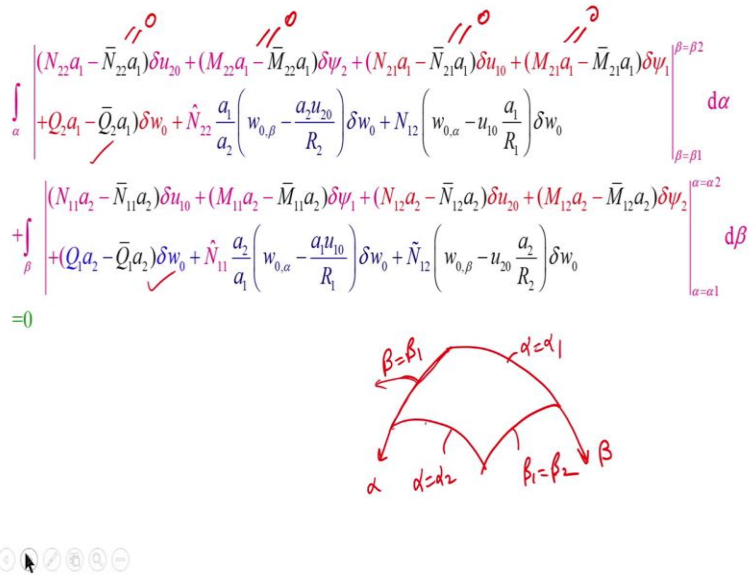

If we do so and collect all the terms under α and under β. In the first one,

$$\int_{\alpha} \left| \begin{array}{l} \left(N_{22}a_1 - \bar{N}_{22}a_1\right)\partial u_{20} + \left(M_{22}a_1 - \bar{M}_{22}a_1\right)\partial \psi_2 + \left(N_{21}a_1 - \bar{N}_{22}a_1\right)\partial u_{10} + \left(M_{21}a_1 - \bar{M}_{21}a_1\right)\partial \psi_1 \\ + \left(Q_2 a_1 - \bar{Q}_2 a_1\right)\partial w_0 + N_{22}\dfrac{a_1}{a_2}\left(w_{0,\beta} - \dfrac{a_2 u_{20}}{R_2}\right)\partial w_0 + N_{12}\left(w_{0,\alpha} - u_{10}\dfrac{a_1}{R_1}\right)\partial w_0 \end{array} \right|_{\beta=\beta_1}^{\beta=\beta_2} d\alpha$$

the boundaries are associated where β is constant. And, in second integral

$$\int_{b} \left| \begin{array}{l} \left(N_{11}a_2 - \bar{N}_{11}a_2\right)\partial u_{10} + \left(M_{11}a_2 - \bar{M}_{11}a_2\right)\partial \psi_1 + \left(N_{12}a_2 - \bar{N}_{12}a_2\right)\partial u_{20} + \left(M_{12}a_2 - \bar{M}_{12}a_2\right)\partial \psi_2 \\ + \left(Q_1 a_2 - \bar{Q}_1 a_2\right)\partial w_0 + N_{11}\dfrac{a_2}{a_1}\left(w_{0,\alpha} - \dfrac{a_1 u_{10}}{R_1}\right)\partial w_0 + N_{12}\left(w_{0,\beta} - u_{20}\dfrac{a_2}{R_2}\right)\partial w_0 \end{array} \right|_{\alpha=\alpha_1}^{\alpha=\alpha_2} d\beta$$

α is constant. This means if we have a patch like this and we are always saying this is α and this is β. Here, α is equal to α_1 and α_2.

This is the edge where α is constant. Over these edges these conditions will be satisfied and, on this edge, β is equal to β_1 and β_2. It may be 0 because you have taken the coordinate system here itself so, it will be 0. These variables need to be satisfied.

I have already told you that we are going to put this term $\left(N_{22}a_1 - \bar{N}_{22}a_1\right) = 0$, this term $\left(M_{22}a_1 - \bar{M}_{22}a_1\right) = 0$, this term $\left(N_{21}a_1 - \bar{N}_{22}a_1\right) = 0$, and so on and ∂w_0 coefficients.

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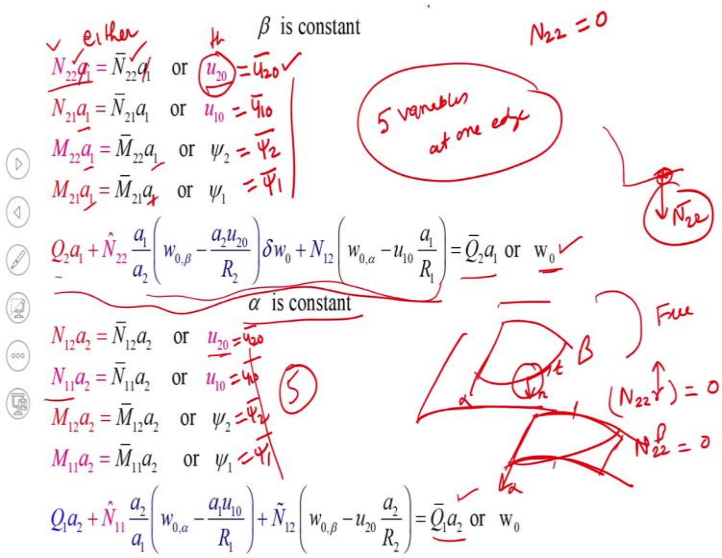

If we do so, we will get an edge where β is constant, either and or conditions are there, either $N_{22}a_1$ or u_{20}, sometimes we can say prescribed. The very reason to write like this we are not saying that N_{22} = 0, but we are saying a_1 will get canceled from both sides. You can cancel it, or take it same, there will be no problem. N_{22} = $\bar{N}_{22}$.

On the edge, N_{22} the external in-plane resultant then at the boundary the internal stress resultant and $N_{22} = \bar{N}_{22}$. Same way at the boundary the displacement $u_{20} = \bar{u}_{20}$, this may be 0, may not be 0.

We should write in a more general form. And, the second reason to put this a_1 is there are some cases where the boundary is free, if we talk about a circular plate, then it will be (N_{22} r) = 0, not just N_{22} .

Then, you may ask from where this ‘r’ is coming. So, this is a_1. That is why I have kept a_1.

In the first case:

$$N_{22}a_1 = \bar{N}_{22}a_1 \quad or \quad u_{20} = \bar{u}_{20}$$
$$N_{21}a_1 = \bar{N}_{22}a_1 \quad or \quad u_{10} = \bar{u}_{10} \quad \text{or } u_{20}$$
$$M_{22}a_1 = \bar{M}_{22}a_1 \quad or \quad \psi_2 = \bar{\psi}_2$$
$$M_{21}a_1 = \bar{M}_{21}a_1 \quad or \quad \psi_1 = \bar{\psi}_1$$

$$Q_2a_1 + N_{22}\frac{a_1}{a_2}\left(w_0,_\beta - \frac{a_2u_{20}}{R_2}\right)\underbrace{(\partial w_0)}_{\substack{need\,to\,be\,\text{mod}ified\\ as\,per\,CST}} + N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\underbrace{(\partial w_0)}_{\substack{need\,to\,be\,\text{mod}ified\\ as\,per\,CST}} = \bar{Q}_2a_1 \text{ or } w_0$$

These are the 5 variables at an edge where β is constant. Only out of these variables we have to choose the edge where β is constant.

And the other edge where α is constant we have another 5 variables. They are:

$$N_{12}a_2 = \bar{N}_{12}a_2 \quad or \quad u_{20} = \bar{u}_{20}$$
$$N_{11}a_2 = \bar{N}_{11}a_2 \quad or \quad u_{10} = \bar{u}_{10}$$
$$M_{12}a_2 = \bar{M}_{12}a_2 \quad or \quad \psi_2 = \bar{\psi}_2$$
$$M_{11}a_2 = \bar{M}_{11}a_2 \quad or \quad \psi_1 = \bar{\psi}_1$$

$$Q_1a_2 + N_{11}\frac{a_2}{a_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right) + N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\partial w_0 = \bar{Q}_1a_2 \text{ or } w_0.$$

These are the cases applicable when the geometry and coordinate axis are matching.

For example, you have taken boundaries like this and your variables are also like this α and β. The normal and tangents are along the same coordinate axis. But there may be a case where you have chosen a coordinate system some like that, but we are getting a boundary like this, a corner like this, instead of this we are getting a curved shape. Then, over this boundary what are the variables to be specified? So, at the boundaries, we have to say in terms of tangent and normal.

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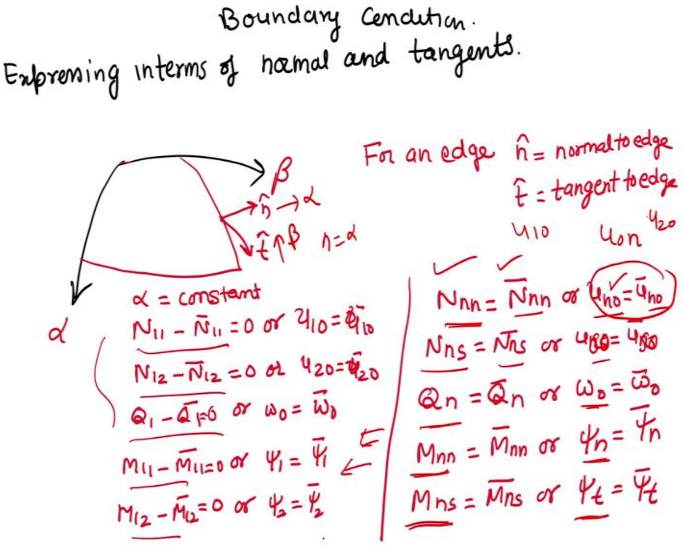

I am going to write these variables in terms of $\hat{t}$ and $\hat{n}$.

$$N_{nn} = \bar{N}_{nn} = 0 \quad or \quad u_{no} = \bar{u}_{no}$$
$$N_{ns} = \bar{N}_{ns} = 0 \quad or \quad u_{so} = \bar{u}_{so}$$
$$Q_n = \bar{Q}_n = 0 \quad or \quad w_0 = \bar{w}_0$$
$$M_{nn} = \bar{M}_{nn} = 0 \quad or \quad \psi_n = \bar{\psi}_n$$
$$M_{ns} = \bar{M}_{ns} = 0 \quad or \quad \psi_t = \bar{\psi}_t$$

Now, we say that if $\hat{n}$ is α then $\hat{t}$ will be β, then you can directly map these things. If $\hat{n}$ is α is equal to 1, then it will be:

$$N_{11} - \bar{N}_{11} = 0 \quad or \quad u_{10} = \bar{u}_{10}$$
$$N_{12} - \bar{N}_{11} = 0 \quad or \quad u_{20} = \bar{u}_{20}$$
$$Q_1 - \bar{Q}_1 = 0 \quad or \quad w_0 = \bar{w}_0$$
$$M_{11} - \bar{M}_{11} = 0 \quad or \quad \psi_1 = \bar{\psi}_1$$
$$M_{12} - \bar{M}_{12} = 0 \quad or \quad \psi_2 = \bar{\psi}_2$$

If, $\hat{n}$ normal is β then you can find the rest of the variable. It is better to write in a more general sense. Depending upon the requirement you can convert it into the explicit form.

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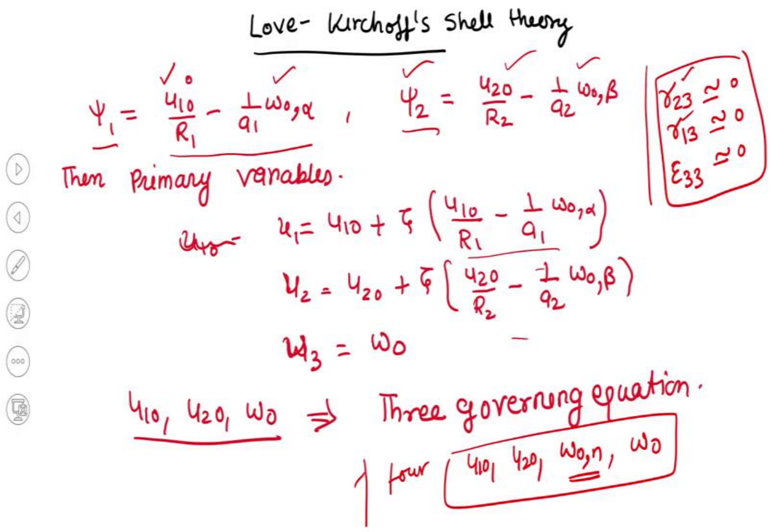

Now, I am coming to the very first shell theory, the Love Kirchhoff shell theory, in that theory γ_{23} and γ_{13} are neglected and ε_{33} is also neglected. Based on the assumptions that the transverse axis remains perpendicular to the reference surface, before and after the deformation, there is no change the angle remains 90˚.

If we pose those constraints, then the rotations will be known in terms of in-plane deformation and transverse displacement.

$$\psi_1 = \frac{u_{10}}{R_1} - \frac{1}{a_1} w_{0,\alpha} \quad \text{and} \quad \psi_2 = \frac{u_{20}}{R_2} - \frac{1}{a_2} w_{0,\beta}$$

If you know all these things, you will get 5 differential equations that can be converted into 3 equations using this concept.

$$u_1 = u_{10} + \varsigma \left(\frac{u_{10}}{R_1} - \frac{1}{a_1} w_{0,\alpha} \right)$$
$$u_2 = u_{20} + \varsigma \left(\frac{u_{20}}{R_2} - \frac{1}{a_2} w_{0,\beta} \right)$$
$$u_3 = w_0$$

The same way you see here that at an edge we need 4 boundary conditions, that we will have 4 variables u_{10}, u_{20}, and one will be a slope $w_{0,n}$ and w_0. First of all, we need to modify the boundary conditions, so, that we will get these 4 variables.

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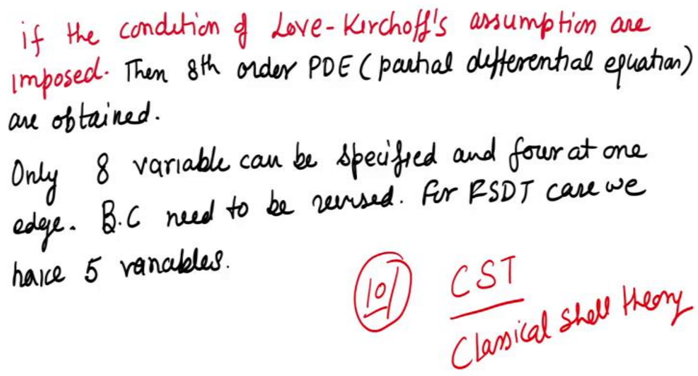

Already, I have said that if you apply Love Kirchhoff's assumptions, it will be 8 order PDE partial differential equation. We can satisfy the maximum or we can solve 8 variables. So, we need 8 boundary conditions. For the case of FSDT, we have 10 boundary conditions, but in the classical shell theory, sometimes we called it CST (classical shell theory), it will have 8 boundary conditions.

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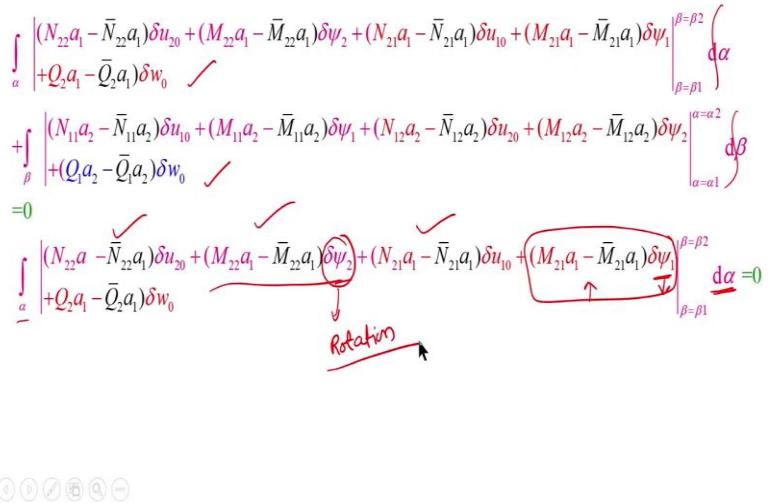

For that purpose, we have to modify the boundary conditions. What are the variables we are going to modify? Right now, I am doing without non-linear terms. So that it will be easy to explain, but if you include the non-linear term one can proceed with that also.

From explanation point of view, in this case, I have deleted the non-linear terms and I am proceeding further.

$$\int_{\alpha} \left| \begin{matrix} \left(N_{22}a_1 - \bar{N}_{22}a_1\right)\partial u_{20} + \left(M_{22}a_1 - \bar{M}_{22}a_1\right)\partial \psi_2 + \left(N_{21}a_1 - \bar{N}_{22}a_1\right)\partial u_{10} \\ + \left(M_{21}a_1 - \bar{M}_{21}a_1\right)\partial \psi_1 + \left(Q_2 a_1 - \bar{Q}_2 a_1\right)\partial w_0 \end{matrix} \right|_{\beta=\beta_1}^{\beta=\beta_2} d\alpha = 0$$

This is the edge, where β is constant

$$\int_{b} \left| \begin{matrix} \left(N_{11}a_2 - \bar{N}_{11}a_2\right)\partial u_{10} + \left(M_{11}a_2 - \bar{M}_{11}a_2\right)\partial \psi_1 + \left(N_{12}a_2 - \bar{N}_{12}a_2\right)\partial u_{20} \\ + \left(M_{12}a_2 - \bar{M}_{12}a_2\right)\partial \psi_2 + \left(Q_1 a_2 - \bar{Q}_1 a_2\right)\partial w_0 \end{matrix} \right|_{\alpha=\alpha_1}^{\alpha=\alpha_2} d\beta$$

This is the edge, where α is constant

$$\int_{\alpha} \left| \begin{matrix} \left(N_{22}a_1 - \bar{N}_{22}a_1\right)\partial u_{20} + \left(M_{22}a_1 - \bar{M}_{22}a_1\right)\partial \psi_2 + \left(N_{21}a_1 - \bar{N}_{22}a_1\right)\partial u_{10} \\ + \left(M_{21}a_1 - \bar{M}_{21}a_1\right)\partial \psi_1 + \left(Q_2 a_1 - \bar{Q}_2 a_1\right)\partial w_0 \end{matrix} \right|_{\beta=\beta_1}^{\beta=\beta_2} d\alpha = 0$$

This is integration along the α

This term $\left(M_{21}a_1 - \bar{M}_{21}a_1\right)\partial \psi_1$ needs to be modified

These terms $\left(N_{22}a_1 - \bar{N}_{22}a_1\right)\partial u_{20}$; $\left(M_{22}a_1 - \bar{M}_{22}a_1\right)\partial \psi_2$; and $\left(N_{21}a_1 - \bar{N}_{21}a_1\right)\partial u_{10}$ don't need any modification.

$\partial \psi_2$ is the rotation. When we are talking about the second direction the rotation will be in that axis. We don't need to modify this term $\left(M_{22}a_1 - \bar{M}_{22}a_1\right)\partial \psi_2$.

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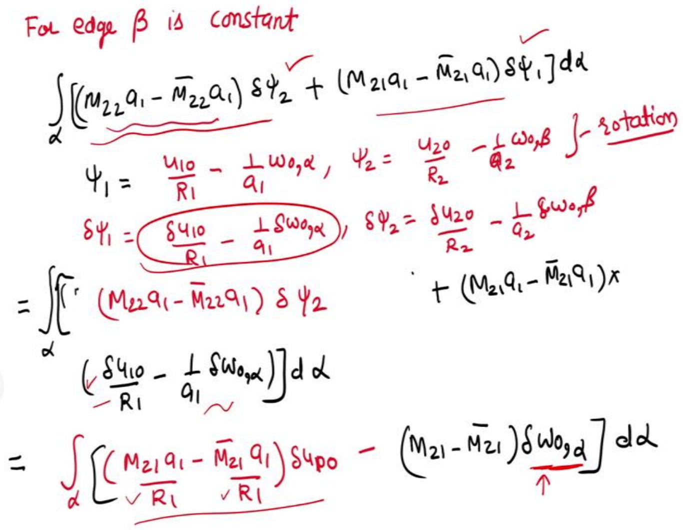

If we substitute the value of $\partial \psi_1$ in this $\left(M_{21}a_1 - \bar{M}_{21}a_1\right)\partial \psi_1$ term.

$$\partial \psi_1 = \frac{\partial u_{10}}{R_1} - \frac{1}{a_1}\partial w_{0,\alpha} .$$

If we put, $\partial \psi_1 = \dfrac{\partial u_{10}}{R_1} - \dfrac{1}{a_1}\partial w_{0,\alpha}$

The term $\frac{\partial u_{10}}{R_1}$ will give a contribution to this $\left(\frac{M_{22}a_1}{R_1}-\frac{\bar{M}_{22}a_1}{R_1}\right)\partial u_{10}$ term.

$\frac{1}{a_1}\partial w_{0,\alpha}$ will give a contribution to this $\left(M_{21}-\bar{M}_{21}\right)\partial w_{0,\alpha}$.

It is a derivative with respect to α. We will further reduce it to ∂w_0.

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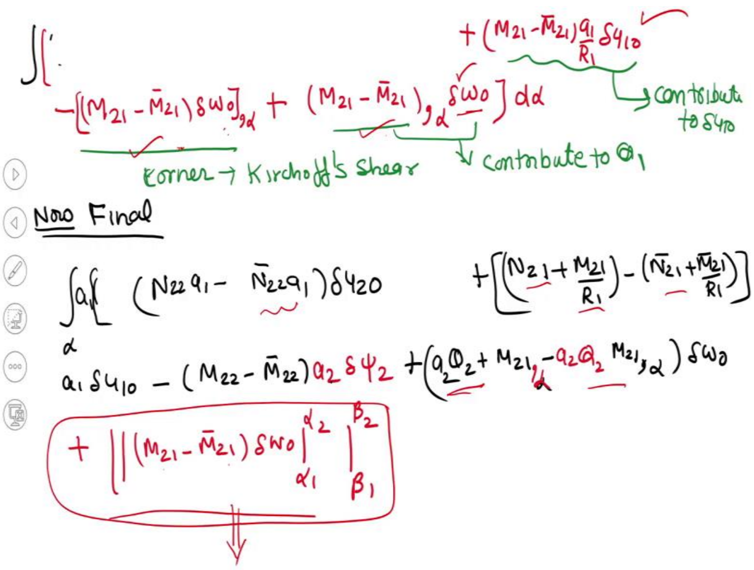

If we proceed further, it will give you two terms:

$\left[\left(M_{21}-\bar{M}_{21}\right)\partial w_0\right]_{,\alpha}$ and $\left(M_{21}-\bar{M}_{21}\right)_{,\alpha}\partial w_0$.

If you integrate this term $\left[\left(M_{21}-\bar{M}_{21}\right)\partial w_0\right]_{,\alpha}$ it will go to the corner

And will look like this $+\left|\left(M_{21}-\bar{M}_{21}\right)_{,\alpha}\partial w_0\right|_{\alpha_1}^{\alpha_2}$.

Generally, we do this derivation before adding the external work done, then $\left(M_{21}-\bar{M}_{21}\right)_{,\alpha}\partial w_0$ will go to the contribution of ∂w_0.

∂w_0 has this $\left(a_2Q_2+M_{21,\alpha}-a_2\bar{Q}_2+\bar{M}_{21,\alpha}\right)\partial w_0$ contribution

∂w_0 contribution has $+\left|\left(M_{21}-\bar{M}_{21}\right)_{,\alpha}\partial w_0\right|_{\alpha_1}^{\alpha_2}$ this contribution known as Kirchhoff shear, corner terms.

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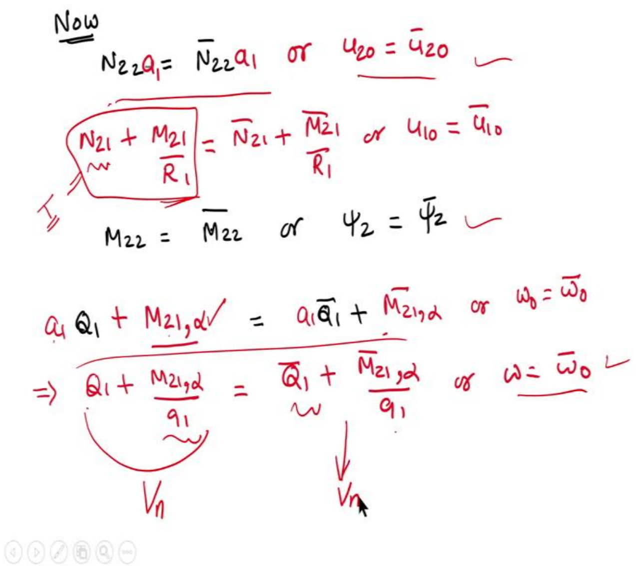

Now, the boundary condition will be either $N_{22}a_1=\bar{N}_{22}a_1$ *or* $u_{20}=\bar{u}_{20}$

The next combination will be:

$$\left(N_{21}+\frac{M_{21}}{R_1}\right)=\left(\bar{N}_{21}+\frac{\bar{M}_{21}}{R_1}\right)\ or\ u_{10}=\bar{u}_{10}$$

$$M_{22}=\bar{M}_{22}\quad or\ \psi_2=\bar{\psi}_2$$

$$a_1Q_1+M_{21,\alpha}=a_1\bar{Q}_1+\bar{M}_{21,\alpha}\quad or\ w_0=\bar{w}_0$$

$$Q_1+\frac{M_{21,\alpha}}{a_1}=\bar{Q}_1+\frac{\bar{M}_{21,\alpha}}{a_1}\quad or\ w_0=\bar{w}_0$$

And, this term $Q_1+\frac{M_{21,\alpha}}{a_1}$ is known as Kirchhoff shear V_n or V_1.

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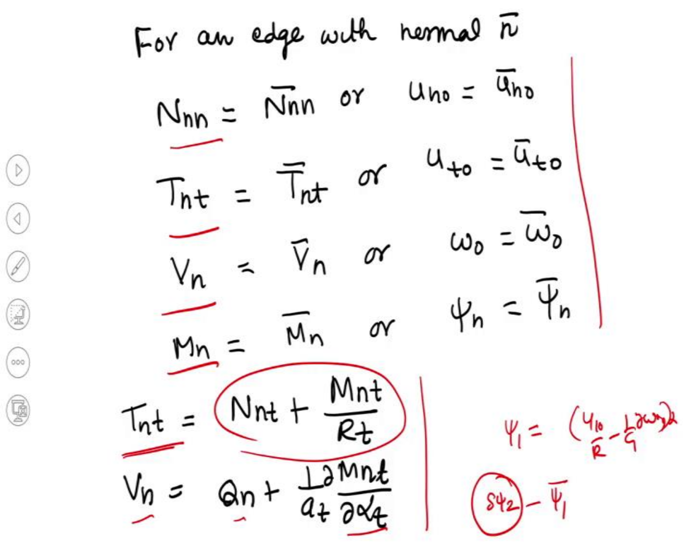

If, I write in terms of $\hat{n}$ and $\hat{t}$ it will be:

$$N_{nn} = \bar{N}_{nn} \ or \quad u_{no} = \bar{u}_{no}$$
$$T_{nt} = \bar{T}_{nt} \ or \ u_{to} = \bar{u}_{to}$$
$$V_n = \bar{V}_n \quad or \quad w_0 = \bar{w}_0$$
$$M_{nn} = \bar{M}_{nn} = 0 \ \ or \ \ \psi_n = \bar{\psi}_n$$

Where $T_{nt} = N_{nt} + \dfrac{M_{nt}}{R_t}$.

For the case of a plate, R is ∞, will be equal to N_{nt}

$$V_n = Q_n + \frac{1}{a_t}\frac{\partial M_{nt}}{\partial \alpha_t}.$$

We have converted 5 boundary variables to 4 boundary variables using the concept:

$$\psi_1 = \frac{u_{10}}{R_1} - \frac{1}{a_1} w_{0,\alpha}$$

Same way, one can go for that edge where α is constant then you have to consider ψ_2; ψ_1 remains the same. In this way, the edge where α is constant, the variables will be found.

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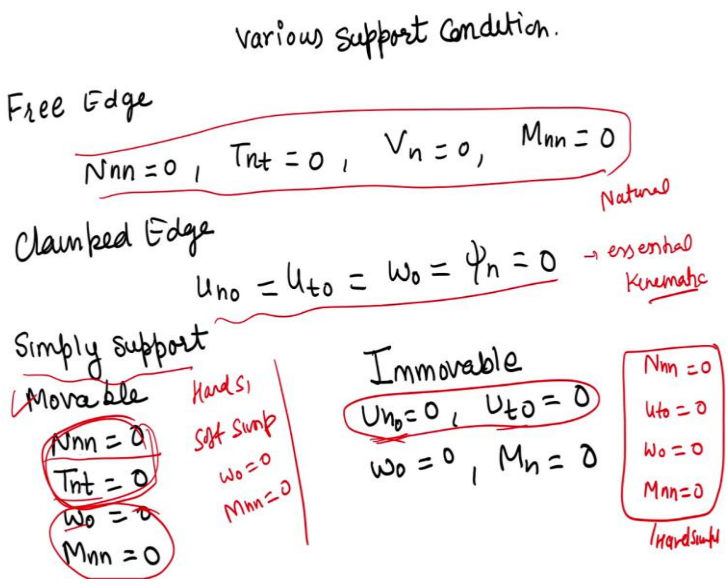

Now, we have various support conditions. If we say that an edge is free, then the natural variables sometimes called stress variables

$N_{nn} = 0$, $T_{nt} = 0$, $V_n = 0$, and $M_{nn} = 0$.

And these variables $U_{no} = U_{to} = w_0 = \psi_n$ are known as essential or kinematic variables. In-plane stress resultant T_{nt} and shear force M_{nn} is going to be 0.

If an edge is clamped then $U_{no} = U_{to} = w_0 = \psi_n = 0$.

Simply supported boundary condition is a mixed type boundary condition and in this, we find several movables simply supported, immovable simply supported, sometimes there is another classification that hard simply supported and soft simply supported concept. If we talk about a movable concept, that a normal resultant N_{nn}, T_{nt}, V_n deflection, M_{nn} the moment is going to be 0. Generally, deflection V_n and moment M_{nn} are fixed.

The transverse deflection V_n = 0 and the normal moment couple M_{nn} = 0, but these two variables N_{nn} and T_{nt} we change with that. If we say that instead of normal stress resultant N_{nn}, we can use U_{no} and instead of T_{nt} we can use U_{to} will be equal to 0, then this type of boundary condition is known as an immovable boundary condition.

Let us say another setup that N_{nn} = 0 is final, instead of T_{nt} we can say U_{to} = 0, w_0 = 0, and M_{nn} = 0. If we choose a variable like this, it is known as hard simply supported.

And, in most cases, analytical solutions are available for hard simply supported boundary conditions. And, this $N_{nn}=0$, $T_{nt}=0$, $V_n=0$, and $M_{nn}=0$ type of boundary condition is also known as soft simply supported boundary conditions.

Already immovable concept is given here. So, we can choose from the variables that are going to be specified.

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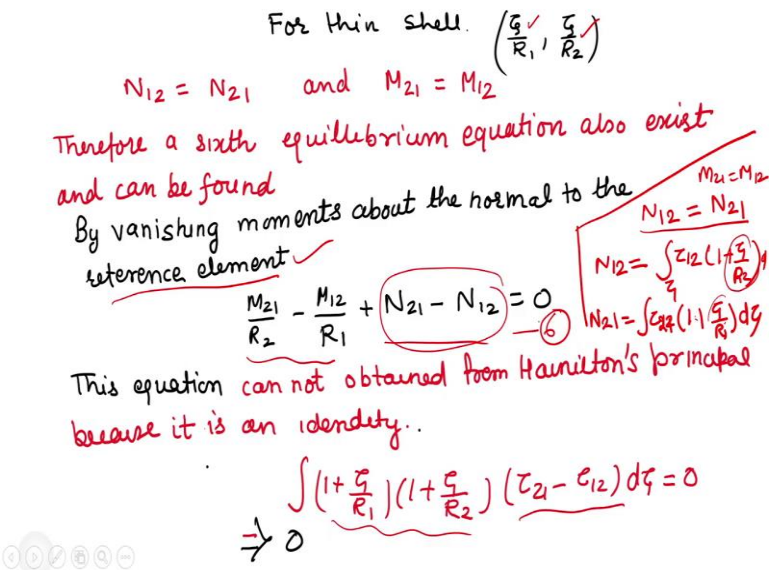

Now, a small concept is also there, if we talk about a very thin shell, $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$ can be neglected. If this is the condition then our $N_{12} = N_{21}$.

The definition of N_{12}:

$$N_{12}=\int_{\varsigma}\tau_{12}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$$

The definition of N_{21}:

$$N_{21}=\int_{\varsigma}\tau_{21}\left(1+\frac{\varsigma}{R_1}\right)d\varsigma \text{ .}$$

Therefore, $N_{12} \neq N_{21}$, because of this term $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$

But, $\tau_{12} = \tau_{21}$.

If we say, the shell is very thin, then $\frac{\varsigma}{R_1}$ and $\frac{\varsigma}{R_2}$ can be neglected and in that case:

$N_{12} = N_{21}$. Same way, $M_{12} = M_{21}$.

The following equation is the sixth equilibrium equation that exists for the case of thin shells:

$$\frac{M_{21}}{R_2} - \frac{M_{12}}{R_1} + N_{21} - N_{12} = 0 \ .$$

If we are talking about a thick or moderately thick shell, this concept is not required and the previous differential equations are valid. But, if you want to analyze a very thin shell, then we need a correction term because we do not get this equation through principle of variations.

We can say that by vanishing moment about the normal to the reference element, that $N_{21} - N_{12} = 0$. This is the 6th equation, and, it cannot be obtained from the Hamilton principle, because it is an identity. If we say that this concept $N_{21} - N_{12}$, if you want to write in that form that is going to be 0:

$$\int_0 \left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)\left(\tau_{21}-\tau_{12}\right) d\varsigma = 0 \, .$$

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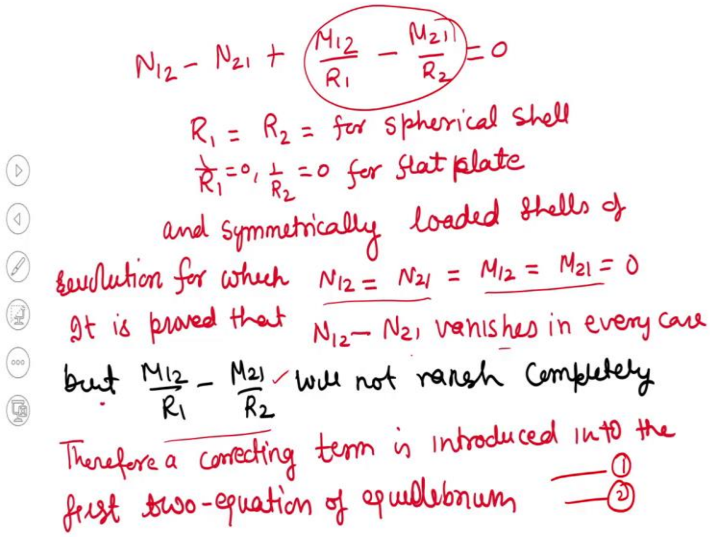

In most of the cases, when we have a spherical shell where $R_1 = R_2$, then :

$$\frac{M_{21}}{R_2} - \frac{M_{12}}{R_1} + N_{21} - N_{12} = 0$$

When we say it is a flat plate, then this equation is also satisfied. And, if we say that shell is symmetrically loaded in that case:

$$N_{12} = N_{21} = M_{12} = M_{21} = 0$$

The shear resultant in-plane displacement and couple is going to be 0.

We have proved that it already has been established, in the literature and in the books, that N_{12} - N_{21} vanishes in every case, but this term $\frac{M_{21}}{R_2} - \frac{M_{12}}{R_1}$ may exist, this may be small. Therefore, a correcting term is introduced in the first two equations. And, the procedure is given in most of the thin shell theory books.

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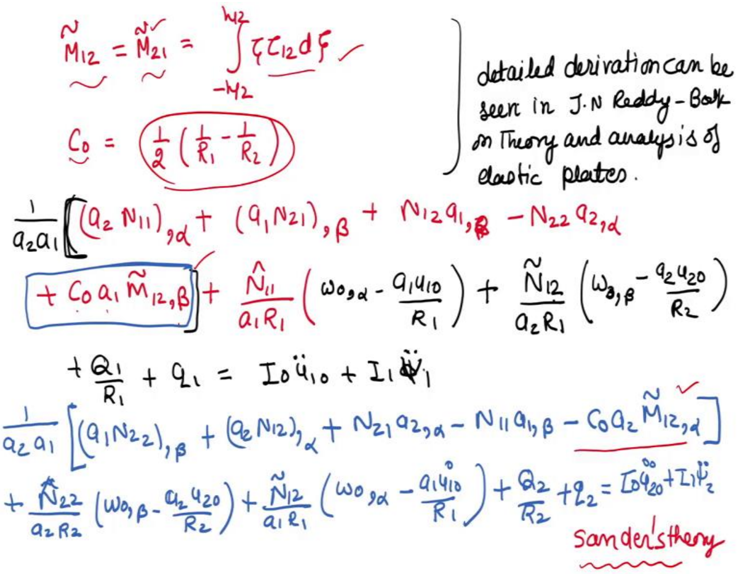

What is that extra term included? Detailed derivation can be seen in a very recent book by J N Reddy on "Theory and analysis of elastic plates" or in the Harry Krauss book, and in other shell theories book also. Now, M_{12} is denoted by $\tilde{M}_{12}$, and $\tilde{M}_{12} = \tilde{M}_{21}$ because they are same $\int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma \tau_{12} d\varsigma$.

Here C_0 is a constant $= \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$C_0 a_1 \tilde{M}_{12,\beta}$ will be added in equation (1)

$$\frac{1}{a_1 a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta} + C_0 a_1 \tilde{M}_{12,\beta}\right] + \frac{Q_1}{R_1} + \left(N_{11}\frac{1}{a_1 R_1}\left(w_0,_{\alpha} - \frac{a_1 u_{10}}{R_1}\right)\right)$$

$$+N_{12}\frac{1}{a_2 R_1}\left(w_0,_{\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1 = \left(I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1\right)$$

. Similarly, in equation (2) $C_0 a_2 \tilde{M}_{12,\alpha}$ with the minus sign is added.

$$\frac{1}{a_1 a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha} - C_0 a_2 \tilde{M}_{12,\alpha}\right] + \frac{Q_2}{R_2} + \left(N_{22}\frac{1}{a_2 R_2}\left(w_0,_{\beta} - \frac{a_2 u_{20}}{R_2}\right)\right)$$

$$+N_{12}\frac{1}{a_1 R_2}\left(w_0,_{\beta} - u_{10}\frac{a_1}{R_1}\right) + q_2 = \left(I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2\right)$$

If we do these corrections, then the exact form of Sander's shell theory is obtained. But this correction is required only for thin shells, if you are interested to derive for moderately thick shells or thick shells, then the previous 5 equations are completely valid.

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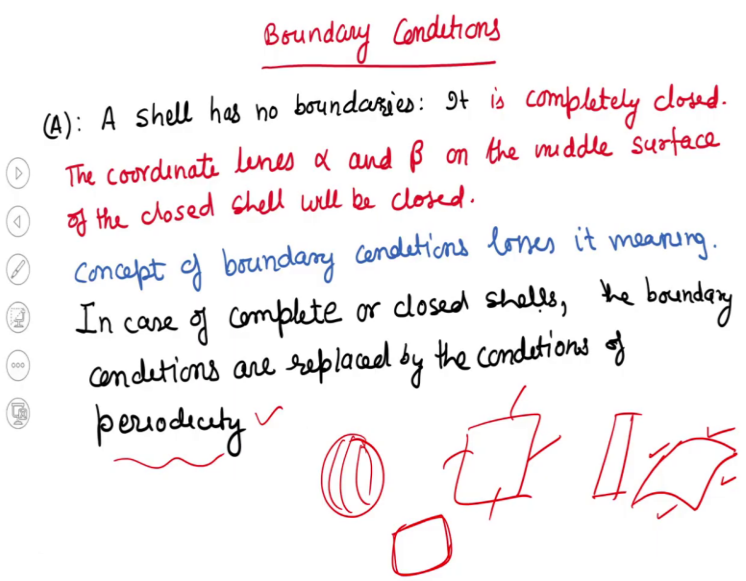

Now, we talk about the boundary conditions. Already, we have discussed that we should specify stress resultant, moments, in-plane displacement, rotations, but where? Because in the shell, in a plate, or in a beam it is clear that there are open boundaries. If you talk about a plate or a beam, their boundaries are available.

But, in the case of the shell, it is not true. If, you talk about a complete closed shell, like a sphere, then where do you provide the boundary conditions? There is no open end similarly for a cylindrical shell, a completely closed shell, it is difficult.

The previous set of equations are applicable when you have both ends, which means corners are available for the boundaries. If a shell has no boundaries, it is completely closed, the coordinate lines α and β on the middle surface of the closed shell, will be closed.

The concept of boundary condition will lose its meaning. In the case of a complete or closed shell, the boundary conditions are replaced by the condition of periodicity means, you have to do periodically to satisfy the boundary condition symmetric concept.

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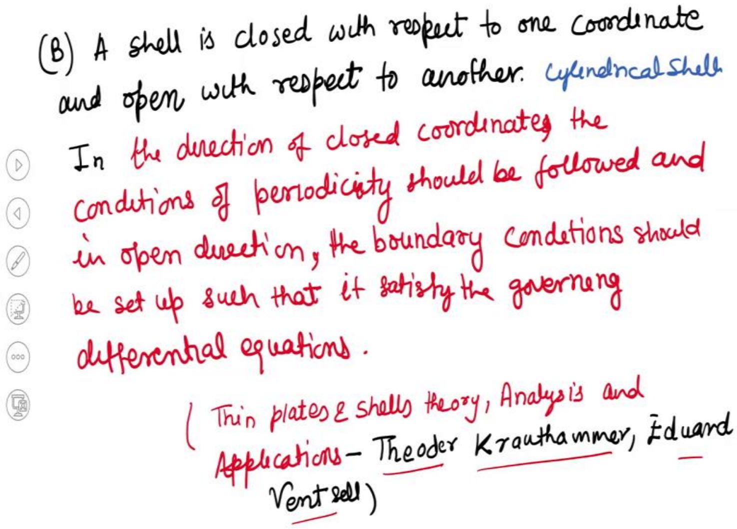

A shell is closed with respect to one coordinate and open with respect to another coordinate. Where the coordinates are closed, the condition of periodicity will be applied, and the coordinate in which the shell is open, we apply the regular boundary conditions, which we have obtained.

This type of information is given in the book thin plates and shell theory, analysis and applications by Theoder Krauthammer and Edward Ventsell. In that book, chapter 10 or 11 is devoted to the shell.

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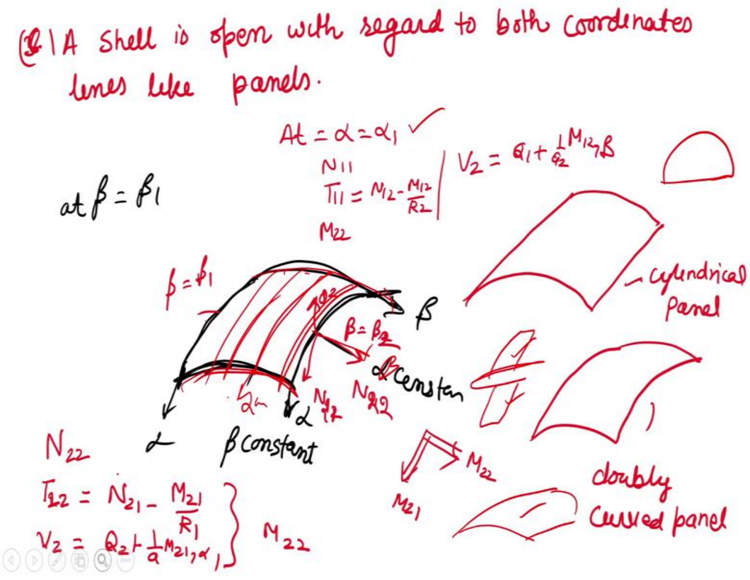

If a shell is open, it is perfect to work concerning both coordinate lines like panels. We can say, sometimes you find in the literature that a shell panel is solved instead of a closed shell. In structural applications, an airplane wing is like a panel or roof of any structure.

If it is closed like a spherical dome then it will be different, but these days you see the roof of metro stations, parking lots, any garden, or greenhouse are having this kind of panel system.

You can apply on both the edges following boundary conditions. Already we have discussed that edge where α is constant. When this is your β; β is increasing over this edge, over this line, β is going to be constant. normal will be β and over this line α is constant.

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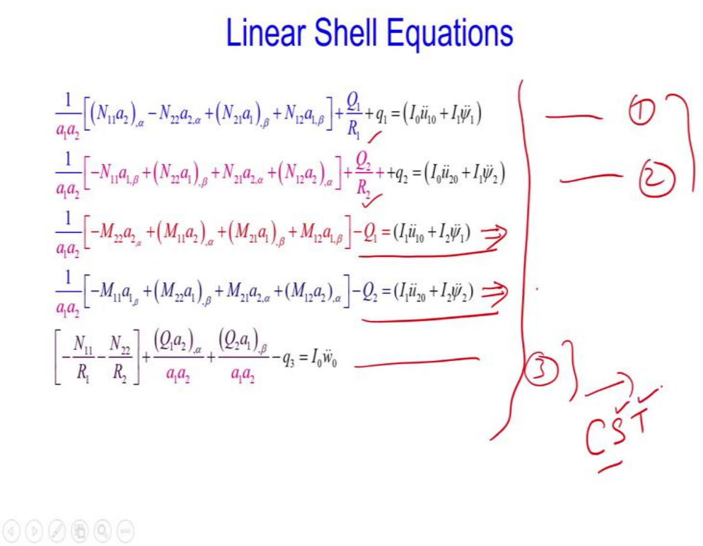

One can specify the boundary conditions. In the Kirchhoff Shell theory, I have already told you that we can obtain love Kirchhoff shell theory by substituting the expression. From this equation,

$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha} + (M_{11}a_2)_{,\alpha} + (M_{21}a_1)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = (I_1\ddot{u}_{10} + I_2\ddot{\psi}_1)$ we will get Q_1 and from this equation

$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta} + (M_{22}a_1)_{,\beta} + M_{21}a_{2,\alpha} + (M_{12}a_2)_{,\alpha}\right] - Q_2 = (I_1\ddot{u}_{20} + I_2\ddot{\psi}_2)$ we will get Q_2.

By multiplying with a_2 and differentiating with α and β, putting it here give you this

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1 a_2\right)_{,\alpha}}{a_1 a_2}+\frac{\left(Q_2 a_1\right)_{,\beta}}{a_1 a_2}=I_0 \ddot{w}_0 .$$ equation (3)

And, same way substitutes the value of $\frac{Q_1}{R_1}$ in this

$$\frac{1}{a_1 a_2}\left[\left(N_{11} a_2\right)_{,\alpha}-N_{22} a_{2,\alpha}+\left(N_{21} a_1\right)_{,\beta}+N_{12} a_{1,\beta}+C_0 a_1 \tilde{M}_{12,\beta}\right] + \frac{Q_1}{R_1}+q_1=\left(I_0 \ddot{u}_{10}+I_1 \ddot{\psi}_1\right) \quad \text{equation (1)}$$

And substituting the value of $\frac{Q_2}{R_2}$ in this

$$\frac{1}{a_1 a_2}\left[-N_{11} a_{1,\beta}+\left(N_{22} a_1\right)_{,\beta}+N_{21} a_{2,\alpha}+\left(N_{12} a_2\right)_{,\alpha}-C_0 a_2 \tilde{M}_{12,\alpha}\right] + \frac{Q_2}{R_2}+q_2=\left(I_0 \ddot{u}_{20}+I_1 \ddot{\psi}_2\right) \quad \text{equation (2)}$$

We will get three equations. These three equations will be valid for CST (Classical Shell Theory).

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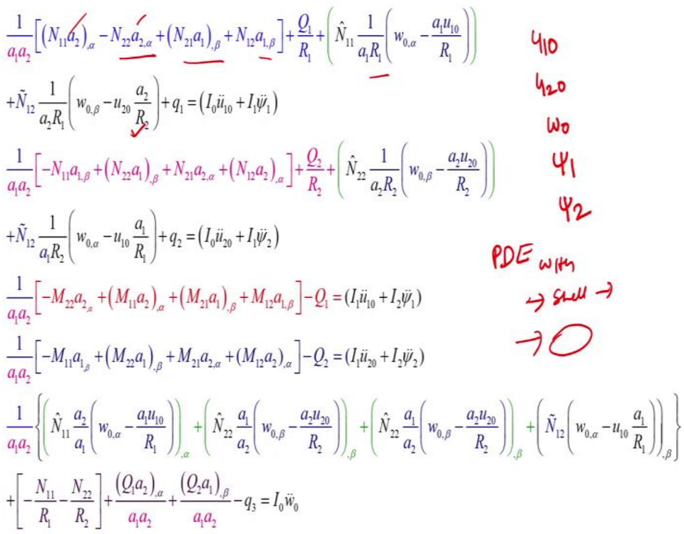

Following are the shell equations which are available to us:

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+\left(N_{11}\frac{1}{a_1R_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)$$

$$+N_{12}\frac{1}{a_2R_1}\left(w_0,_{\beta}-u_{20}\frac{a_2}{R_2}\right)+q_1=\left(I_0\ddot{u}_{10}+\underset{\substack{\text{need to be modified}\\\text{as per CST}}}{I_1\ddot{\psi}_1}\right)$$

$$\frac{1}{a_1a_2}-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}+\frac{Q_2}{R_2}+\left(N_{22}\frac{1}{a_2R_2}\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)$$

$$+N_{12}\frac{1}{a_1R_2}\left(w_0,_{\beta}-u_{10}\frac{a_1}{R_1}\right)+q_2=\left(I_0\ddot{u}_{20}+\underset{\substack{\text{need to be modified}\\\text{as per CST}}}{I_1\ddot{\psi}_2}\right)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+\underset{\substack{\text{need to be modified}\\\text{as per CST}}}{I_2\ddot{\psi}_1}\right)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+\underset{\substack{\text{need to be modified}\\\text{as per CST}}}{I_2\ddot{\psi}_2}\right)$$

$$\frac{1}{a_1a_2}\left[\begin{array}{l}\left(N_{11}\frac{a_2}{a_1}\left(w_0,_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}+\left(N_{22}\frac{a_1}{a_2}\left(w_0,_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}+\\\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}+\left(N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\end{array}\right]$$

$$+\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}=I_0\ddot{w}_0$$

But can we work with this? Can we get the solution just by solving these equations?

We have to first convert them to the primary variables. What are the primary variables? u_{10}, u_{20}, w_0, ψ_1 and ψ_2. We have to convert this set of equations into these primary equations then only we can solve it.

Even today, solutions to these equations are very difficult. Why is it difficult? You see that these are the partial differential equation with variable coefficients. Here the coefficients are varying with respect to α and β.

These are the PDE with variable coefficients. These are difficult to handle. From the day when the first shell theory was proposed, since then we have specialized in development or solution techniques for a general shell theory.

Let us say a shell is subjected to membrane loading, not any transverse loading or bending loads. There is a lot of application in the industry where a shell is acting as a membrane. And, there are some applications where it will take the flexural bending, and there are some which have axisymmetric loading, by doing so one by one some terms get eliminated and solutions become easy.

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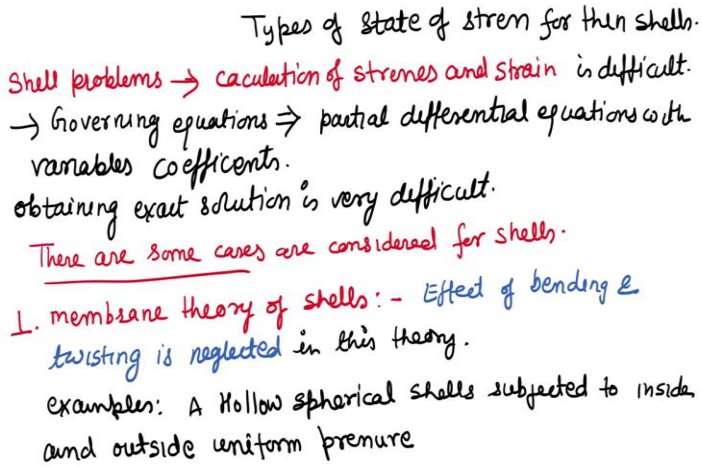

The types of state of stress for thin shell-like membrane theory of shells: Effect of bending and twisting is neglected in this theory. For example, a hollow spherical shell subjected to inside and outside uniform pressure will be covered under a membrane theory of shells.

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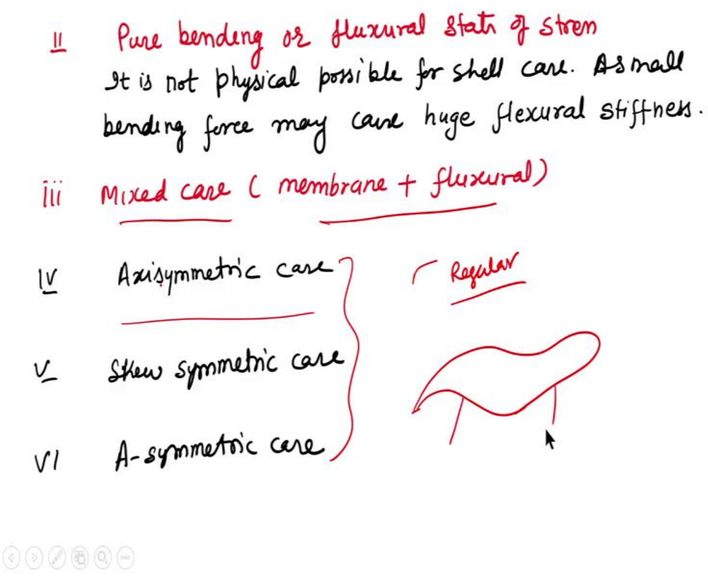

We have a case of pure bending or flexural state of stress: some studies have been done or the equations are solved, but, from an actual physical point of view this condition is very dangerous, it is not possible for the case of a shell. because a small bending force may cause huge flexural stiffness. Bending and stretching cannot be decoupled, if, there is bending there will be some stretching effect also.

Then we have the mixed case (membrane +flexural) which is the more complicated one. And then the case of Axisymmetric, then the loading, because of the loading also the equations get simplified, then the case of skew-symmetric, and Axisymmetric case. To date the most generalized shell is not solved, they are mostly regular shells, formed by the revolution of surfaces like cylindrical shells or spherical shells; the structures are made out of that.

In 90% of literature or books, the cylindrical shells, conical shells, and spherical shells are solved, their governing equation is slightly less complex. But the other shells may have a completely different profile or doubly curved, even today, shell equations are difficult to solve for those.

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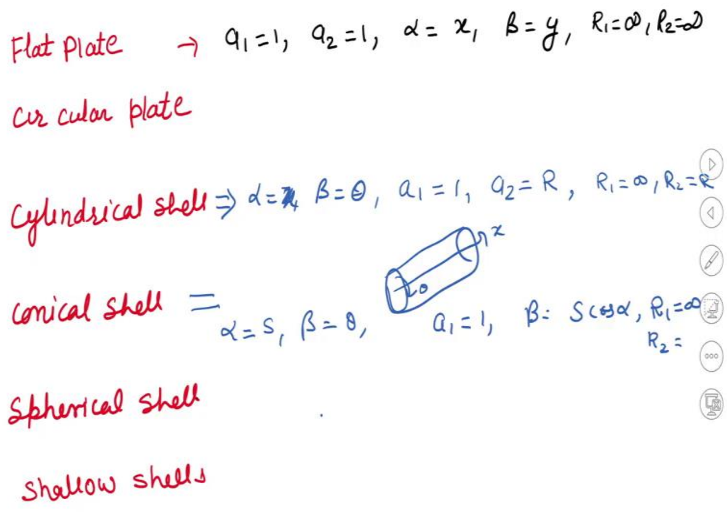

In the next lecture, I will derive the equations for a plate, cylindrical shell, spherical shell, for the given governing equations. First, we will try to find that can we get the governing differential equations for those special cases? I will explain that first and then we will proceed further.

Thank you very much.