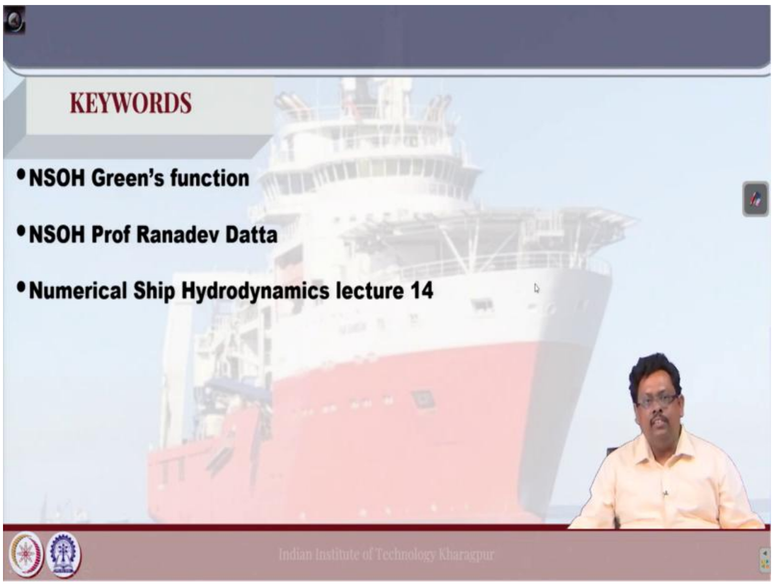
Numerical Ship and Offshore Hydrodynamics
Prof. Ranadev Datta
Department of Ocean Engineering and Naval Architecture
Indian Institute of Technology, Kharagpur

Lecture - 14
Introduction to BEM

(Refer Slide Time: 00:20)

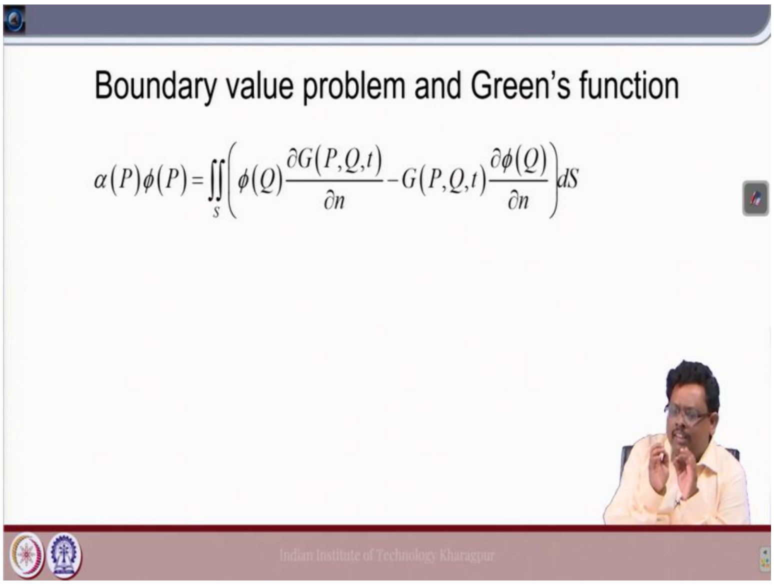
Welcome to the Numerical Ship and Offshore Hydrodynamics, in lecture 14. Today, we are going to discuss a very important concept which is Green's function.

(Refer Slide Time: 00:26)



And this is the keyword that you are going to use to get this lecture.

(Refer Slide Time: 00:31)



Now, what is the boundary value problem of the Green's function method? Now, in the last class, I discussed that that typically this boundary integral method is looks like this. And we have derived that ϕ, any point of the phi can be explained in the distribution of the singularities over the surface.

Now, here, the same thing that G is nothing but that 1/r that we are going to take in the last class however, in our case, like our problem also not very simple. Now, 1/r is very good if you take like a find out that the pressure of a body, absence of free surface. And nowadays people are also taking 1/r, which is popularly known as Rankine panel method, they try to find out the pressure of a body over the surface everything considering the G= 1/r.

However, there are other methodologies also where instead of 1/r they are taking 1/r plus some other functions. So, we are going to discuss all these things in future. So, that is why instead of 1/r, I am writing a general function G, ok.

So, here you can see that this integral equation, you can call this a integral equation, it is $\alpha(P)$ ϕ(P) equal to integral over the surfaces, then ϕ(Q), ∂ G, ∂ n, P, Q, t etcetera.

Now, we need to understand what is the all quantities over here. Now, this $\alpha(P)$ basically you can call this as a solid angle. Now, what is the solid angle? We are going to discuss later on. Here we are just going to discuss about the two things one is P and another is one Q.

Now, what is the P? P is the point anywhere in the fluid domain, ok. However, Q is the point on the surface. Now, you can see over here that any point on the fluid (Refer Time: 02:54) the velocity potential at any point on the fluid domain can be determined if I know the velocity potential over the surface, right. Also, the normal that del phi, del n also on the surface. So, that is so, you can see over the integral this is basically where going through the Q, not P.

So, you can understand that P is the point on a fluid domain; however, Q is the point on the surface. So, therefore, in order to get; the interesting part is that that in order to find out the velocity potential at any point on the fluid domain we really no need to discretize the entire fluid domain, right. We have to discretize the surface. So, that is a very interesting thing, right, ok.

But here, you can see that there are two things that actually we need to find out. The first thing that what would be the G, G is the, as I said the Green's function then, but what is the value we have to know the value of G to solve this problem, right. This is one thing. And second thing we need to know what is the value of φ at the surface, right. And also what is the value of $\frac{\partial \phi}{\partial n}$ on the surface.

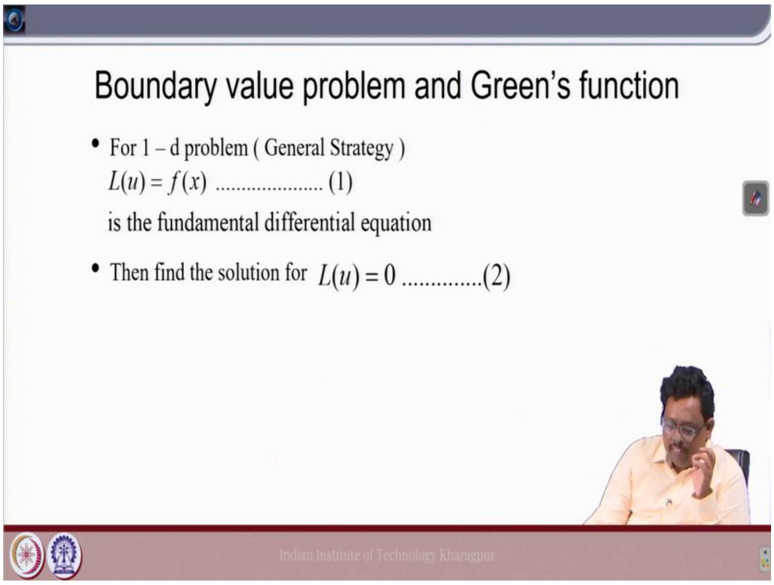
But you know I know from my boundary condition, if you remember my previous classes we discussed $\frac{\partial \phi}{\partial n}$ should be equal to the v n over the body, right. So, and then we have the kinematic free surface conditions. So, therefore, somehow I understand del φ del n also known to me. Definitely, we are going to discuss everything later on. But now here just I am just to give you some kind of feeling, ok.

And then major part now boils down to how to find out the φ at Q, this is the number 1 problem. And number 2 problem is how I can get the G. So, today, actually let us try to find out how we can get the value for G. And it is extremely complicated thing, but you know just to give you some kind of feeling, some essence like how we get the value for

G, we are taking a very elementary problem where that finding out the Green's function is easy.

However, in our class of problem this G is already known to me. However, getting G is one problem, that we suppose we have the G still it is very difficult to solve this integral equation. So, in coming days numerically how we are solving this, we are going to discuss extensively, right. Now, let us try to find out how we can get the Green's function, how we can find out the G for simplified problem, ok.

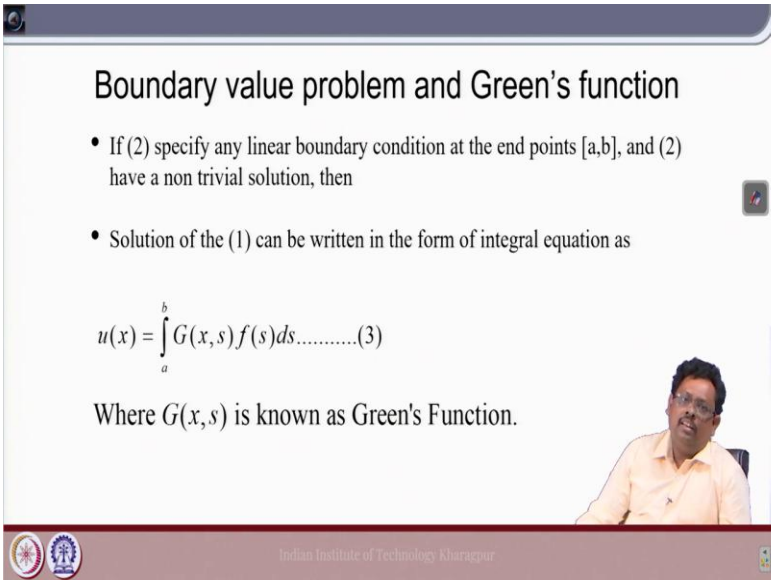
(Refer Slide Time: 06:12)



So, now let us take one-dimensional case; in that is the general strategy to find out the G. So, first thing let us we have a differential equation L(u)=f (x). Now, L we called as a linear operator. Now, what is the definition of linear operator? Let us not go into this, but you know accept that that Laplacian is a linear operator. So, $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$ that is a that operator is a linear operator; that means, ∂^2 operator is a linear operator, ok.

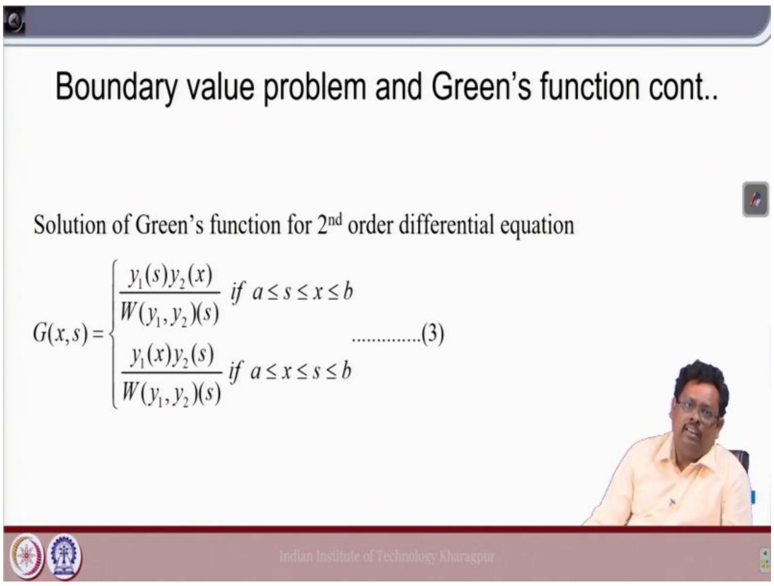
Now, suppose L(u)=f (x), if this is your differential equation, so then the solution, first you find out the solution that L(u)=0 which is called the homogeneous problem, ok.

(Refer Slide Time: 07:16)



So, once you know this L(u)=0. So, then actually your Green's function solution can be given as this expression. So, see, thing is that, you can find out a function, Green's function if L(u)=0, then you can find out some function, and then I mean we can call this a trial function. And then actually, we can manipulate this trial function and we can obtain the Green's function. And once you know the Green's function, from direct integration we can find out the solution. So, this is the strategy, ok.

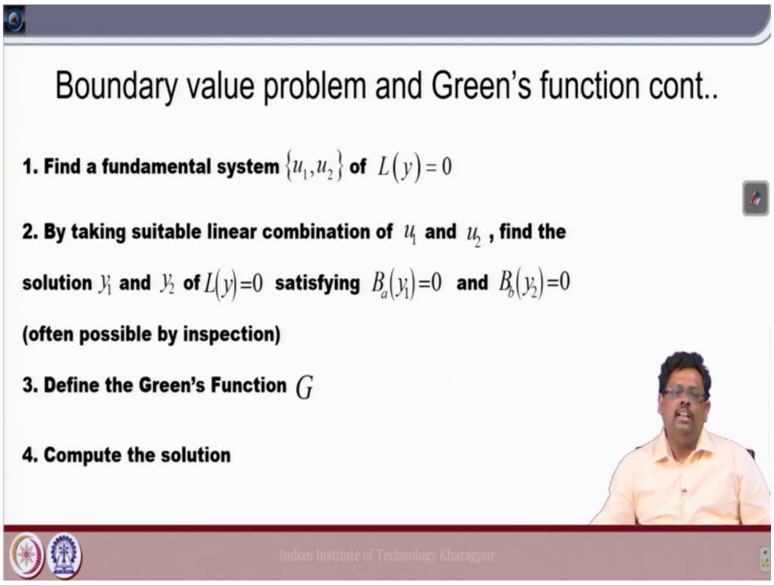
(Refer Slide Time: 07:57)



Now, this Green's function for a second order differential equation can be found out from this equation. Now, you can see it is very format that was, right. You have some what is y_1(s), what is y_2(x), what is W, nothing is known to me, right. Also, but I know that that boundary is from a to b because it is a one-dimensional problem, it is a length. So, limit is from a to b. But I really do not understand what is the s, what is the x, etcetera right.

So, we will get to know by solving this problem, ok.

(Refer Slide Time: 08:37)



So, now let us find out the general strategy. The first, we have to find out the solution u_1 and u_2 for L(y)=0. So, that means, the differential equation anything, any linear differential equation; if you make the right hand side equal to 0 and if you try to solve this problem, if you able to solve this problem then you can get two solution u_1 and u_2.

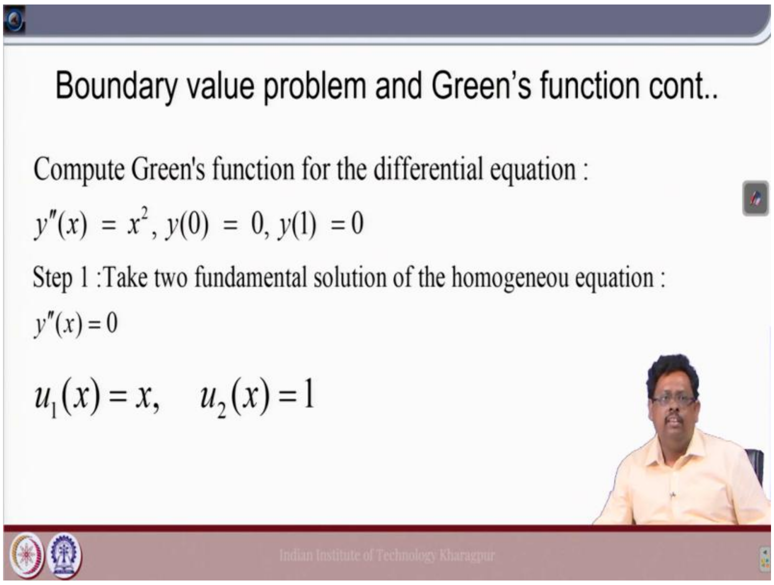
Then, we have to next step. What we need to do is suppose I have two solutions, if it a second order differential equation for example, like your $\ddot{y} + y = 0$, right, if this is a classical second order equation and you have the solution is sin t and cos t, right. So, $\frac{\partial^2 y}{\partial t^2} + y = 0$, the solution is sin t cos t.

So, you know that if you have a second degree differential equation, definitely you can have two solutions. So, here also we have two solution, let us see u_1 and u_2. And then, we need to find out another solution y_1, which satisfy the first boundary condition and again we are finding out the another function y_2, which satisfying the second boundary condition.

And actually this, actually we are going by the trial method, ok. I mean there is lot of other things, but for the simplified problem we can find out using the trial method. Now, once we find out this y_1 and y_2, then you know that from my this expression we can find out the Green's function G.

So, this is the strategy. I have a differential equation. So, I make the right hand side equal to 0. And then, I find out true trial solution u_1 and u_2, and from that I can find out y_1 and y_2, y_1 satisfying the first boundary condition and y_2 satisfying the second boundary condition. And once I find out this y_1 and y_2, definitely I know what would be my Green's function. So, this is the idea, ok.

(Refer Slide Time: 10:57)

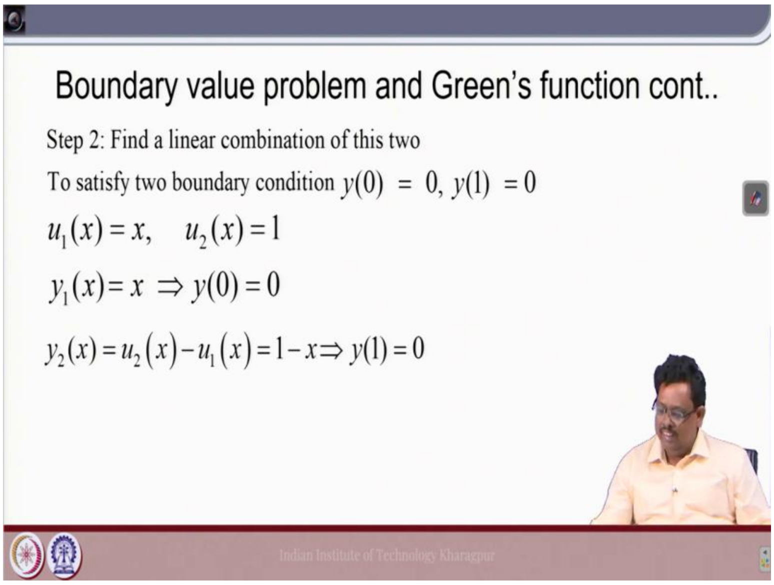


Now, let us take one problem. Now, you see here this problem is very simple, $y''(x) = x^2$. You really do not have to do all this Green's function technique to get this solution, right. Well, now let us go also, I mean this boundary conditions given is y(0)= 0 and y_1= 0. So, then what is my first objective to find a trial solution $y''(x) = 0$.

Then, what would be the solution? You can see like by you know, you can get by trial you can get many solution, right. So, we have to have two solution, u_1 and u_2. So, I take that first solution u_1= x and second solution u_2 equal to; as I said that I do it you know trial method by anticipating this is the solution, right. So, of course, this is the solution if you differentiate u_1 two times plus 0, you differentiate u_2 two times it is 0.

So, my next job is find out y_1, one function with the linear combination of u_1 and u_2 such that y_1 satisfy the first boundary condition which is y(0) = 0. And similarly, with the linear combination of u_1 and u_2, we have to find out another solution y_2, ok such that this second boundary condition also satisfies which is y_1= 0, ok.

(Refer Slide Time: 12:33)

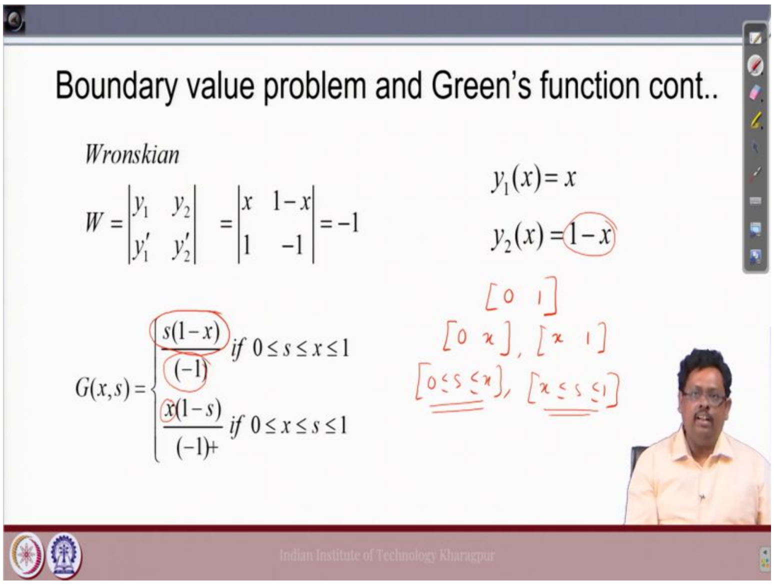


Let us find out. Now, I have u_1= x and then I have u_2= 1. So, then I can find out y_1= x and then that satisfy by the first boundary condition y(0) = 0, right. Because you may put here 0, then it is 0, so y = 0.

Now, what would be the u (y_2)? Now, y_2 again you can see that y_2 also find out in the linear combination of the u_1 and u_2, which is u_2-u_1 that is 1-x. Now, you see this y_2 is now satisfying my second boundary condition, right. So, u_2(x) is 1-x, you put x=1, then y_2=0. So, therefore, now I am ready. I have my y_1 which is x and I am having the y_2 which is 1-x.

Now, how do I write the Green's function for this particular problem? Ok.

(Refer Slide Time: 13:45)



Now, this two my solution, y_1(x) and y_2=1-x; now, first I need to find out the W, right remember and this W is nothing but the Wronskian. And this is the formula to get the Wronskian which is y_1, y_2, y_1 dash, y_2 dash. So, it is x, 1-x, if you differentiate this you will get 1, and if you differentiate the second term, we will get minus 1 and then finally, we get the value is minus 1. So, I get W equal to minus 1.

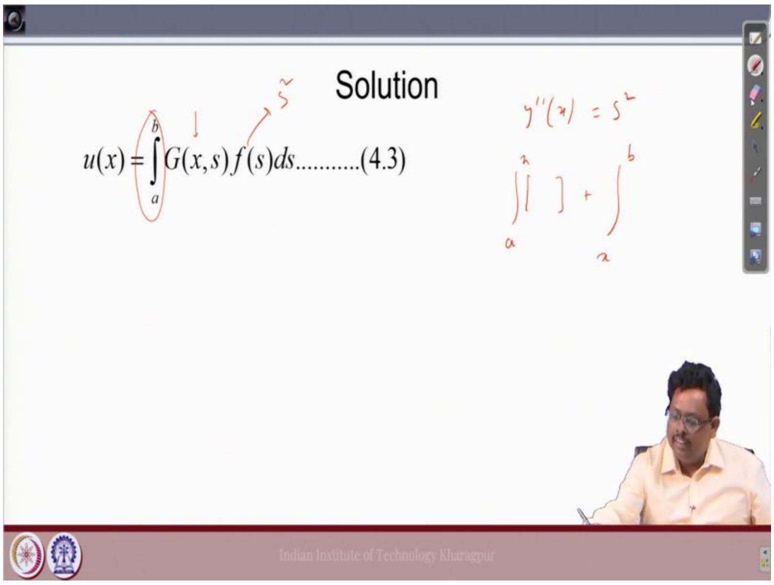
Now, the question is how do I write the Green's function, ok. Now, let me explain this carefully. Now, here this 0 to 1, I split in between there is a two point, one is a variable point s, we call s, s is a variable point, and finally, you know the x should be another variable, but at this particular moment I can consider this as a fixed point between 0 to 1 so, therefore, and this s this variable s running from 0 to 1, ok.

Now, here in the first time when this s, let us say x as I said that x right now is a fixed point between 0 to 1, so therefore, this domain 0 to 1 now I can split into two different domains, right. Now, this domain 0 to 1, so this 0 to 1, I can split into two sub domain one is 0 to x, and second domain is x to 1. And then I can tell that this is firstly, vary over 0 less than s, less than x, and in the second case this s is varies along x to 1.

So, once it is for the first case, so then I replace this y_1 by the variable point s and I keep y_2 as it is, ok. So, that is why we call this function is s into 1-x. And I need to divide by the Wronskian. So, I divide it. So, I get it is minus 1. In case of a second, in case of a sorry in case of this second interval, I keep this y_1 with the x is a fixed one, and then I vary this one, y_2 as 1-s. So, I replace x by s in the next domain and of course, I divided by the Wronskian minus 1. So, once we do that, so once we do that we will get the expression for the Green's function, clear, ok.

Now, once we get this Green's function then remaining part is elementary.

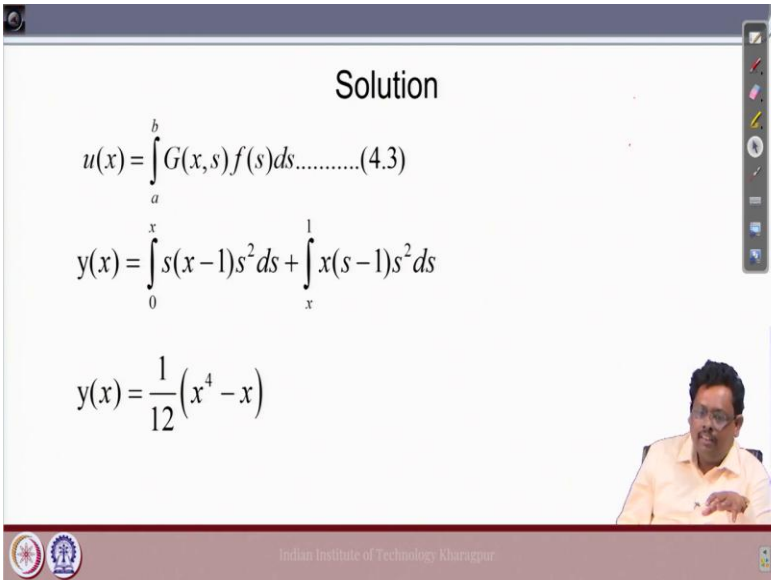
(Refer Slide Time: 17:08)



So, remaining part is as follows. This is your solution, $u(x)=\int_a^b G(x,s)f(s)ds.$. Now, in our case, my problem is what? My problem is y double dash x is equal to s square. So, this f s is nothing but my s square and this Green's function actually I can write this Green's function; of course, this a to b. Now, actually I split it into two domains, one is integration of a to x, this is the first domain, and then second domain integration of x to b, and I perform the integration, ok.

So, if I do this, if I do this, let us see what we can get.

(Refer Slide Time: 18:01)

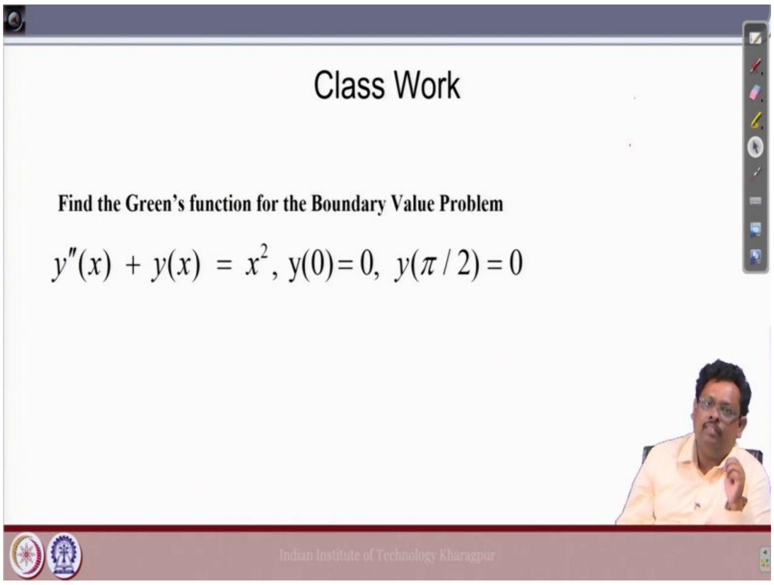


So, now you can see that I have now in my case it is 0 to 1, so I have 0 to x. And then this is s and this is the first Green's function, s (x -1) s^2. And then in between x to 1, I am using the second part of the Green's function which is x (s -1) s^2. Now, you see that this is the solution for x, right.

Now, if you solve it, you will get $y(x)=\frac{1}{12}(x^4-x)$. You see that you know it is actually very simple. Now, if; now what I get is from here the thing I am getting is, if my solution, I mean this the differential equation is simple and if the boundary condition also simple, so I can really find out the Green's function. So, it is not that difficult to find out the Green's function, let us say. And then the solution again become very simple.

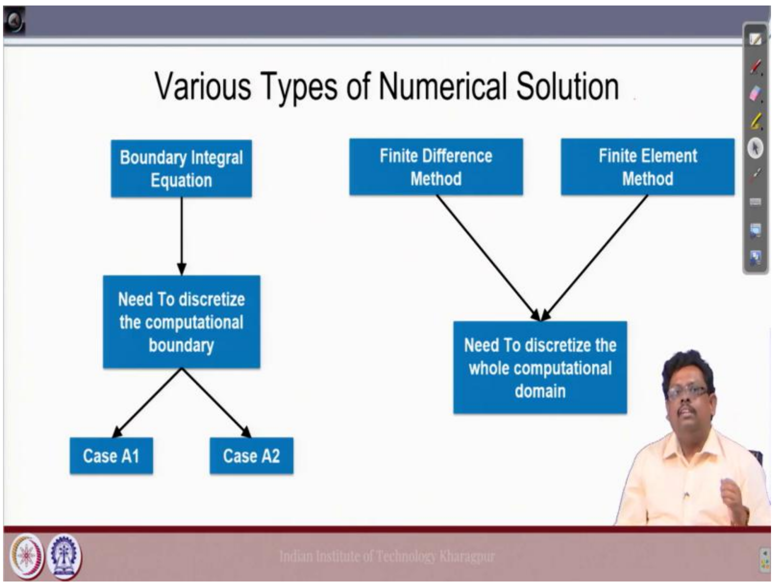
So, this exercise will tell you for a simple problem how I can find the Green's function, right.

(Refer Slide Time: 19:19)



So, now this is the class work for you. You try to solve this problem using the boundary using the Green's function; first you find out the Green's function and then you solve this problem, ok.

(Refer Slide Time: 19:36)



So, now, so till now we actually solve one thing, we get one integral equation which have a part which is called the Green's function and then some part called the velocity potential. So, I know right now that I have two unknowns, one is the Green's function and second one is the velocity potential over the surface, right. I am hammering that part over the surface, right because this gives us the enormous advantage over the you know other type of numerical technique.

Now, let us see that what other type of numerical techniques are available to solve this you know ship structure, I mean ship wave interaction problem or fluid stat interaction problem. So, we are now discussing with the bound integral equation which is here, ok.

However, we can use the finite difference method also or we can use the finite element method also. These two are also very popular method, specially, finite element method for the structural analysis is very popular. And then typical, you know to find out the any classical CFD problem, we are using the finite difference method or finite volume method. So, these methods are very popular of course.

Now, the difference between the boundary integral equation method or this finite difference method or finite element method here the domain discretization. Now, you see here in case of a finite difference method or finite element method, we need to discretize the whole computational domain.

I must tell you this boundary integral equation is popular only, especially in this you know this hydrodynamic numerical ship, hydrodynamics this domain, and also in very popular to solve this fluid problem I mean if you consider this potential theory. But it is not very popular to solve you know that generic problem.

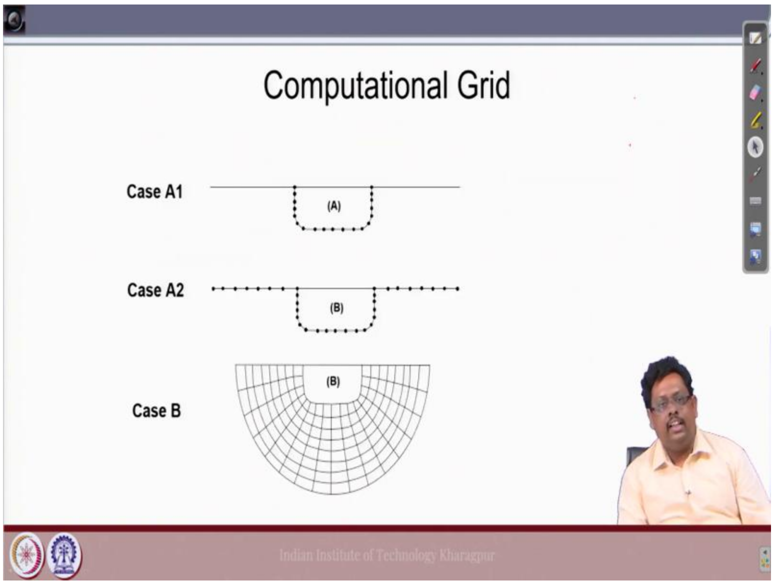
Mostly, people use the finite difference method, and finite element method, or finite volume method to handle all sort of you know fluid structure problem or fluid stage interaction problem definitely. However, we prefer boundary integral equation method over this. Why? Because the main problem is the domain discretization. Here our entire fluid domain is basically the ocean.

Now, this discretize the ocean surface or the total volume, it is not easy. And it is not you know worthy also. So, therefore, if you use the finite difference or finite element method, you have to discretize the whole computational domain, right. And this is you know in fact, all these studies started during the 70s or 80s that time, we really cannot think of discretize the whole ocean using this some finite element grid or finite difference grid.

However, in case of a boundary integral method, you only need to discretize the computational boundary. Remember, I said the velocity potential at any point in the fluid domain can be found if we solve the velocity potential that is distributed over the surface. So, therefore, in case of a boundary element we only need to discretize the boundary.

Now, you can see here we have the case A1 and case A2. So, let us see what is the case A1 and what is the case A2.

(Refer Slide Time: 23:28)



Now, here you can see the computational grid for the case A1, which basically I am only discretizing the body. See, now, this entire the fluid domain it is only become the you know the body, you see. I really do not have to discretize the whole fluid domain. Even I do not have to discretize the free surface also. Only I need to discretize the body. So, the mesh is only for my body.

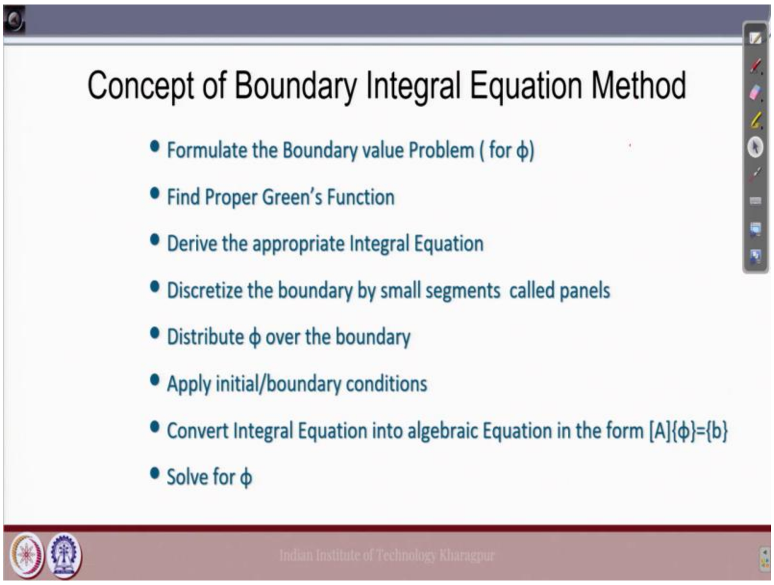
Now, you can see the computational advantage over the finite difference and the finite element where you need to discretize the entire domain. Now, as I see in the case B, you can see here. I am discrete; it is a two-dimensional problem of course; I am discretizing the whole fluid domain.

But in case of case A2, ok in case of A2, actually I only discretize the body along with the free surface. So, here of course the discretization is more compared to A1, right.

However, still it is much lesser than the case B. So, this is actually very advantageous to use the boundary element over the finite element and finite difference. And second point, mostly this boundary element it is started after some kind of you know mathematical analysis. We discretize things, and we are getting that computational domain, and we are finding the results.

However, this finite difference scheme or you know finite volume scheme, we started with the Navier-Stokes equation and started discretized at from that particular point. So, therefore, normally it is we can we are we observe that this boundary elements somehow numerically more stable comparison to the finite difference or finite volume method, ok.

(Refer Slide Time: 25:37)



Now, what is the concept of the boundary integral method? Right now, I am just listed out this the strategy. We really today maybe we are not getting everything what we are listing out, maybe in coming future again slowly I discuss the all the points, ok. So, just listed out those point.

The first you have to formulate the boundary value problem for φ, which is the $\nabla^2\phi = 0$ with the boundary conditions. So, once you find that then you have to find out the proper Green's function.

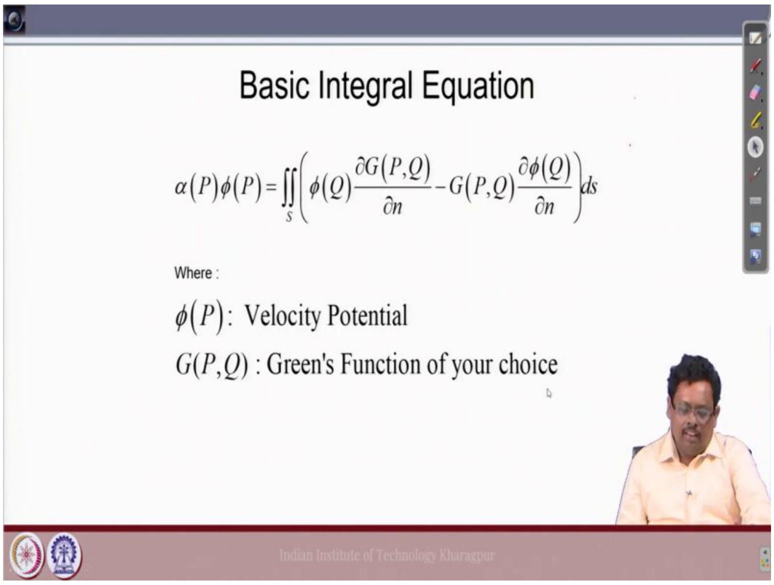
Now, right now we just find out the Green's function for the simple problem, but for this Laplacian also you need to find out the Green's function and this is difficult. And actually, we really not you know finding out the Green's function, rather we are using some available Green's function.

Then, derive the appropriate integral equation. I will come into this point later, when you say that derive the appropriate integral equation. Next point is that discretized the boundary. Now is meshing, I am talking about that. We need to mesh the surface, not the not the whole volume, but the surface, ok.

And then we have to distribute the φ over the boundary. What is the meaning of the distribution of the φ over the boundary? Definitely, we are going to discuss in future lectures. So, right now just I am listing out here that what is our job, ok. Next one, this is the apply this boundary conditions. Of course, we are going to apply the boundary conditions. And then this convert this integral equation into the algebraic form.

Now, this is when you are solving some numerical method, either it is a differential equation or its integral equation. It has to finally, a system of linear equation and from this system linear equation, we are going to φ, and once I get the φ we can find out this pressure, the force everything. So, this is the concept of boundary integral equation method or panel method. This is the steps that we are discussing; when. From the next class onwards maybe we are every point, one by one we are going to discuss, ok.

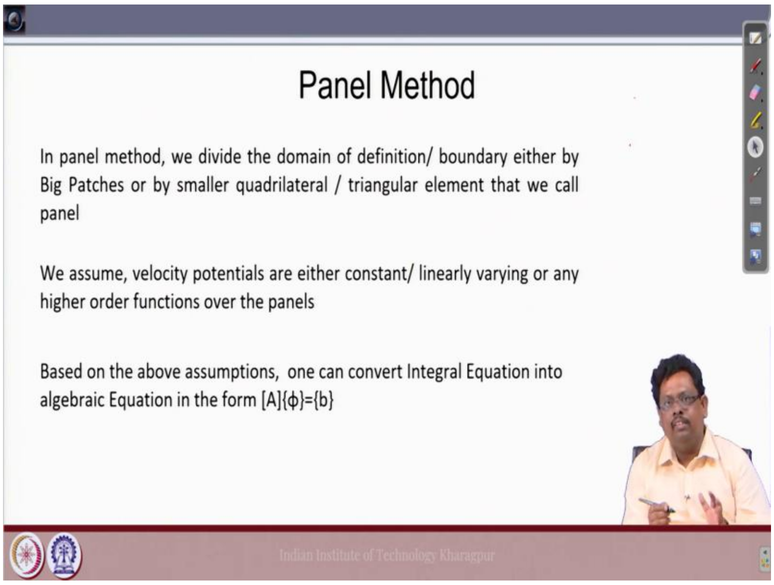
(Refer Slide Time: 28:04)



So, this is the basic boundary integral equation. Again, I am showing you, right. Here this Green's function G is my choice and then we have to; once I choice I make my choice of the Green's function where the G is known to me. I can find out that φ that is distributed over the surface, ok.

What is $\alpha(P)$? We have not discussed today. We are going to discuss later on, ok.

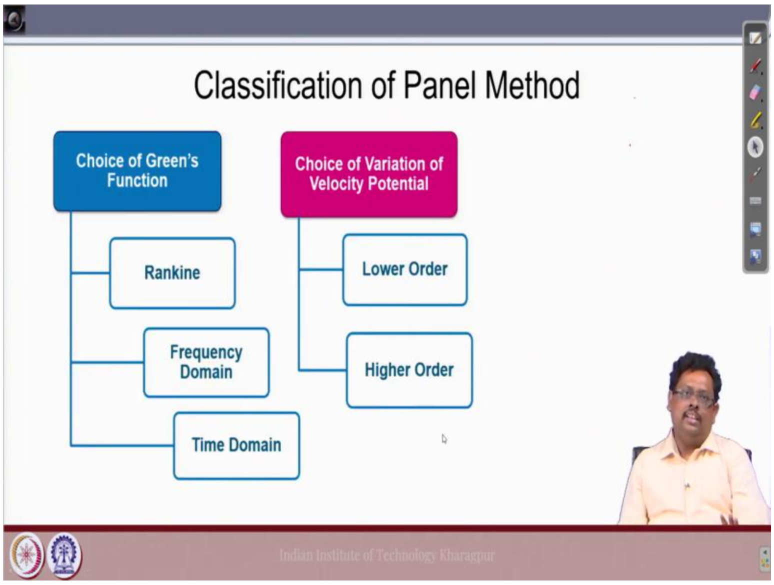
(Refer Slide Time: 28:34)



Now, so basic strategy for the panel method is followed. That, in panel method, we divide that domain of definition with some quadrilateral or triangular panel that means, that ship, that ship actually I discretized with number of quadrilateral or triangular element, ok. And we call that as a panel.

And then, actually we assume that velocity potential either constant or varying over this panel. And then, when I distribute the velocity potential, I apply the Green's function, Green's identity and solve $[A]\{\phi\} = \{b\}$, and get the solution for the φ. Again, at present, this is I know it is abstract. But in coming class we are going to slowly we are going to find out how I discretized that domain, how I get the solution for φ.

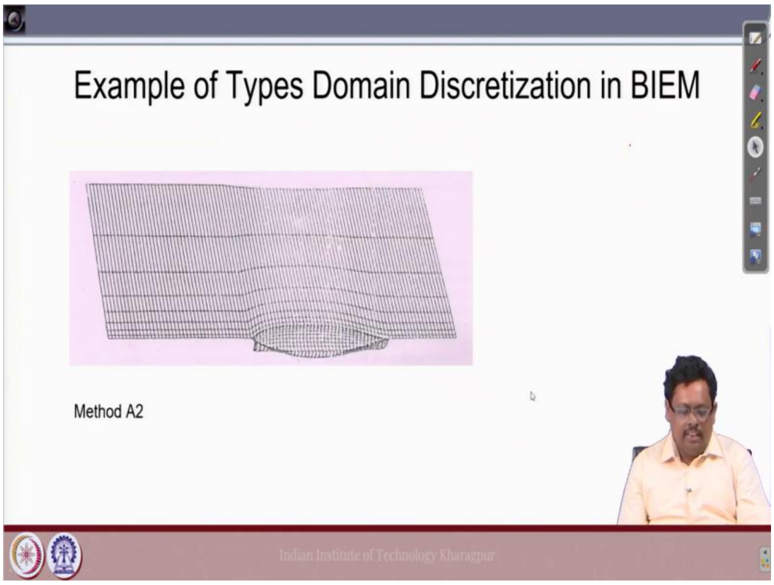
(Refer Slide Time: 29:34)



Now, this is the classical slides like that the choice of the Green's function, this panel method can be you know 3 types, you can find out. One is the Rankine panel method, one is frequency domain panel method, and one is time domain panel method. And in choice of the velocity potential, it can be lower order and higher order.

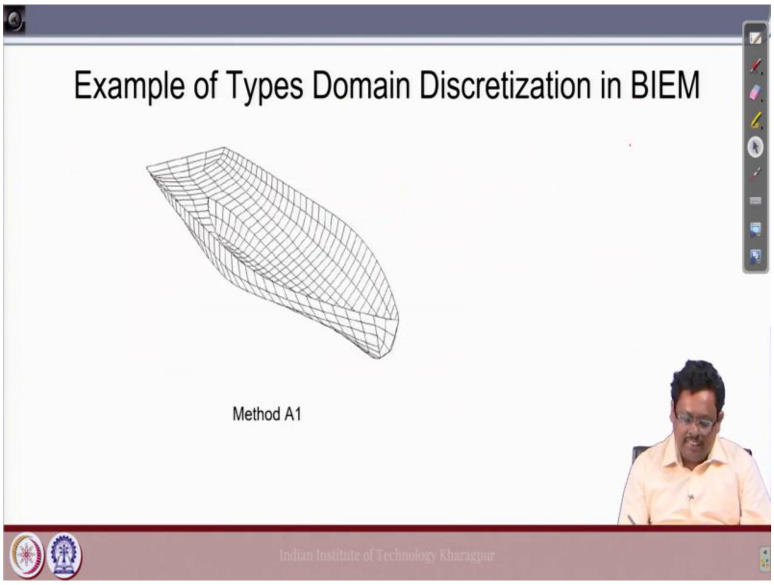
So, all this different method actually we are going to discuss again our coming classes, ok. Just, you know you can see that Green's function there are 3 choices. In first choice, you can have Rankine, in the second choice you have the frequency domain in the third choice you have the time domain, ok.

(Refer Slide Time: 30:18)



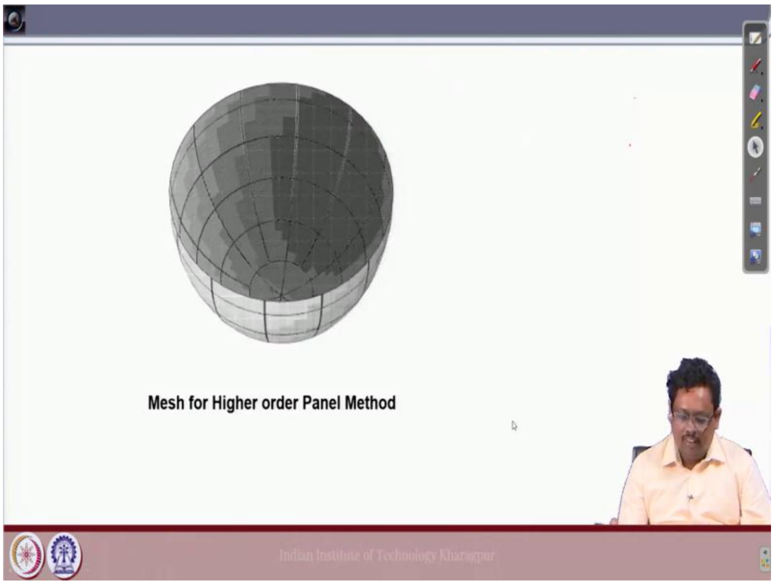
Now, this is basically the method A 2, which is the Rankine panel method. And this is the computational grid for that, ok.

(Refer Slide Time: 30:26)



Now, you see the method A 1 and this is the computational grid for that, ok.

(Refer Slide Time: 30:32)



And this is the grid for the higher order panel method. You can see that this is how we can discretize the surface. Now, in case of higher order, we have the big curvilinear patches. In case of other two, we are having the quadrilateral patches, right ok; some more exciting things coming of course, in future classes.

So, till then thank you.