Theory of Mechanisms

Lecture – 37

Introduction to the Kinematics of Spatial Mechanisms

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Kinematics of spatial chains

Matrix method

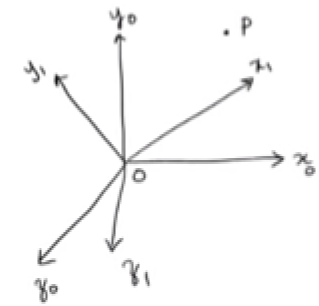
A cartesian coord. system attached to each link following certain conventions

D-H method

Denavit - Hartenberg

The coords. of a pt. in space expressed in two such systems are related through a 4x4 matrix → homogeneous transformation matrix

→ Loop closure equation is expressed in matrix form



Point P is located by vector p^0 in the absolute coord. system $Ox_0y_0z_0$ & by p^1 in the rotated coord. system $Ox_1y_1z_1$

$$p^0_x \hat{x}_0 + p^0_y \hat{y}_0 + p^0_z \hat{z}_0 = p^1_x \hat{x}_1 + p^1_y \hat{y}_1 + p^1_z \hat{z}_1$$

Both are descriptions of the same point

So, a detailed study of, kinematics, of spatial chains, especially robots, you would find in any robotics, course. Typically uses a matrix method, but we will just look at some of the basics, of this method, applied to special mechanisms. So the, basic idea is, you will have, a Cartesian coordinate system, attached to every link. And usually you follow certain, conventions. One of the most common conventions, for attaching, for defining these systems is the, 'Denavit- Hartenberg method', which is, you know the book that you are that we are using for this course, those two authors, are very famous in the robotics literature, for this, the Denavit-Hartenberg coordinate system, method for assigning, coordinate systems, so. So once these are defined, in this manner, the two coordinate systems are related, or, the coordinates, of a point in space, expressed in two such systems, are related, through a four by four, matrix, which is called the, 'Homogeneous Transformation Matrix', so we'll see how that is done. And then, once you do this, the loop closure equation, is expressed in, matrix form. Okay? through these transformation Matrixes. So first let us look at, two coordinate systems, two Cartesian coordinate systems that are rotated with respect to one another but share a common origin. Okay? Two coordinate systems. So let's say there is a point P, located in space and we want to express, this point P, so it has a certain set of coordinates, in the X naught, Y naught, Z naught, coordinate system and another set of coordinates, in the x1, y1, z1, coordinate system and we want to relate, the two sets of coordinates. So let's say, point P, is expressed or located, by vector, P naught, let's say the X naught, Y naught, Z naught, coordinate system, is the absolute or fixed coordinate system.

So it's probably the coordinate system, attached to the fixed link ,of the mechanism, in the absolute coordinate system, O, X naught, Y naught, Z naught, and by P 1 in the, rotated coordinate system, for X 1, Y 1, C1. Okay? so both are descriptions, of the same point position. So, if it's described as, say in p0, I have p naught x, along the unit vector, x naught, plus p naught y, along the unit vector, y naught, plus p naught z, along the unit vector, so the Caps denote the unit vectors. Okay? So this, is the same as, because they describe the same point, ax 1, along unit vector along x 1 plus p1, the z component, along the z1, direction, because both are descriptions of the, same point. Okay? So now, I'll dot-product both sides by, x Naught. So I have, let's see.

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$$\hat{x}_0 \cdot \left(p_x^0 \hat{x}_0 + p_y^0 \hat{y}_0 + p_z^0 \hat{z}_0\right) = \hat{x}_0 \cdot \left(p_x^1 \hat{x}_1 + p_y^1 \hat{y}_1 + p_z^1 \hat{z}_1\right)$$

Dot product both sides by $\hat{x}_0$

$$p_x^0 = (\hat{x}_1 \cdot \hat{x}_0)\, p_x^1 + (\hat{y}_1 \cdot \hat{x}_0)\, p_y^1 + (\hat{z}_1 \cdot \hat{x}_0)\, p_z^1$$

Similarly, doing the dot product with $\hat{y}_0$, then $\hat{z}_0$, we'll get

$$p_y^0 = (\hat{x}_1 \cdot \hat{y}_0)\, p_x^1 + (\hat{y}_1 \cdot \hat{y}_0)\, p_y^1 + (\hat{z}_1 \cdot \hat{y}_0)\, p_z^1$$

$$p_z^0 = (\hat{x}_1 \cdot \hat{z}_0)\, p_x^1 + (\hat{y}_1 \cdot \hat{z}_0)\, p_y^1 + (\hat{z}_1 \cdot \hat{z}_0)\, p_z^1$$

In matrix form

$$\begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & \hat{y}_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & \hat{y}_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix} \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix}$$

$$p^0 = R_1^0\, p^1$$

rotation matrix

Direction cosines of $\hat{x}_1$ w.r.t frame 0

I have, plus p1 y, along Y 1, cap, plus p1 z along z 1 cap and I dot product, both sides? X naught. Okay, now what do I have? All the, the, these two are, Cartesian coordinate systems, so the vectors are orthogonal. So in a particular system, x naught t naught z naught, are orthogonal to each other. So the dot products will be 0. Right? And x naught, dot, x naught, will just be 1. So I have, I eliminate all the other terms, on the left-hand side and I will get, this is equal to, x 1, dot, x naught, pz 1.

Similarly instead of, with x naught if I do the dot product, on both sides with y naught, in succession if I do, with, y naught, then, z naught, will get, Py0, equals, Py1 plus, sorry, z1 dot, y naught, Pz1 and this is by, multiplying the dot product, with z naught, I'll get, pz naught, equal, to x 1 dot, Px 1, plus, y1 dot, dot, z naught, Py1, plus, z1 dot, dot, z naught. Pz1. Okay? So in matrix form, I can write this as, is equal to, X 1, dot, X naught, X 1, do,t y naught, X 1 Dot, z naught, y 1, dot, x naught, y 1, dot, y Naught, y 1, dot, z naught and here, z 1, dot, x naught, z 1, dot, y naught, z 1, dot, z naught ,into px 1, py 1, we say Pz1. In a compact form, I can say that, the coordinates of P, in the zero frame, equal a matrix R 1 0, times, the coordinates of P, in the frame 1. Okay? This matrix here, is called, 'A Rotation Matrix.' It really it relates the, Orientation, of the frame, x1, y1, z1. It Shows, how that is rotated, with respect to, the frame, 0, frame 1, with respect to, frame 0, so this is the notation, rotation of frame 1, with respect to 0. And if you look at the columns, of this matrix, then you see that. This, column here, gives you what? If you look at, x 1, dot x 0, x 1 dot, these are the direction cosines, of x 1, with respect to, frame zero. These are similarly, the direction cosines of, y1. so the columns of this rotation matrix, are basically the, direction cosines of, the three unit vectors, defining frame 1, with respect to frame 0. So, there, these are nine quantities in this, matrix. Right? So do you need nine Quantities? To describe the orientation, of one coordinate system, with respect to the other. So how many independent quantities are these? You have only three, the, the access are coincidence, sorry, the origins are coincident. You only have three independent quantities and that's because, you already have the, the, x1, y1, z1, are orthogonal, you know that. Right? So that gives you three relations, three relations. In addition you have each column, the sum of the squares, the direct of the direction cosines, equal to one. Okay? So it's only three independent quantities that you have in this matrix.

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Out of the nine quantities describing the DC^s of the axes of $ox_1y_1z_1$ w.r.t $ox_0y_0z_0$, only 3 are independent

$\because$ $\hat{x}_1, \hat{y}_1, \hat{z}_1$ are orthogonal

& sum of the squares of the $DC^s = 1$

$$R_1^0 = \begin{bmatrix} \hat{x}_1 \cdot \hat{x}_0 & \hat{y}_1 \cdot \hat{x}_0 & \hat{z}_1 \cdot \hat{x}_0 \\ \hat{x}_1 \cdot \hat{y}_0 & y_1 \cdot \hat{y}_0 & \hat{z}_1 \cdot \hat{y}_0 \\ \hat{x}_1 \cdot \hat{z}_0 & y_1 \cdot \hat{z}_0 & \hat{z}_1 \cdot \hat{z}_0 \end{bmatrix}$$

$$R_0^1 = [R_1^0]^{-1} = \begin{bmatrix} \hat{x}_0 \cdot \hat{x}_1 & \hat{y}_0 \cdot \hat{x}_1 & \hat{z}_0 \cdot \hat{x}_1 \\ \hat{x}_0 \cdot \hat{y}_1 & \hat{y}_0 \cdot \hat{y}_1 & \hat{z}_0 \cdot \hat{y} \\ \hat{x}_0 \cdot \hat{z}_1 & \hat{y}_0 \cdot \hat{z}_1 & \hat{z}_0 \cdot \hat{z}_1 \end{bmatrix} = [R_1^0]^T$$

Orthogonal matrices

So nine quantities, describing, the direction cosines, of the axis, of only three, are independent, equal to 1. Okay? So you have, R1 0, equal to, x 1, dot, x naught, x 1, dot, y naught, x 1, dot, z naught and then you have y 1 dot, x naught, y 1, dot, y naught and y 1, dot, z naught, z 1, dot, x naught. Okay? Say if I want, the rotation matrix, how is this? So I want to look at it from the angle of the, x naught, y naught, so I want to see what is the rotation of x naught, y naught, z naught with respect to, x1, y1, z1. Okay? So this is the inverse of, this matrix. But I can, from the direction cosines, I know that, it has to be, x naught, dot, x 1, x naught, dot, y 1, because this is my reference now, so I want the direction cosines, with respect to, x1, y1, z1, y naught, dot, y 1, y naught, dot, z 1 and z naught, dot, x 1, z naught, dot, y 1 and z naught, dot, c1. Okay? This is the inverse of this. And if I look at these two, how are they related? This is nothing but, the transpose, of the original rotation matrix. So such matrices, where the inverse equals, the transpose, they are called, 'Orthogonal Matrices'. So the rotation matrices are orthogonal matrices. Yeah, right. So the sum of the DC, is equal to one. So the sum of each row, each column, equals sum of the squares. Each row and each column equals 1. So let's do some basic rotation matrices, just so, we get familiar.

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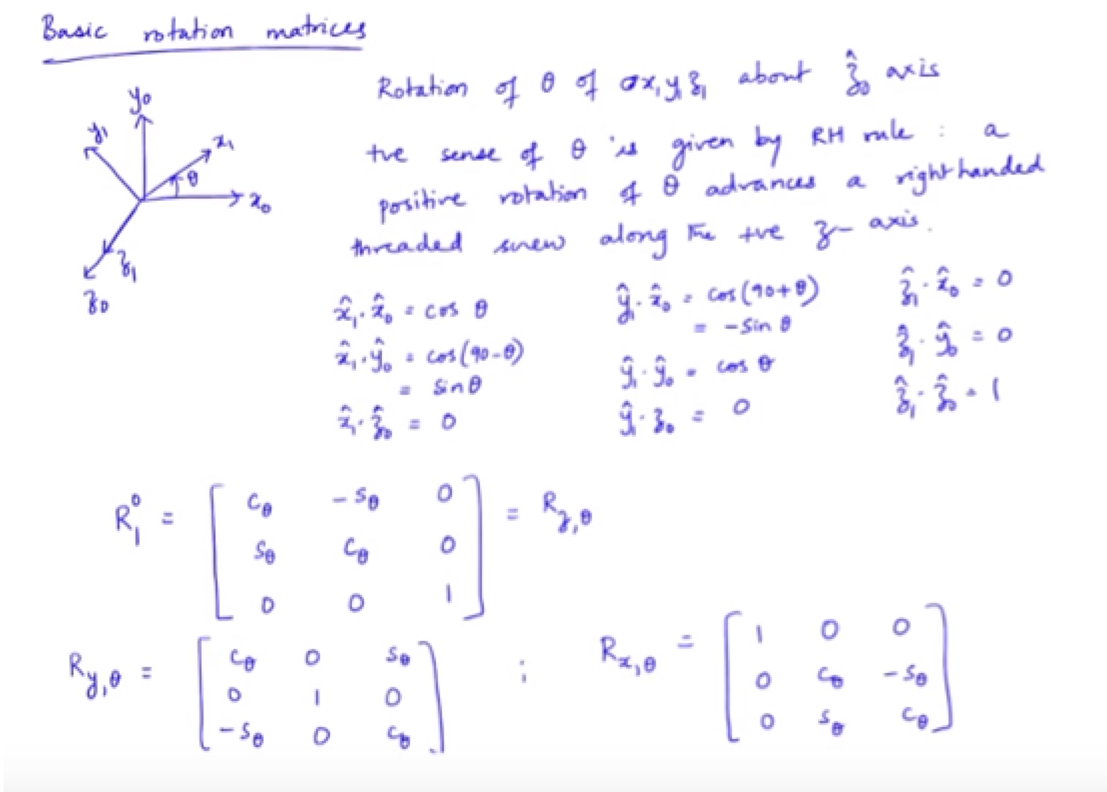

So let's say, we have, and I have another coordinate system, x1, y1, z1, where the rotation happens about, z naught, so that, z naught and z1, are aligned. Okay? So I have a rotation, of theta of, ox1, y1, z1, about the z naught, axis. Okay? Typically we follow the right-hand rule. So if there is, the positive sense, if you have a, a positive rotation, of theta, is when, if you have a screw, you have a right-handed threaded screw and if you move along, the direction of theta, then, the screw advances, along the positive axis, in this case the z-axis, it's a positive sense of theta, is given by the right hand rule. Okay? So a positive rotation of theta, advances, axis. So construct the rotation matrix, for this. What are your direction cosines? You have, what is x1 dot x naught? Cos theta. X1, dot, y naught, is the angle between, x1 and y naught is, 90, minus, theta, which is sine theta. Okay? And dot, z naught. What is the angle between, z 1 and z naught, are parallel, which means, it's coming out, if this is your plane. Okay? Which means x naught and z naught, x 1 and z naught, are perpendicular, so you have cos 90, which is, 0. Similarly y 1 dot, X naught, what is y 1, dot X naught? Cos of 90, plus theta. Okay? Minus sine theta, then you have, y1, dot y naught, cos theta, y1 dot, z naught, equal to 0. z1 dot, X naught, is zero. Z1, dot z1 is 1. Okay? So my rotation matrix, relating, 1 with 2 is, so a short form, will write cos theta is, C subscript theta, sine theta, 0, minus sine theta, cos theta, 0, 0, 0, 1. So we can, we designate this, since this is a special rotation, we'll call this Rz theta. Okay? So these are considered, the basic rotation matrices, where.

Similarly I can write, Ry theta would be, cos theta, 0, minus sine theta, 0, 1, 0, sine theta, 0, cos theta and I have Rx theta, is, 1, 0, 0. Which one? Ry theta, yeah? Which term I you. You have a question about the Ry theta matrix? What are you saying? Are you saying its one theta? Minus sine theta here, oh, sorry. No this is fine, this is fine, this is Yeah, its fine. So this is, what is this term corresponding to? You are talking about a rotation, about the y axis. Right? Here itself you can change this and see. Okay, I can take this as yz and x, no I want this is, y, so this will be xyz, change it like that. Okay? I want a rotation about, the y axis. Do the direction cosines. X1, dot, z naught, when you are rotating about the, y axis. Okay? Let's do another, I want you to, try this.

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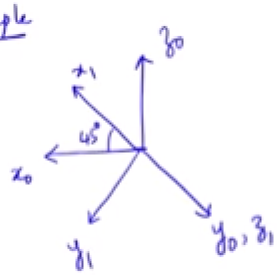
Example



Find the rotation matrix R_1^0 specifying the orientation of $ox_1y_1z_1$ relative to $ox_0y_0z_0$

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Let's say, I have x naught. Okay? still the right handed system. And let's say, y naught is coinciding with z1, and I have, x1, this angle is 45 and I have Y1, orientation of. Okay? Take a few minutes, to try it out, do the direction cosines. x1 dot, x naught, is what? what is x1 dot, x naught? 1 by root 2, x1 dot, y naught. This is coming out. Right? So it is, 0. y naught is, parallel to z1, which is 0. What is x1 dot, z naught? Hmm? 1 by root 2, yeah, then y1 dot, x naught Y1 dot, y naught, y1 dot, z naught. Then z1 dot, z naught, 0, sorry, z1 dot, x naught, yeah, z 1 dot, y naught, 1, z 1 dot, z naught, is 0. Okay? You just apply the, same direction cosines. Okay?

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Compositions of rotations

3 frames $Ox_0y_0z_0$, $Ox_1y_1z_1$, $Ox_2y_2z_2$ with origins coincident

$$p^0 = R_1^0 p^1$$

$$p^1 = R_2^1 p^2$$

$$p^0 = R_1^0 R_2^1 p^2 = R_2^0 p^2$$

$$\therefore R_2^0 = R_1^0 R_2^1$$

Rotation occurs w.r.t frame 0, then w.r.t frame 1

Order of rotations matters!

Now let's look at, if you have, multiple rotations happening, how these rotations are, composed? So let's say we have three frames. O, x2, y2, z2, so the origins are coincident. I have p naught, equals r10, p 1. Let’s Say, I have p 1 can be described as, this. Okay? With respect, 1 in 2 can be related, in this fashion, then p 0, is, R1 0, R 2 1, p2, so you can see, it goes like this. Okay? You can see that, you can think of it, as the, these terms cancelling out. Right, 2 & 2, 1 & 1, cross terms, cancelling out. And this, I can write as, R 2 0, p 2. Okay? Therefore, R 2 0 is. Okay? And remember, matrix multiplication, is not commutative, so this is different from. So the order of the rotation matters. Okay? So we take it as a rotation, occurs first, first with respect to frame 0. That’s the first matrix. Then frame 1. Okay? so order of rotations, matters. Okay?

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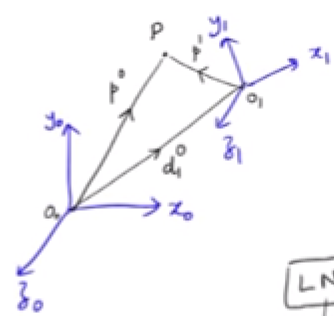
$$\underbrace{p^0 = R^0_1 p^1 + d^0_1}_{\text{general rigid body motion}}$$

Bcos p^0 & p^1 are expressed w.r.t non-parallel coordinate systems

Multiplying p^1 by R^0_1 expresses it in a coord. system that is parallel to p^0



LNCS

If we have 2 rigid motions

$$p^0 = R^0_1 p_1 + d^0_1$$

$$p^1 = R^1_2 p_2 + d^1_2$$

Then $p^0 = R^0_1 R^1_2 p^2 + R^0_1 d^1_2 + d^0_1 = R^0_2 p^2 + d^0_2$

$\therefore R^0_2 = R^0_1 R^1_2$ & $d^0_2 = d^0_1 + R^0_1 d^1_2$

So this, these are the cases, where, you have, the origins of the two coordinate systems, coinciding. Okay? In a more general case, it's not going to happen.

So you typically choose, say coordinate systems, attached to the CG, of each link. Okay? The chances of the CGS, of all the links coinciding is, very low. Right? So a more general case is, when you have a base coordinate system. And then you may have another coordinate system, which is both, rotated, as well as, translated, with respect to, the first coordinate system. So let's say, the origin, so o1. This is o 0, is displaced by, some d 1, so this d1 0 represents the, displacement of the origin o 1, in the reference frame, 0. And then let's say, we again have a point p, which has a certain description, described by this vector, p1, and by a vector, a0, in this. Okay Typically, when you have, two coordinate systems, if the coordinate systems are parallel, then, I could say that, p 0 equals d1 0, plus p1. However, when the coordinate systems are rotated with respect to one another, you cannot directly add, the two vectors. I cannot add a vector expressed in coordinate frame, 0, directly to a vector, that is expressed in, coordinate frame 1. So this would be expressed as p 0, equal to, so suppose I had, that coordinate system shifted to this origin. Okay? Then the description, of that, would be, this, plus, d1 0. So I cannot directly add p1. I have to transform everything, to the same coordinate system, so this vector transformed into, a coordinate system, that is parallel to, x naught, y naught, z naught, would be this, term here. You multiply it, by the rotation matrix because we saw, in the previous case, when the origins were coincident, that those two representations are equal. So I've converted everything into, this base coordinate system, before I can add these vectors. Okay? Because, p0 and p1, are expressed, with respect to, non parallel, coordinate frames.

Do you remember when we did the force analysis? One of the things that, I insisted on or I mentioned repeatedly, is the local non rotating coordinate system. We expressed all those position vectors, for the, you know, points of application, of the forces etcetera. So in each case there, we took a coordinate system, with the origin with the CG, but we said it remains parallel to the, global coordinate system. Okay? So that is the case, where you have a non rotating coordinate system. So that that's when, you can add those vectors, directly. If they were expressed in coordinate systems that are rotated, with respect to one another, then you have to apply the appropriate, rotation matrix, in order to, be able to, add vectors in these two coordinate systems. Okay? So basically multiplying, p1 by r10 expresses it, in a coordinate system that is parallel, p0. Okay? In this case, these vectors can be added. You can do the, translation first. So this is a description, of a general rigid body motion, this represents, where you have a rotation and a translation, this represents, a general rigid body motion. Can be represented as a translation, followed by a rotation or the you can do the rotation first and then do the translation, so it doesn't, as long as they're both with respect to the same frame, the order in which you do the translation and rotation do not matter. So if we have two rigid motions, we have, p0, equal to r10, p1 plus d1 and say, we have p1 equal to r21, P 2 plus d21, then I can substitute, this into this and I will get, p0, equal to r1 0, r2 1, into P 2, plus r1 0 and this, I can represent as, R 2 0, P 2, plus d2 0. Therefore if I compare the two, I get, R20, equal to R10, R21, which we saw earlier, when we compose the rotations and d20, is d10, plus, R 1 0, d2. Okay? So this represents the translation of the second coordinate frame. Again, so when you are working with manipulators or special mechanisms, these transformations have to happen to, express everything in the same coordinate frame. Okay, we're.

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In matrix form

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix}$$

Rigid motion is expressed in a matrix of the form

$$H = \begin{bmatrix} R_{3\times3} & d_{3\times1} \\ 0 & 1 \end{bmatrix}_{4\times4}$$

Matrix representation of a rigid motion

Homogeneous transformation matrix

$$p^0 = R_1^0 p^1 + d_1^0$$

$$\begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \\ 1 \end{bmatrix} = \begin{bmatrix} (R_1^0)_{3\times3} & \begin{matrix} d_{1x}^0 \\ d_{1y}^0 \\ d_{1z}^0 \end{matrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \\ 1 \end{bmatrix}$$

$$[P^0]_{4\times1} = \begin{bmatrix} p^0 \\ 1 \end{bmatrix}; \quad [P^1]_{4\times1} = \begin{bmatrix} p^1 \\ 1 \end{bmatrix}$$

$$P^0 = H_1^0 P^1$$

So in matrix form I can write this as, R10, d10, 01. So we say, the rigid motion, of the form, we call it H, for homogeneous transformation matrix and we say R0d1. Okay? So this will be a 4 by 4 matrix. This is a 3 by 3, the rotation matrix, this vector, is a 3 by 1. Okay? This is, the matrix representation, of a rigid motion, is called the,' Homogeneous Transformation matrix'. So we saw here, so we have this, p1 equal to r1, plus d10. Right? So essentially what we've done is, you have, and I put a 1, a fourth column and, and here, I have the R10, which is a 3 by 3. Okay? And this is d1, x, d1y, so the origin. Right? This is a 3 by 1 and then I add 0, 0, 0, 1 into p, sorry, this should be 0. Did I do this wrong here? Nobody corrected me. This is zero and this is, 1. Okay? It's basically this equation in, in this form. Okay? So we call, this, so we typically call this, I in, in matrix form, we will write this, as P, capital P 0. We define this as P0 1. Capital P 1, is P 1, 1. Okay? It becomes a 4 by1; this is a, 4 by 1. And then, so we can write this as, P0, capital P0, is equal to, H10, P1. Okay? The loop closure, let's see. So you can, use certain conventions, to set up, these matrices. Okay? The Denavit-Hartenberg convention is one. So where you choose this coordinate system. So if you have n links, in the mechanism, you choose the coordinate systems that you attached to each of these links. Okay? And the coordinate systems rotate with the links. As opposed to the earlier case, where we said we are choosing, only the origin at the CG of the link, but the coordinate systems remains parallel to the ground coordinate system or the fixed links coordinate system.

So here you would attach, these coordinate systems, to the individual links, so that they rotate, with the link and then you would relate, the, coordinate systems, to one another, through these transformation matrices. So that takes into account, the rotations of the links. Because you are now expressing everything, in terms of these, transformation matrices, which contain the information, about the relative orientations and displacements, of these links with respect to one another. Therefore, now if I want, for any specific point on the link, if I know, the description of that point, with respects to that links own coordinate system, I can easily, relate it and find its spatial location, with respect to the, ground coordinate system. So that's the, that sort of the idea behind, doing it in this systematic fashion. That way any point, on any link, that I want, once I know the relationship, between the coordinate systems, I can determine the kinematics of that. Okay? So this seems like a good point to close this course. Okay? So this would be The, because doing it in a geometric or using geometry or trigonometry will become very complex, when you try to look at special mechanisms. So this is, I hope this has given you an introduction, to special mechanisms, but predominantly our course was about, synthesis, of planar mechanisms and I hope you have gotten a good idea, of that, over the course of this semester.