Foundations of Classical Electrodynamics
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Lecture - 10
Divergence operator, Divergence Theorem

Hello students to the foundation of classical electrodynamics course. So, today we will be going to continue the divergence operator and the related theorem that we started in the last class.

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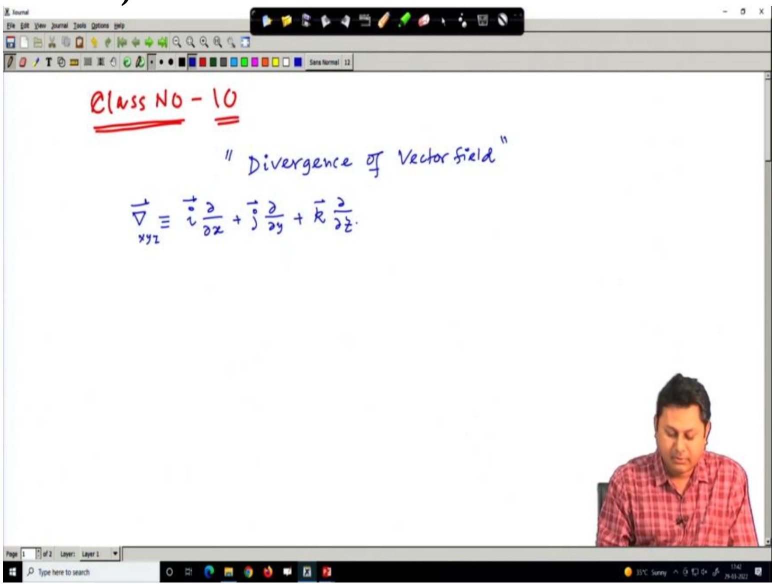

So, today we have class number 10. So, in the last class, we started the divergence of vector field. So, we define this operator, which we call the del operator ($\vec{\nabla}$) and this is a vector operator in Cartesian coordinate system xyz it should have this form.

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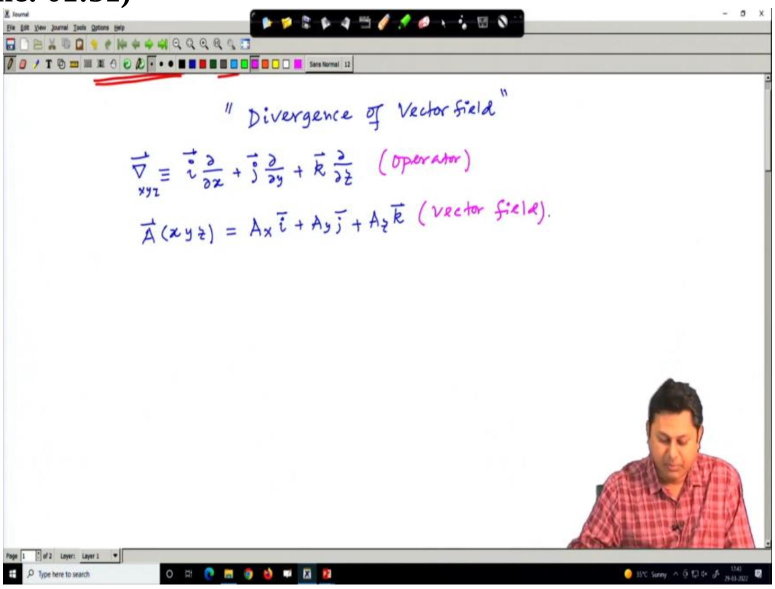

And if it operates over a vector field suppose now, I am having a vector field here A, function of x y z. In component wise I can divide it like A_x i + A_y j + A_z k. So, this quantity is an operator and it will be going to operate over the vector field.

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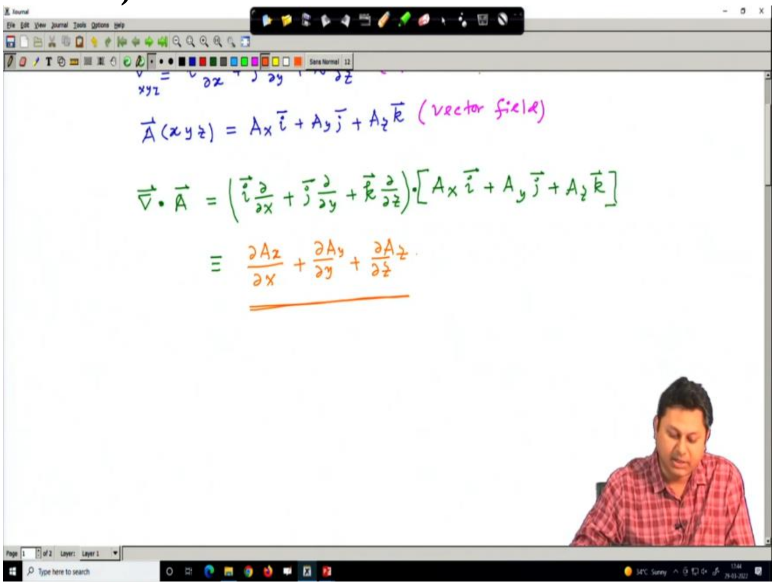

And the operation, the mode of operation is something like this like a dot product. I am having this operator and I am making this over A. So, eventually I am having i $\frac{\partial}{\partial x}$ + j $\frac{\partial}{\partial y}$ + k $\frac{\partial}{\partial z}$, this is the operator and it will be going to operate over the entire vector field A. If I write in component form it should be simply this. Now, here we should have a dot operation such that the output look like this.

So, finally, we will get this will operate over A_x, $\frac{\partial}{\partial y}$ we are going to operate over A_y and $\frac{\partial}{\partial z}$ going to operate over A_z. So, this is the recipe of the operation. Well, that thing is fine. Now, we try to understand that this is the mathematical form how the divergence operator can operate over a vector field A. Now try to understand what is the physical meaning or the geometrical interpretation of this operator.

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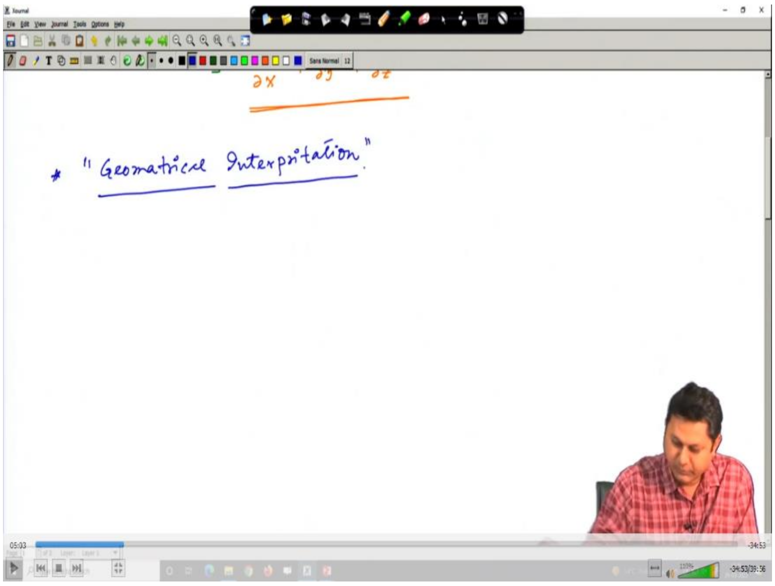

So, the next thing is geometrical interpretation.

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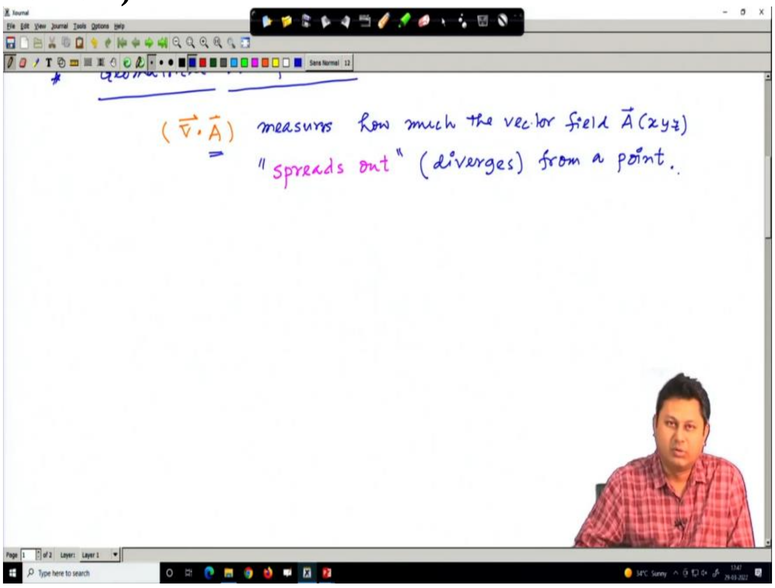

So, this operator basically measures, so this quantity measures how much the vector whatever the vector I am having here, this vector A or the vector A, how much this vector A, vector rather vector field A, a function of x y z or ρ φ z it for the time let us consider this is in Cartesian coordinate system we are doing that so, it basically measures how much this vector field A spreads out or in other way diverges from a point.

So, it basically this is a measurement, this operator basically measures if a vector field A is given to you how much it spreads out and or the diverges from a given point. So, let us try to do a few examples and then it will be clear and also I like to show few typical vector fields then things will be much easier to understand pictorially and conceptually.

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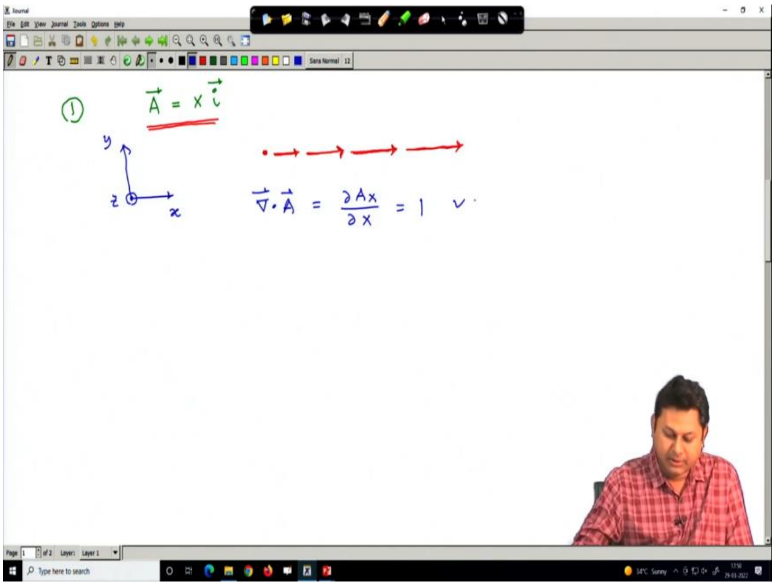

So, case 1, so, now, I am making a vector field here. Suppose, my vector field is the simplest one A = x i, this is the vector field that is given. Only 1 component is there. Now, if you plot this vector field in the xy coordinate system. So, let me draw the xy coordinate system first, because this is a 1-dimensional vector, but let me plot it in at least in 2-dimension.

So, this is x, this is y and this perpendicular z. So, it will now in this coordinate system if I plot this vector field, so, this vector field will be something like that, suppose this is the point and over this x direction I have this line, then I have this line, then I have this line and I have this line, you can see that the line is increasing here. Why it is increasing? Because with the increment of x; this vector, magnitude of this vector is going to increase.

So, first, whatever the value of x for example, I am taking x = 1, so, I am having a unit vector along the direction i, then the value of x is 2 I am having the unit vector, a vector, which is having a magnitude twice the previous one and so on. So, it will be going to evolve like this over x direction, if I put -x it should go in this direction. So, this is the way it is changing. And the form of the vector field is this one.

If you plot it, you are going to get this value. Now, the question is, is it diverge? And the answer is yes, because for a given point, you can see that the length of this vector is changing. So, there is a feeling that this thing is diverge. I can also check this by applying this operator over that. So, this quantity, if I now apply over A, which is given like this. I will simply get $\frac{\partial A_x}{\partial x}$, only 1 component will be there and I will get the value 1 and this is a non-zero value.

That means, this vector field is having a spread out or divergence and that divergence I measure with this. Next example.

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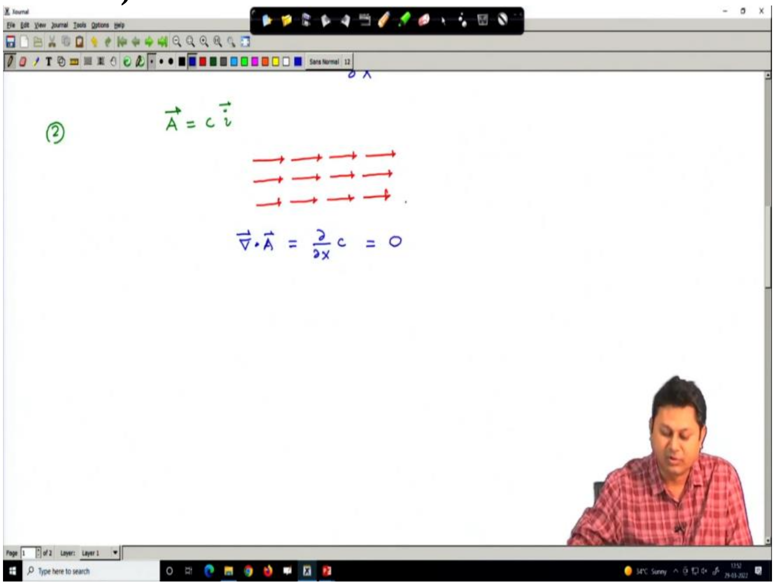

So, this is example 2 we find it I am having a vector field and it is gradually increasing along x direction or also -x direction is going this side, but if I want to find out what is the divergence what is the spread out then this value is a scalar value and this value is 1 here. Now, another example I like to give that A constant vector if A is given but this is a constant vector, some constant multiplied by i.

If that is the case you can see this is not depending on the x or y. So, in 2-dimension if I want to plot this the vector field look like a constant line. So, whatever the value of the c magnitude that should be the magnitude of this all the line will be of same length and I am having something like this a uniform flow of something for example now, the question is, is this vector whatever I depicted so, is it spreading out?

And answer is no, it is not spreading out. It is a constant, it is moving and also if I calculate the divergence of this given vector A, I will find this $\frac{\partial}{\partial x}$ over a constant and that is 0. So, this is 0 divergence that means, it is not spreading out.

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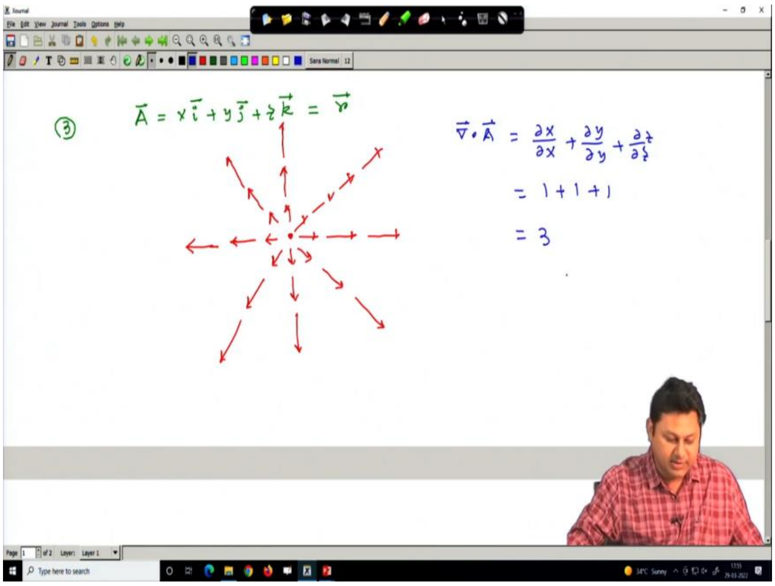

What about a more well-known vector field? The vector field I am giving now A, which is x i + y j + z k, now, this is a 3-dimensional vector field I am giving. This quantity is nothing but vector r, a very important vector field indeed, vector r, the simplest 3-dimensional vector field one can imagine where xyz both all components are there and, in that case, this if I now draw this vector field.

How this vector field will look like, in 3-dimensions it is difficult to draw, but I would like to show you a figure later along x direction it is increasing, along y direction also these lines will be going to increase, -x also it is increasing, -y it is also increasing, along this direction it is also increasing, because it is a 3-dimensional and I can have all the directions, in this direction, in this direction, so, overall it is increasing like that.

So, from the figure it is quite obvious that this is really spreading out. So, if I calculate also, we will see that it is indeed giving a non-zero value. If I calculate this value is $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$, which is simply 1 + 1 + 1 = 3, which is the higher value that we get that we get in the first example. So, here we can see that it is also diverging like the previous one the first one example 1.

But it is diverging in a more rapid manner, the spread out in a more, higher value with higher value. You can see previously it is 1 and now it is 3. This is the way you can understand the divergence, but now, I like to show you some typical vector field.

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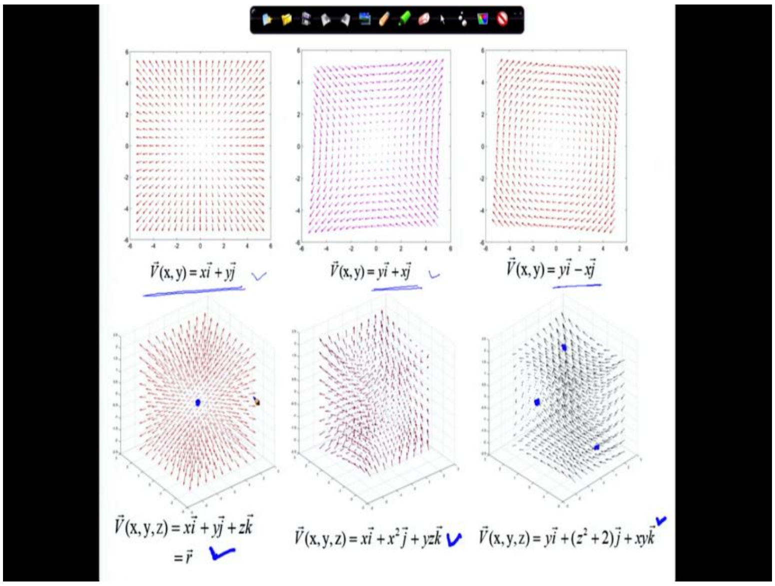

So, that you can have an idea that if somebody is saying it is diverging, so, how it looks like in the upper panel you can see that I just plot into 2-dimension I just plot the vector field and below I just wrote down this expression of this vector field V (x, y), this is a function of x y. So, it is a, I just if I add plus z k then it should be r, but forget about the z component and you can see that how this vector field is spreading here.

So, obviously, we should have a divergence here without doing anything we can say that, but there are other kind of vector fields also and where it is difficult to measure whether this is spread out or not you need to calculate it by using this operator. So, here we have another kind of vector field where the vector field the value of this vector field or the expression of the vector field is this, y i + x j and as soon as I change this coefficient.

Now, the V_x is y and V_y is x. So, you can see that how this changing the vector field is completely different, it shows that each and every point we have a vector and the direction of the vector is mentioned. So, that is why it is forming a vector field here 2-dimension in order to understand this, we make it 2-dimensional. Here it is a very interesting kind of vector field third one, which is (y i – x j).

Now, if you calculate the divergence of this quantity, you will find 0 also, even though we have non-zero x and non-zero y, this is not a constant, but still it is not diverging rather you can see that there is some sort of rotation is going on a very interesting thing, we will find later that another operator curl is there, if you want to find out how curly a vector field or how a vector field is spiral.

Then this value, this vector field will be coming non-zero. But here you can see that it is a spiral kind of vector field, it is just rotating and if somebody wants to find out that how divergence it is whether it is spreading out or not, obviously, it is not spreading out, because the vector field is something like this. In the lower panel, more general, you know the representation of the vector field I try to plot.

So, here you can see that the first plot is the vector field r I already mentioned that this vector field is very important one of the important vector field one can have this one r. And here in the 3-dimensional picture, it is clear that how it is spreading out form a given point. So, if I have 0 0 point here, so, this is the 0 0 0-point origin and at the origin the vector, value of the vector is 0, it is a null vector in the origin.

And then if I increase the value x y and z, then you can see that how it is changing, it is evolving it seems like that it is coming out from a point it is going outside. So, obviously, this vector should have a divergence and we already checked that this divergence value of this divergence is 3. However, we can also consider a typical kind of different kind of vector field as per your choice.

Here you can see that we have a different kind of vector field with x i + more complicated one and also the vector field looks very complicated, at each and every point in the space, this picture shows that at each and every point of the space, how the vector will look like and this is the magnitude and the direction is also plotted here simultaneously and that is why it looks like this.

Also, here, another complicated kind of vector field you can see where y i + (z^2 + 2) j + xy k, this is the form of the vector field and you can see that some sort of flow is going on here from one point to another point in some certain directions. So, also it is interest to find out whether it is having A divergence for a given point or not when you calculate the divergence you sometimes you will see that the divergence itself is a function of x y z.

It has to be, because, you want to find out the divergence of certain points you need to put the point and check the value whether this divergence is a function of xyz or constant for example, in this case, in the first case, here, the divergence is constant it is diverged with a uniform value and this value is 3, on the other hand here in these 2 examples, it depends on which point you are talking about in order to find the divergence.

If you are having this point, you have 1 divergence, if you have this point you have another divergence, if you have these points, so, it depends on the value of the xyz and there is a possibility that at some point the divergence is 0. So, these values these vector fields this is the way one can appreciate that how the vector fields are distributed over space.

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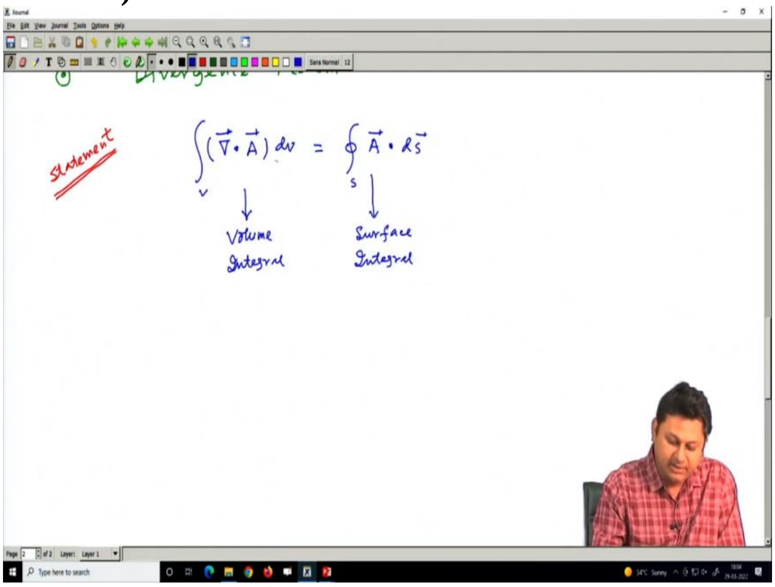

So, the next important thing that I would like to discuss is this very important thing that is the divergence theorem. So, the statement of the divergence theorem is so, this is a statement I will prove that, but let us try to understand the statement. The mathematical statement suggests that if I have a divergence of a given vector field and want to find out the volume integral of this quantity.

So, first a vector field is given I make a divergence over so I will be going to get a scalar quantity a scalar field and then I want to find out the volume integral of that scalar field, which we did in earlier classes this is a volume integral. So, that value should be exactly identical if I have a vector, the same vector field and if I perform a close surface integral of that vector field such that this surface is closed the volume we took.

So, this is the volume integral left-hand side and this is a surface integral and the point is this surface whatever the because this is a closed surface, so, this is a closed surface integral and this close surface should be such that it encompasses this given volume. So, this is the mathematical statement.

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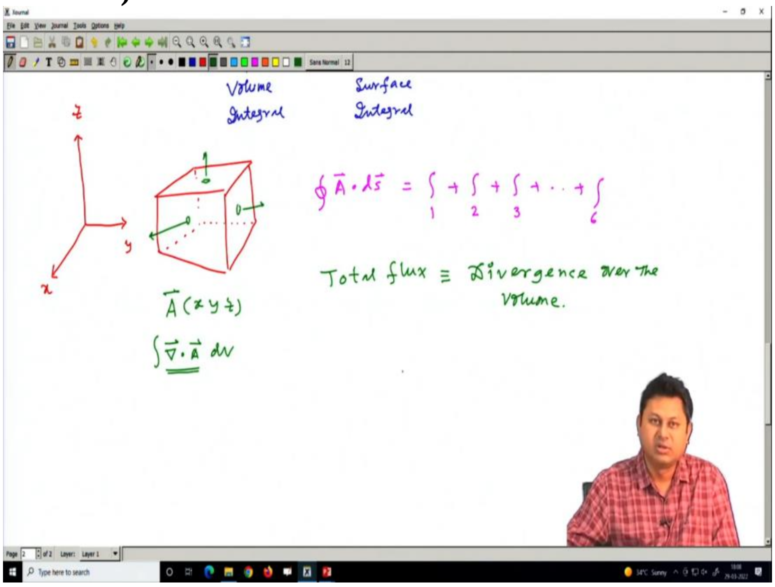

And if I want to find out these things before that if I want to show you how it looks like. Suppose this is a volume element that is given and it should have 6 surfaces in Cartesian coordinate system we considered this volume element so, I can have x, y and z coordinate here x y and z coordinate and a volume element is given. So, I can and also a vector field is given in Cartesian coordinate system.

This is x y and z function of, so, it can have different values as I just before I show you that how the vector field is changing. Now, if I want to calculate the volume integral of the quantity this, so, this basically gives you the divergence of the given vector, and then you make the volume integral over this given volume, you are going to get a value, but you can also make the surface integral of this enclosed volume.

So, there will be 6 surfaces these are the direction of this area and you can have a close volume integral, closed surface integral for this system as well. So, this is the volume integral you calculate one hand and also for this given volume you have a closed surface integral and this closed surface is nothing but you need to calculate the flux of all the points. Flux means the vector field multiplied by the area integration.

So, whatever the value we are getting that is called the flux. So, that statement is saying that this is with 6 surfaces. So, there are 6 surfaces. So, if you calculate this entire volume for this particular volume, so, you need to calculate all surface integral of all the 6 surfaces and then add to find the closed surface integral. So, the statement eventually tells us that the total flux. That is that means the vector field that is coming out through this entire surface that is inclosing the volume is equivalent to the divergence over the volume.

So, that means, if you have a vector field here and if you calculate the amount of divergence and then integrate over the entire volume, the same value is equivalent to if you calculate the total flux that is coming out from the system and if you calculate the flux. The closed surface integral is introduced and this closed surface integral gives the same value that you are getting by calculating the divergence over the volume.

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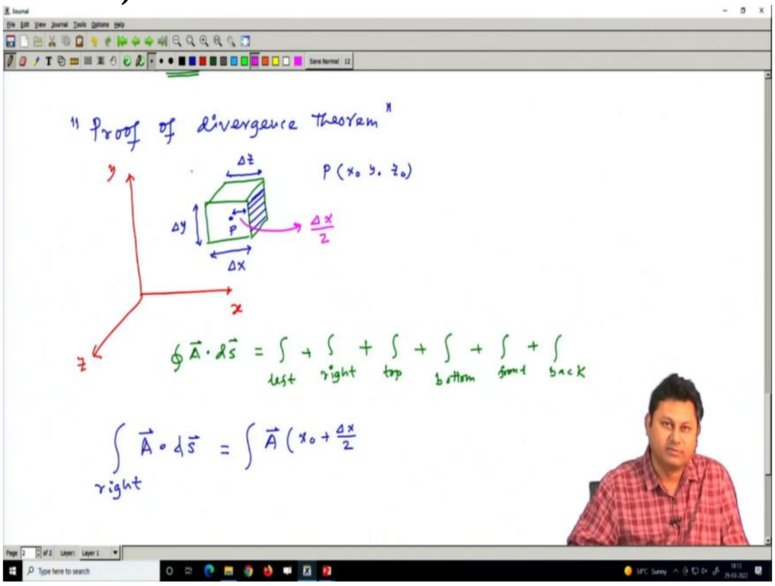

Let me now quickly prove this, which is not that complicated so, proof of divergence theorem. So, in order to prove we are going to use this Cartesian coordinate system, but it is true for other coordinate system also. In Cartesian coordinate system, it is easier so, I am having a coordinate system x y z and the small volume element here somewhere such that let me write in a different way.

Because, if I use this is my x, it will be easier. Suppose this is my x direction, y direction and z direction, and this length from here to here is Δx, from here to here it is Δy and here to here it is Δz. So, these are the 3 lengths we are having and exactly in the middle point here suppose, this point is P, the coordinate of the P say is x_0 y_0 z_0. Now, I want to calculate first the total surface integral now, I try to calculate the total surface integral.

So, that means, first I calculate for this given volume I calculate A•ds. So, now A•ds is, this is a cube, so, I can have 6 surfaces that is enclosing the entire volume. So, these 6 surfaces I write it in this way, one is left surface another is right surface. Left surface means I am talking about this one, right surface means this one, another is you know, top surface and another is the bottom surface and another is the front surface and another is the back surface.

So, these 6 surfaces we are having and we name it, this is my left surface, this is my right surface, this is top, this is down or bottom, this is front and this is back. So, these are the 6 surfaces I need to calculate. So, let us calculate with these left and right surfaces first and then we understand what is the value. So, let us calculate the right surface first. So, right surface if I want to calculate A•ds, the ds element is known, and the value of A is also known.

So, over here I am calculating the integration for these surfaces try to understand, this is my this surface, I called my right surface. So, for right surface my value of the A vector field will change because this point it is x_0 y_0 z_0. So, what is the point here? The point here the value of the A is $x_0 + \frac{\Delta x}{2}$ because this length is Δx from here to here this length is Δx, from P to here this length is should be $\frac{\Delta x}{2}$, this side is $\frac{\Delta x}{2}$, if you add you will find Δx.

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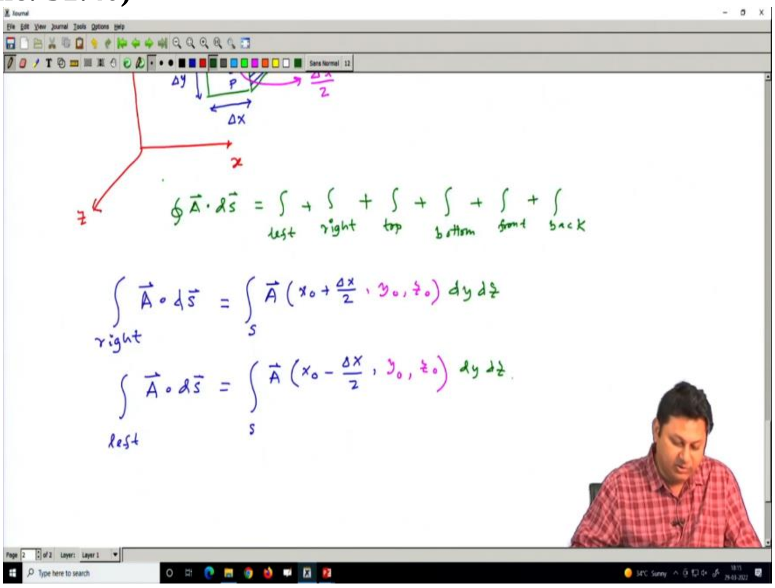

And also, we have the other dependency y_0 and x_0, which is there and then dx is simply, dx simply dy dz because this is the surface where dx is dz and dy, which is this one. Now, what about the left surface? If I calculate for left surface the same quantity A•ds I will have this is a surface integral, so, I should put a surface here. So, for left surface simply I have A and now the value is other surface.

So, this value should be executed at this x point, y point and the z point is like same there is no change in this and finally, we have dy and dz. Mind it the left side and this side the unit vectors are in one case the unit vector is plus and another case it is the minus. In this side the unit vector of the surface is i. So, I should mention that the unit vector of along this direction of the surface is i but this side the unit vector is -i.

So, that means, when you add these 2 eventually there should be a minus sign involved with this left integral because of the unit direction is now in opposite.

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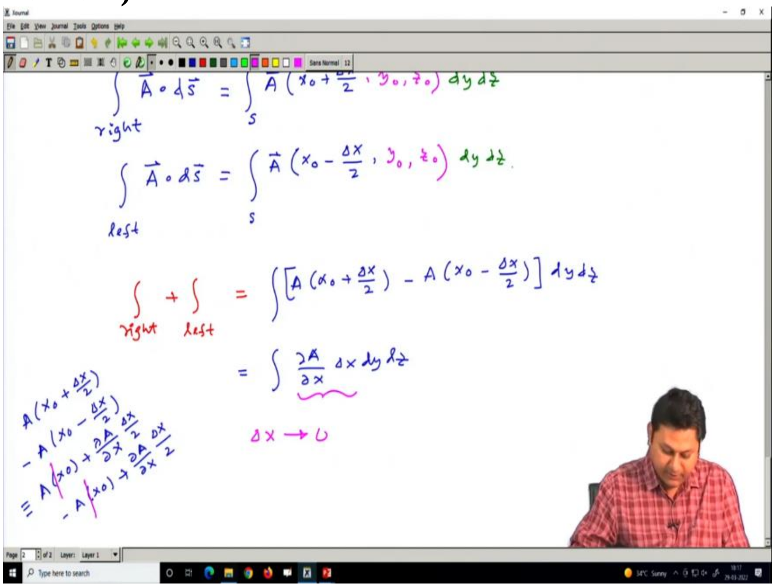

So, I calculate the right plus left and then right hand side is simply this one, integral A ($x_0 + \frac{\Delta x}{2}$), I am not writing y_0 and z_0 because there is no change, - A ($x_0 - \frac{\Delta x}{2}$) and dy dz, making a Taylor series expansion can give you simply this quantity, simply you will get $\frac{\partial A}{\partial x}$ dx dy dz because A ($x_0 + \frac{\Delta x}{2}$) - A ($x_0 - \frac{\Delta x}{2}$) is simply equivalent to A of $x_0 + \frac{\partial A}{\partial x}\frac{\Delta x}{2}$ - A of $x_0 + \frac{\partial A}{\partial x}\frac{\Delta x}{2}$.

So, this term these 2 terms are going to cancel out. And if you add these 2, you are going to get this one. Now, with the limit $\vec{\nabla}$x tends to 0 that means for every small amount of volume element.

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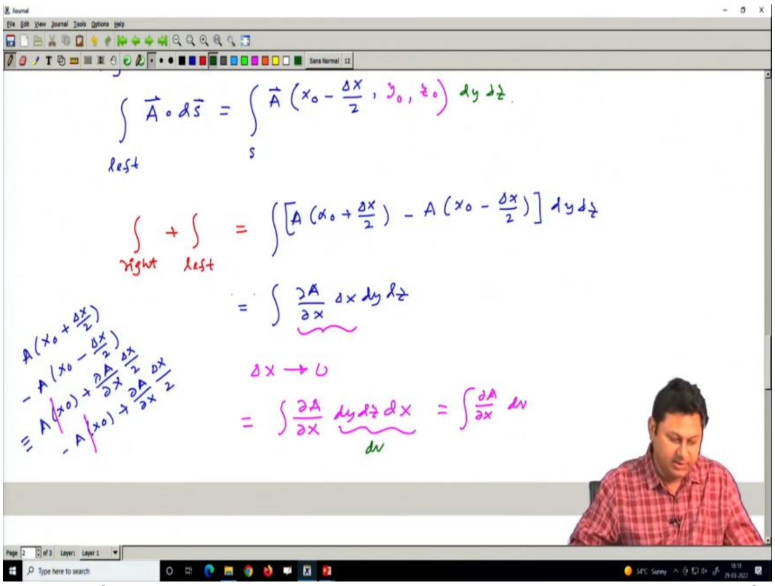

This integral is simply $\frac{\partial A}{\partial x}$ dy dz dx, which is nothing but integration of $\frac{\partial A}{\partial x}$ dv because this quantity is simply dv. So, that is for the pair of left and right. Now, if I want to calculate for all the pairs.

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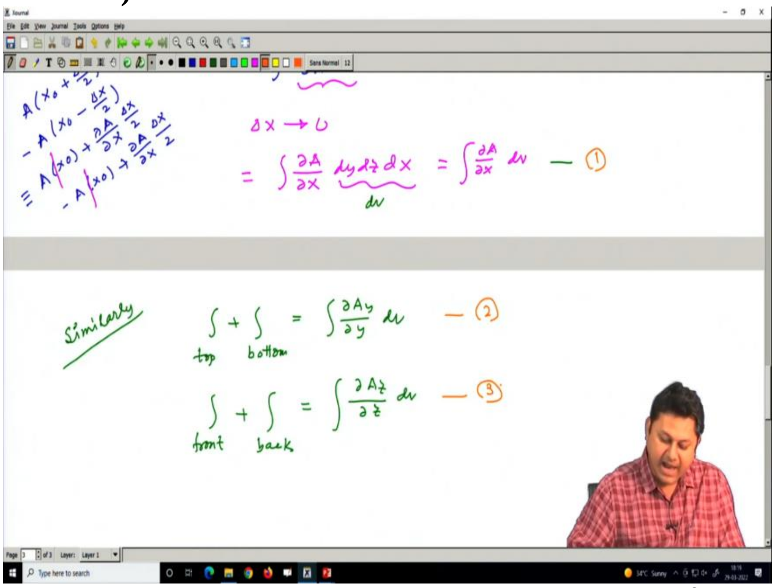

Similarly, for top plus bottom exactly in the similar way you will have $\frac{\partial A_y}{\partial y}$ dv and for front and back you simply have, now, if I add all these 3, this is my say equation 1, this is equation 2 and this is equation 3, if I add all this equation.

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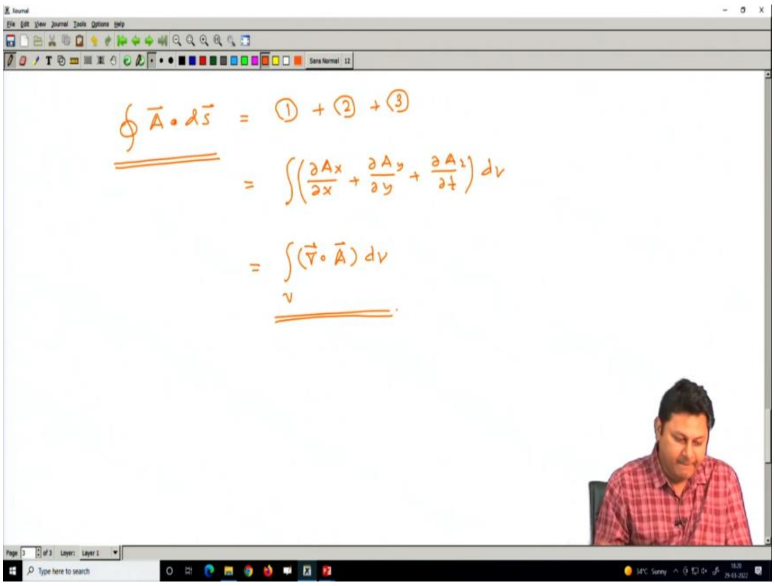

So, that basically gives me the total surface integral volume total closed surface integral that is 1 + 2 + 3 and that value is integration $(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z})$ dv, which is nothing but the volume integral of the $(\vec{\nabla} \cdot \vec{A})$ and that is the statement of divergence theorem. So, the right-hand side you can have the closed surface integral and left-hand side and right-hand side you have the volume integral. Well before ending today's class.

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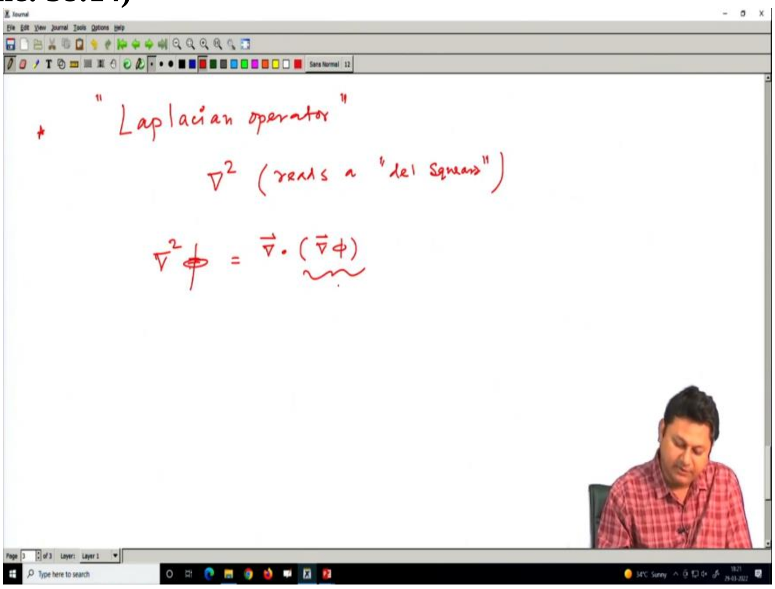

So, quickly I like to mention this is the last point today. So, another important differential operator and that is called the Laplacian. The Laplacian operator normally defined in this way square, we call this is reads as del square, simply ∇^2. What is ∇^2? So, this operator when it operates over a scalar field suppose scalar field ϕ, it looks like this. It basically gives us this.

So, first I can have a gradient of a scalar field then this is a vector quantity or it becomes a vector field and then if I operate over this divergence.

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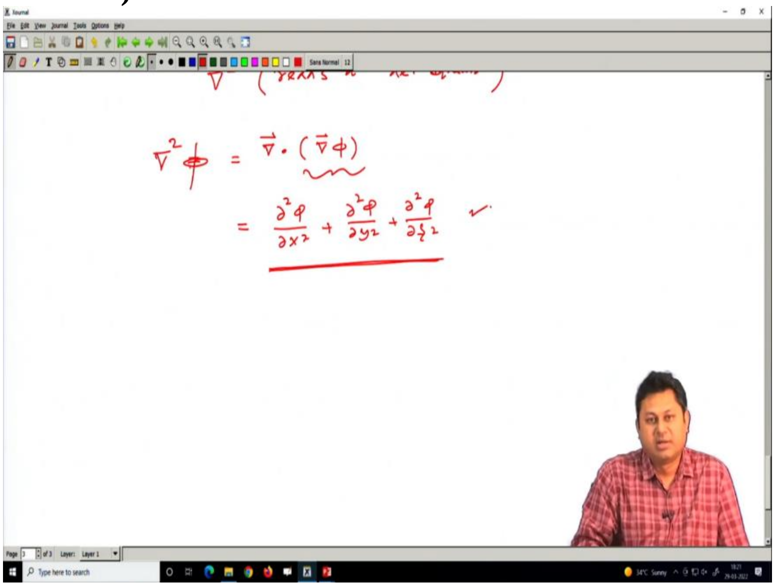

I will be going to get this. So, if I expand it should we simply $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$. I am giving a small homework please check by yourself this after doing this operator, you are getting this expression. So, with that note, I would like to conclude today's class. Thank you very much for your attention. See you in the next class.