Foundations of Classical Electrodynamics
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Lecture – 12
Gradient, Divergence and Curl (A recap), Vector Identities

Hello students to the foundation of classical electrodynamics course. So, today we have lecture number 12 and today we will make a small recap of gradient, divergence and curl operator and the related theories that we have developed in last few classes and also like to prove few important vector identities.

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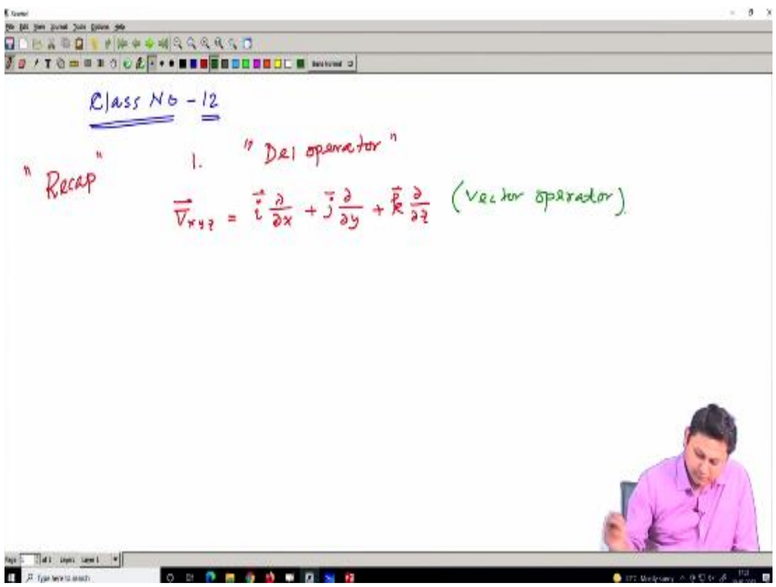

So, let us start the class. So, today we have class number 12. So, as I mentioned today I will have a recap so, the first thing we should mention that this is del operator ($\vec{\nabla}$). We define this $\vec{\nabla}$ in Cartesian coordinate system in this way. This is a vector operator and it is defined in this way. And this is a vector operator.

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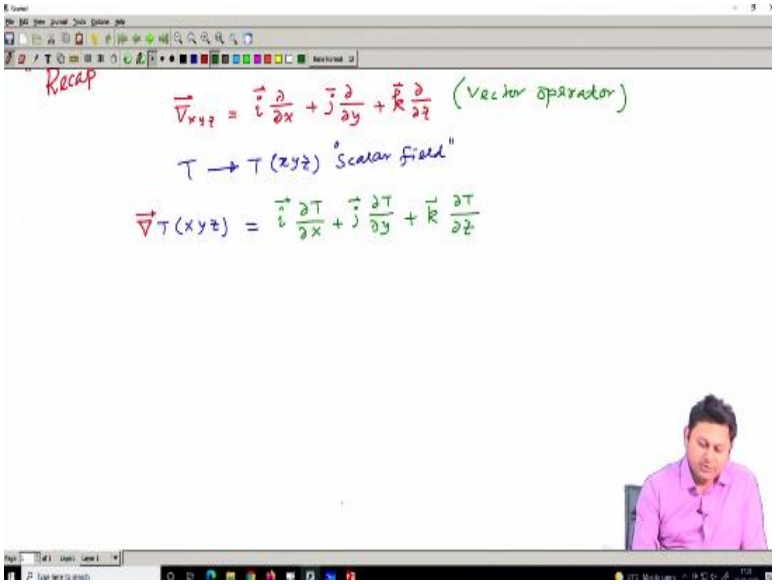

Now, the next thing how to operate these things over some function, so, we mentioned that T, which is a function of x y z it is a scalar field. Now, I can operate these things over that this operator is operating over this T (x y z) and give rise to something like i $\frac{\partial T}{\partial x}$ + j $\frac{\partial T}{\partial y}$ + k $\frac{\partial T}{\partial z}$. It will operate and give us this.

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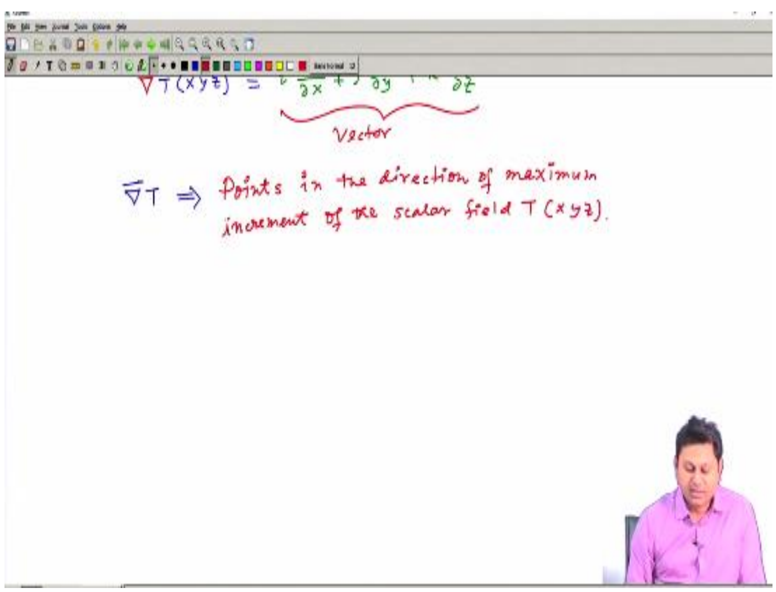

Now, this quantity is a vector quantity and what does it mean? What is the meaning of that thing? So, these things eventually tell us that this is the direction basically this is points in the direction of maximum increment of the scalar field. So, T is a scalar field, which is changing over x y z and

I am putting this operator over that scalar field that basically gives us the direction along which it is changing in the maximum way.

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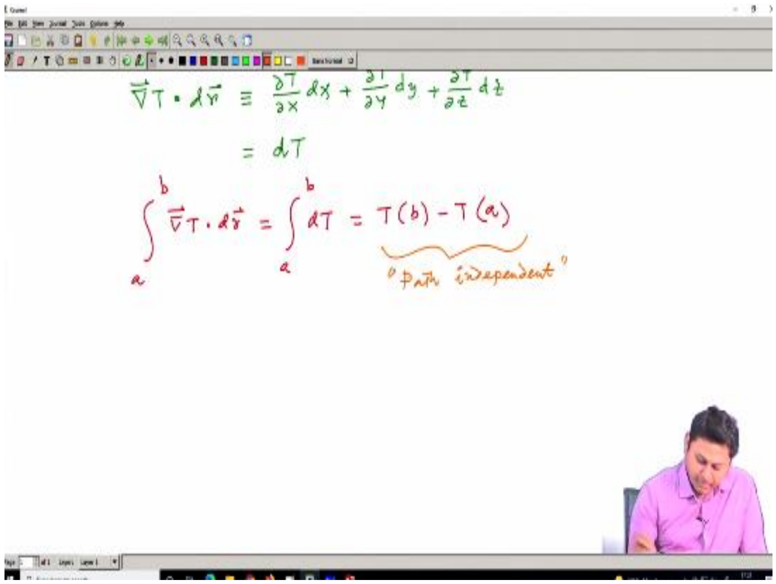

Now, the next important thing is this quantity is eventually this. And these things I can write as dT that is all. So, if I write this $\vec{\nabla}T \cdot d\vec{r}$ in this form, then the next thing I can write is if I want to calculate do the integral line integral because it is essentially give us a line integral where $\vec{\nabla}T \cdot d\vec{r}$ so, this is a line integral. So, that eventually means that I am making a line integral of this thing dT, which gives me a simple result T (b) – T (a), so, that is path independent.

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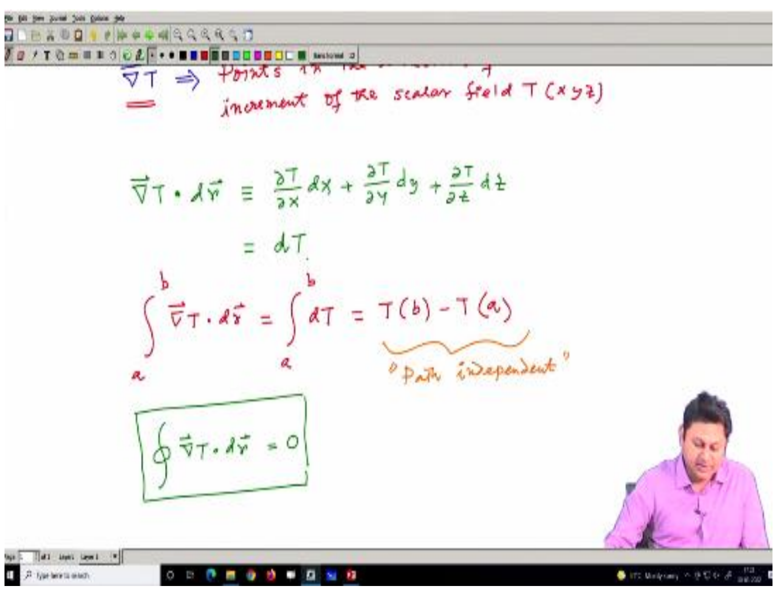

Now, in after writing this we can extend the idea and make a closed line integral for the same quantity and when I write a closed line integral, it should be simply 0 that is a very important information that we should note. So, that is whatever we have learned about this is a gist of this $\vec{\nabla}$ and how it operates over a scalar field and what is the working principle and working mechanism everything is shown here.

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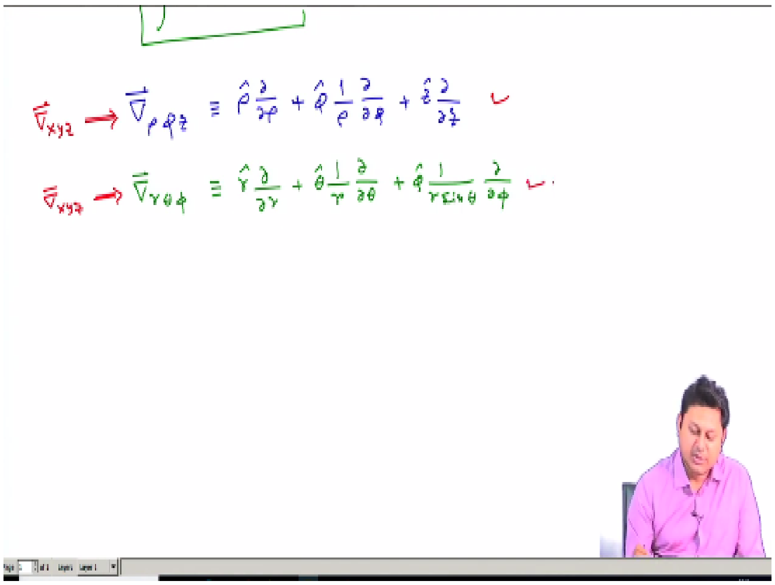

Now, the next thing I mentioned that what is a form of this operator in other coordinate system. So, at ρ φ z you can see the operator looks different. Unlike the scalar, unlike the Cartesian coordinate it is showing something like this. This is the form of this vector, form of this vector operator over it as a function of in cylindrical coordinate system. What about the spherical coordinate system? In spherical coordinate system, I should write it r then θ and then φ.

And the operator reads like this then $\hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$ and we derived this how one can transform from this to this, this is the original Cartesian coordinate and from Cartesian coordinate I derived and few classes ago that how to you know convert these things to here. In the same way, you can convert that by using the relationship between x y z and r θ φ you can derive that it is coming like this.

Later we will see that for curvilinear coordinate system we can generalize this and try to understand that how should be the general expression. After that the next thing we learn is, this is 1 and this is 2.

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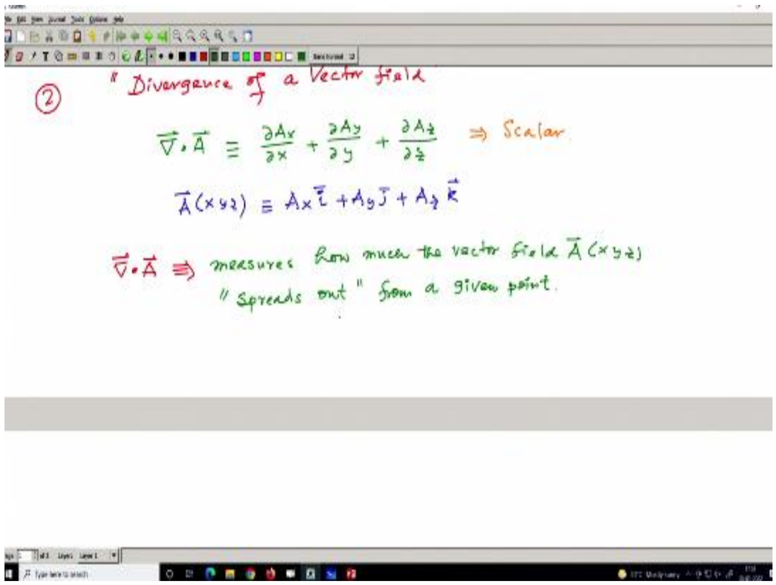

And that is the divergence of a vector field. And what was that how it operates? The operator $\vec{\nabla}$ is same, but now the operation mechanism is like I am working A dot product kind of things and what is the outcome here? Outcome is $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ and this quantity is a scalar quantity where A is having a component A is a vector field should have a component like this. The next thing is what it measures, what is the meaning of these things? If I have a quantity like this what is the physical meaning of that will also explain this stuff.

So, it basically measures how much the vector field A spreads out from a given point. So, with this divergence we basically find out that for a given point how this vector field is spreading out how much.

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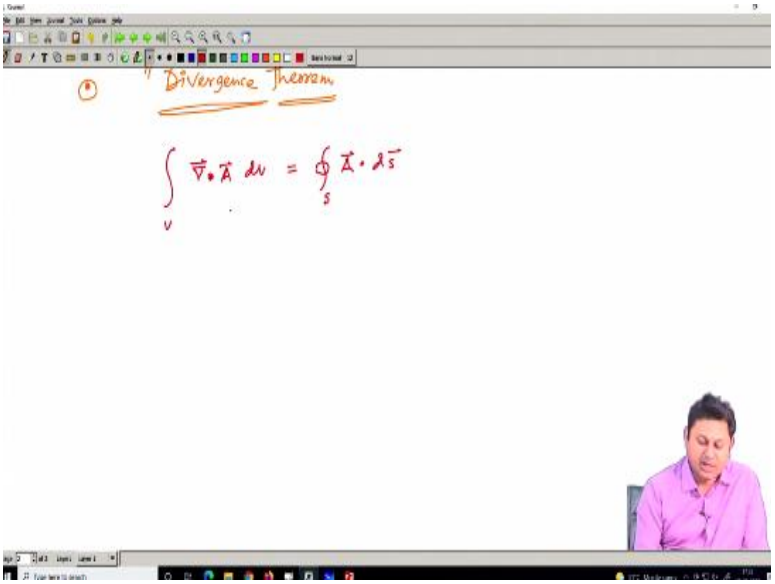

The next very important thing we learn, very important theorem we learn and that is the divergence theorem. This divergence theorem the statement of this theorem is like that if I have a volume integral of the divergence of A then that value is the total flux over the surface that is enclosing the entire volume. So, this surface is in enclosing the entire volumes. So, I am calculating a volume integral here so, that means I should have a volume. So, this surface should be such that it is enclosing the entire volume then this identity holds.

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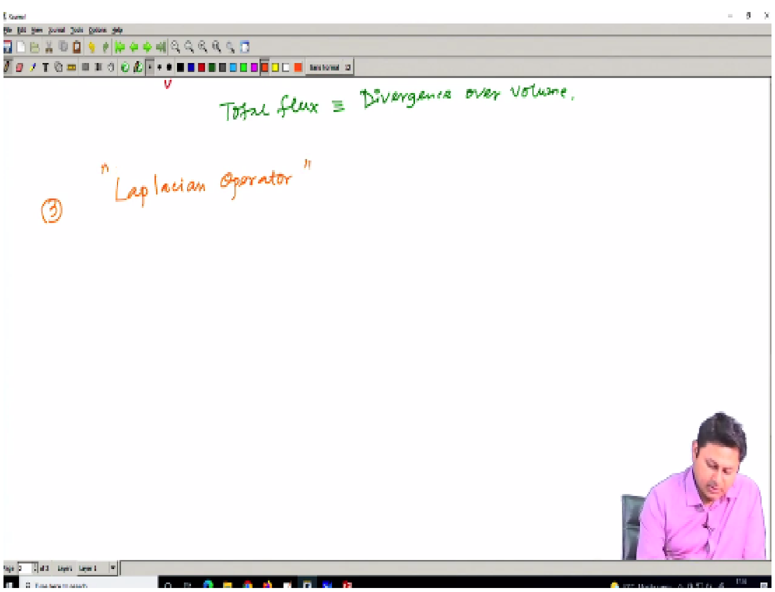

Well, what is the physical meaning? That physical meaning that this is the total flux. So, total flux is same as divergence over volume, total flux is divergence over volume. The next thing we mentioned is also having some significance and that is 3 is called Laplacian operator. What was Laplacian operator?

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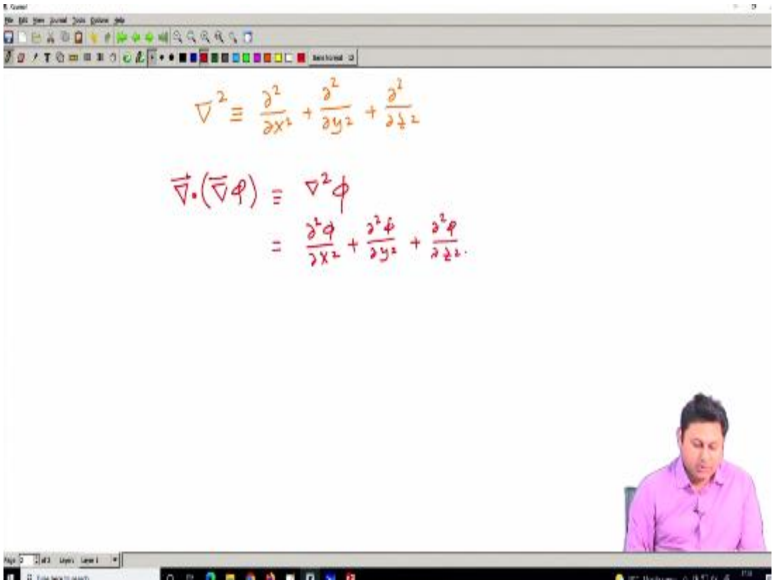

The Laplacian operator was something like this, this is a scalar operator and it defines like a second order partial derivative of x, y and z, which like sum of it. And how it one can consider these things as a first I am having a scalar function and first make the gradient of the scalar function φ and you are getting a vector and then if you make a divergence of that thing, so, what you are getting is simply the Laplacian of that function.

So, first you are making the gradient and then you are getting a vector quantity that vector quantity if you want to find out how diverge it is, so, we are now making a divergence and you are getting this and this simply gives us this stuff.

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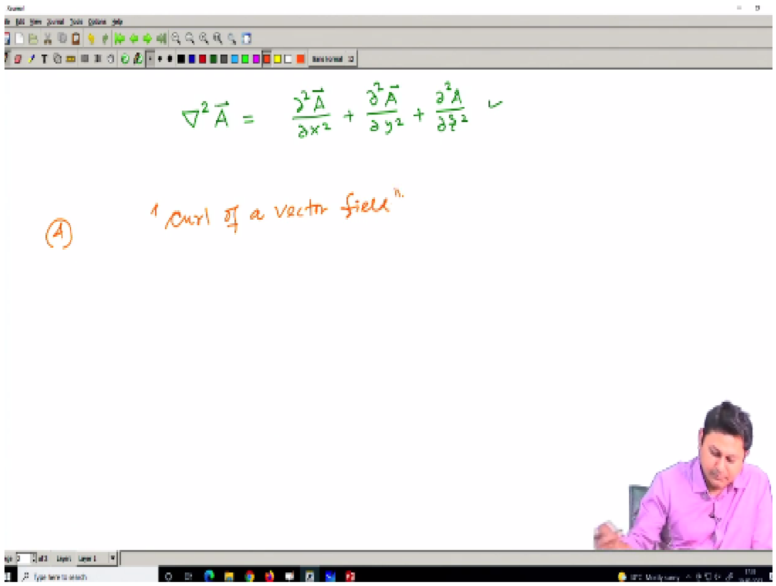

And also this operator can operate over a vector field that is very interesting. So, this operator can operate over a vector field and we can have the form the right-hand side like this. So, this is a scalar operator it can operate over a scalar field as well as on a vector field. This is the way one can define. So, the next thing we discussed was this is the third point, fourth point rather that the curl of a vector field.

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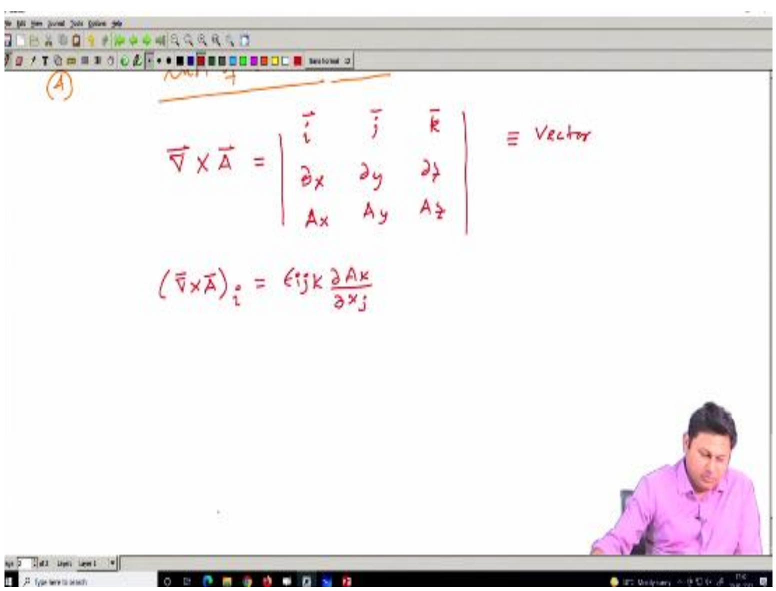

The curl of a vector field so, how I calculate the curl of a vector field? So, this is my operator. Now, instead of having divergence I am taking curl that means, I am making a cross product, which results this shorthand notation of partial derivative and then A_x A_y A_z bracket close. So, this mind it, this is a vector quantity and component-wise if I want to find out what is this. So, the ith component of this is simply $\varepsilon_{ijk} \frac{\partial A_k}{\partial x_j}$ so, that is the way we can define.

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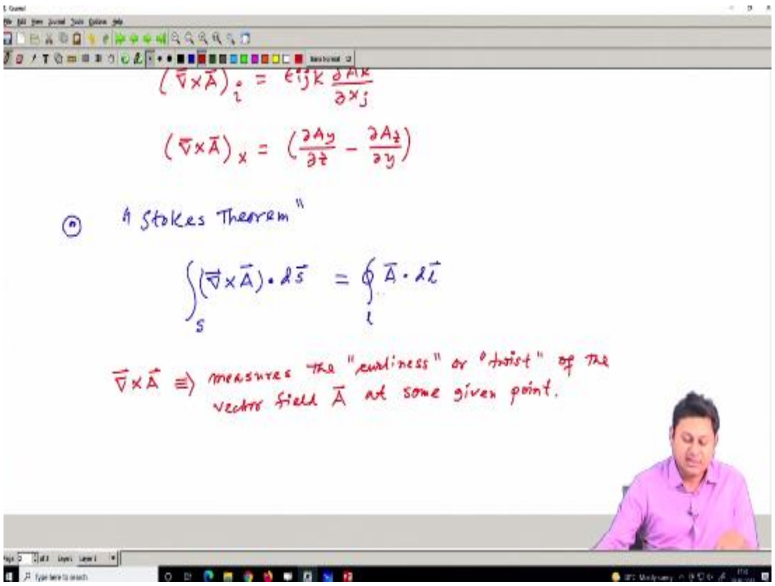

So, x component if I want to find out what is the x component is simply gives us this. Also there is a very important theorem that is associated with this curl stuff and that is the Stokes theorem. What is the statement of the Stokes theorem? The Stokes theorem tells us if I have a curl and then from this curl if I calculate the surface integration of a given surface whatever the result I get that should be identical with this quantity where we calculate the closed line integral and this closed line is encircling this surface.

So, that was the statement. And what is the physical meaning I must mention that also what is the physical meaning of these things what is physically what it measures it basically measures the curliness or twist of the vector field A at some given point. So, this is all over the recap in last couple of classes whatever we have done, so, I make it in a single class so that you can appreciate that how $\vec{\nabla}$ is there.

And how it operates over the scalar field giving a something called the gradient and then how you calculate the divergence for a given vector field and what is the meaning of divergence, what is the important theorem related to divergence, which is the Gauss's theorem or divergence theorem, which tells us that if you calculate the volume integral for a divergence that should be identical to the closed surface integral.

And the surface should be such that it is enclosing the entire volume over which you are calculating the volume integral, then the Laplacian operator is a very important operator in later part of the course, these operators will come very frequently. Then curl of a vector field how you can calculate the twist, twist nature of a vector, a vector field is a field that is evolving over the space and for a given point if you want to find out how curly it is, how twisted it is then you can calculate this curl and if it is a nonzero value that means, these vector field is twisted there.

And then the Stokes theorem suggests that if you calculate the curl and then make a surface integral, then that value should be identical if you have the vector field and calculate the line integrals, but the line should be such that it is encircling the given surface over which you are doing the surface integral. So, this is overall the recap I wanted to do quickly today.

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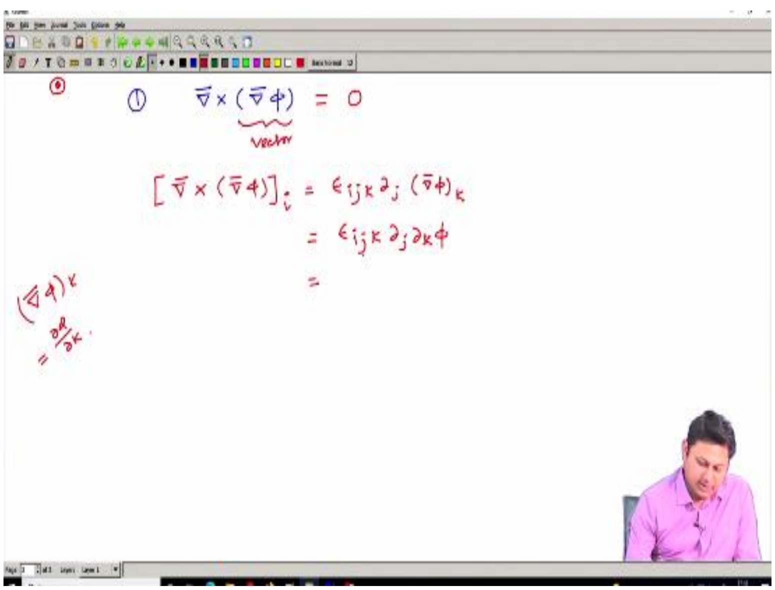

Now, the next part of the class I will going to prove some vector very important vector identity and that we should be very careful of. So, the first vector identity I like to show is this. So, I want to calculate the curl of a gradient of some vector what should be the value. So, I know this is a vector quantity, this is a vector quantity the gradient, which is giving us the direction. Now, if I want to find out the curl of that thing, you will see this is identically 0 always you will be going to get 0.

So, how you prove what is the proof that I will be going to do here quickly by using these symbols useful Levi-Civita symbols that we know. So, let us try to find out first the ith element of this vector because at the end of the day this gives us a vector quantity. So, ith component if I want to find so, how you write it in terms of ε, it is ε_{ijk} and then ∂_j and whatever we have here the kth component I should write it. That is the way and this is the derivative with respect to this j.

So, here then further I can write it ε_{ijk} these things and these things what is φ and what is the kth component of the φ? It is simply $\frac{\partial \varphi}{\partial k}$ simply that. So, the kth component is k is 1 2 3. So, that means x y z so, the x component is $\frac{\partial \varphi}{\partial x}$, y component is $\frac{\partial \varphi}{\partial y}$, z component is $\frac{\partial \varphi}{\partial z}$, so, instead of writing xyz if I write k it should be $\frac{\partial \varphi}{\partial k}$. So, then it is $\partial_k \varphi$ that we are having. So, now, what I do is here how many indices are there, you can see that j k and j k are repeated index. I am writing in an Einstein notation.

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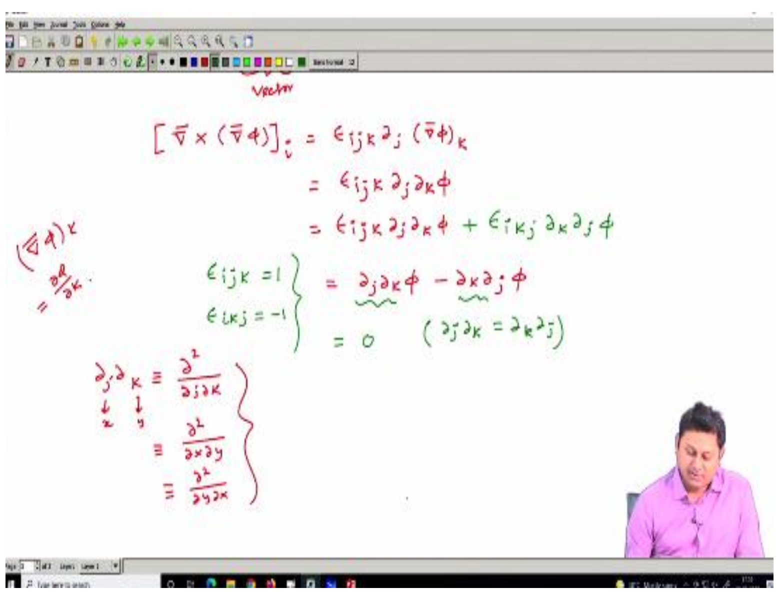

So, if these 2 are repeated index then if I expand this, so, one thing is ijk $\partial_j \partial_k \varphi$ that is one and another thing I should write and that thing is another option is ikj and then φ k $\partial_k \partial_j \varphi$. Now, we know that ε_{ijk} it is 1 but $\varepsilon_{ikj} = -1$. So, this we know so, we can write in the next line that this is ε_{ijk}, I do not need to write because I am now putting the value. So, ε_{ijk} is not required anymore, because I am going to put this value.

So, this is eventually $\partial_j\ \partial_k\varphi$ and here we have a minus sign because this is -1 $\partial_k\ \partial_j\varphi$. Now, you can see that this operator what is the meaning of $\partial_j\ \partial_k$ this means, I am having the operator in this form if j is x and k is y. So, it is eventually $\frac{\partial^2}{\partial x\,\partial y}$, but I know that this is also equivalent to $\frac{\partial^2}{\partial y\,\partial x}$ so, these things are same. So, since these and these are same these operator over φ and this operator over φ same, so, I can have simply this equal to 0.

As $\partial_j\ \partial_k = \partial_k\ \partial_j$ I am making the partial operation but I have changed the order, but even if I change the order, then it is commutative. So, at the end of the day we will get the same result and this gives us 0. So, that means, we always have this is in general.

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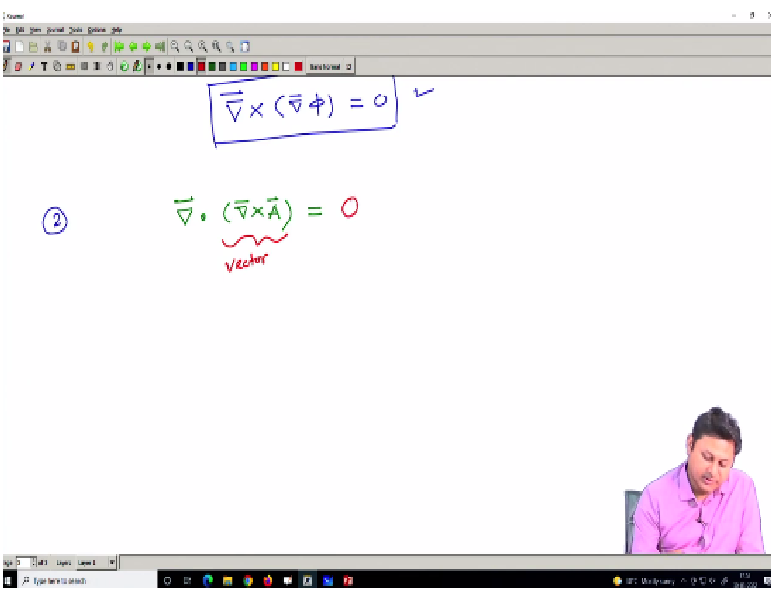

So, we should note that we always have the curl of a gradient of a scalar field to be 0, this is a vector identity, this identically 0 without any condition for all the cases this is 0. So, that is the first identity I like to prove. Now the second one second identity, the second identity is if I want to find out the gradient of a curl of a vector field then what should I get the right-hand side? So, I find an A curl first. So, divergence of a curl of a vector, so, this is A vector it gives rise to a vector field.

Now, I try to find out what should be the divergence of this quantity and again you can see that this is identically 0 always we will get this is 0. How to prove that?

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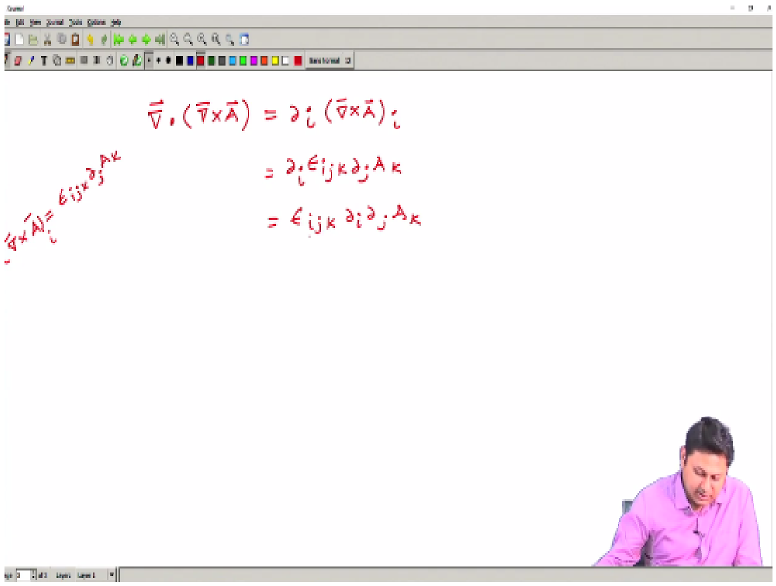

So, let us start with the expression. So, the expression is saying that divergence I need to calculate and this is curl. So, this quantity if I write I should write ∂_i and $\vec{\nabla} \times \vec{A}$ and i because this is a divergence. This is a divergence operator both the cases it should be i. Now, I expand this ∂_i how I expand this ith component because we know that curl this is $\mathcal{E}$ ith component $\mathcal{E}_{ijk}$ ∂_j A_k. So, that I will be going to use $\mathcal{E}_{ijk}$ ∂_j A_k. So, this I can write this $\mathcal{E}_{ijk}$ ∂_i ∂_j A_k.

And now, I will play with that you can see that how many combinations are there ijk, ijk this is a repetitive so, that many combinations you can have.

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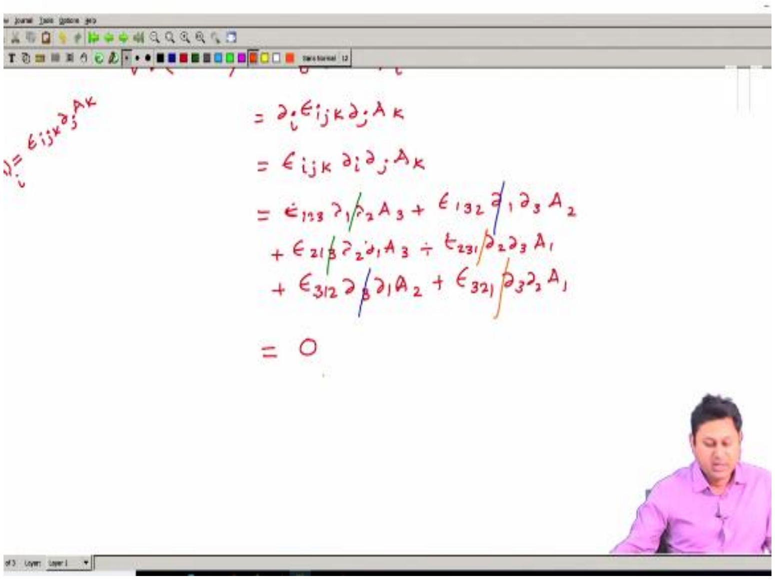

So, now, you need to you know start putting the value, so, the first value should be the first value should be say 123. Then ∂_1 ∂_2, then A_3 + ε_{132} then ∂_1 ∂_3 A_2 then + ε_{213} ∂_2 ∂_1 A_3 + ε_{231} then ∂_2 ∂_3 A_1 and finally, we have ε_{312} and then ∂_3 ∂_1 A_2 + ε_{321} ∂_3 ∂_2 A_1. So, for each value you have a value with the negative and if you calculate you will find this is identical is 0.

For each, for example ε_{123} is here and ε_{321} is here and sorry, ε_{123} is there ε_{213} is there. So, these 2 will cancel out because these are the negative of this one. In a similar way, 123 A_2 is here, then 321 A_2 is there. So, these 3, these 2 will be going to cancel out. So, I am just showing that this thing will cancel out with this one. This thing will cancel out with this one and this thing will cancel out with this one to give rise to a 0 value. Finally, this is my last, today's last identity that I am going to prove but very important identity.

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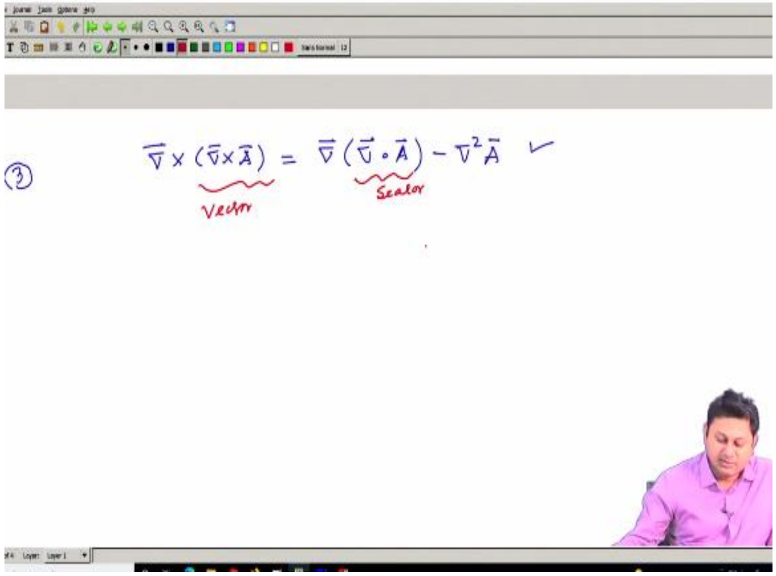

This is the third identity I am going to prove. And it is saying that what is the value curl cross A I need to prove that. And the result first let me write. This is a very famous identity, and you should remember this identity, because this will be going to help us a lot in future. And this is the identity, I am just blindly writing. I will prove that later. So, this is curl vector quantity. And I will then calculate the curl over that. So, at the end of the day, whatever I am getting is a vector quantity, right-hand side what I am getting this is a scalar quantity.

But I am taking the gradient of that thing. So, this should be a vector and this is the Laplacian operator operating on a vector field. So that should also give you the vector. So, right-hand side and left-hand side both will give us both give the vector quantity. So that means at least this is fine.
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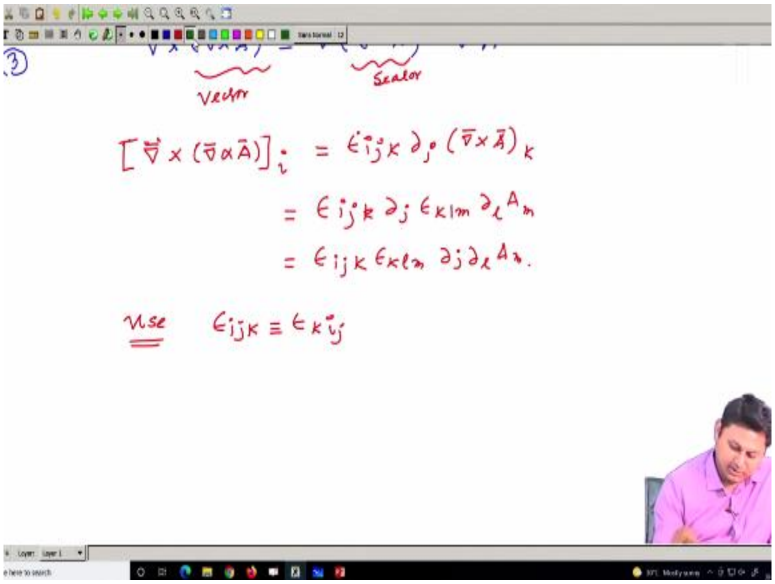

Now, I am going to prove this stuff quickly. So, $[\vec{\nabla} \times (\vec{\nabla} \times \vec{A})]$ ith component, if I calculate then it should be simply ε_{ijk} and then ∂_j and kth component of this quantity, if I elaborate it should be ε_{ijk} ∂_j, this quantity is how much ∂_j sorry this quantity is ε_k and then I need to put 2 new lm and then ∂_l A_m. Now, I will put this together ε_{ijk}, I put this ε where ε_{klm}, then it is ∂_j ∂_l A_m. So, now I use this relation ε_{ijk} is equivalent to ε_{kij} so, that I am going to use here.

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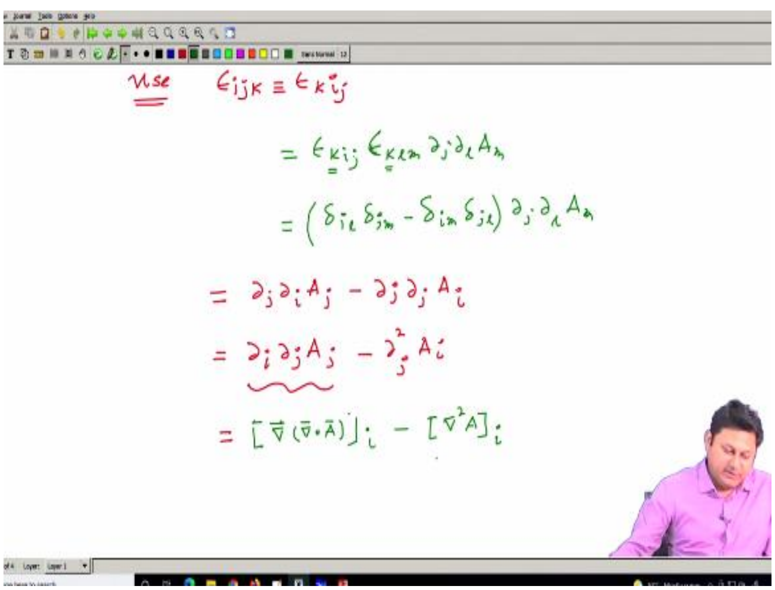

So, after that, I can write it this ε_{ijk} I replace to kij because both are same then ε_{klm} then ∂_j ∂_l A_m as usual now, why I write this because in both cases if I write this in a k form, then I can use this form delta identity and that is δ_{il} δ_{jm} - δ_{jl} sorry δ_{im} δ this one to this tells one im and δ_{jl}, then ∂_j ∂_l

A_m whatever is here. Now, these things I need to be careful. So, when i = l then it is non-vanishing and j = m then it is non-vanishing.

So, keeping that I can write it δ_{jl} become i so, it is ∂_i and then A_j because m = j so, then only it is non-vanishing $-\partial_j$ because when j is l = j then it is non-vanishing then ∂_l, so I should write it as ∂_j and m i so write it is i. So, this quantity now, if you look carefully, this is I can write it ∂_i ∂_j and then A_j minus this is double so, I simply write $\partial_j^2 A_i$. Now, this quantity is very interesting and I can write it, please check it, I am not going to do the elaborate thing here you can take it as a homework.

Then the gradient of this quantity and ith component is simply this one and minus this is simply the ∇^2A and the ith component.

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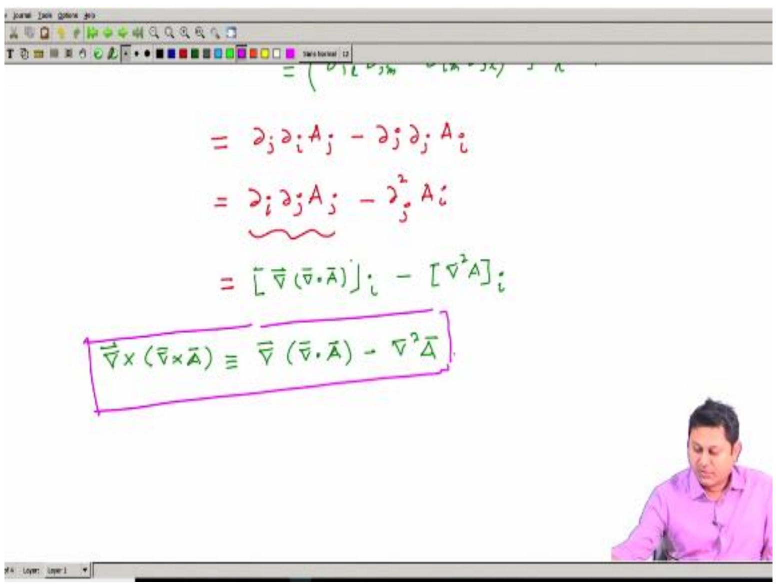

So, eventually I prove whatever I wanted to prove that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$ the left-hand side is equivalent to this is the $\vec{\nabla}\left(\vec{\nabla} \bullet \vec{A}\right) - \nabla^2 \vec{A}$ this is the thing I wanted to prove, and very important identity I want you to please practice this proof check very carefully whatever is done today's class and try to do by your own. So, with this note I like to conclude because I do not have much time today. So, thank you for your attention and see you in the next class.