Foundations of Classical Electrodynamics

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology - Kharagpur

Lecture - 13
Curvilinear Coordinate System

Hello students to the foundation of classical electrodynamics course. So, today we will have lecture number 13 under the module 1 mathematical preliminaries and today we will be going to learn the curvilinear coordinate system.

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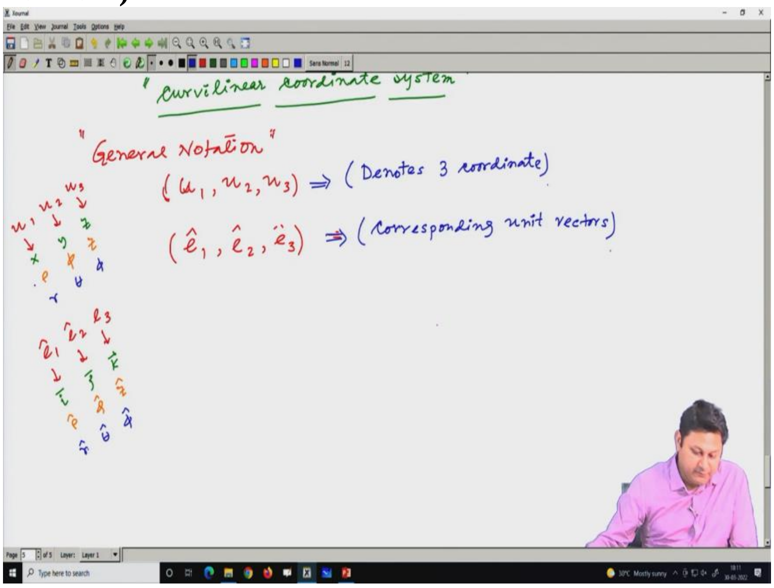

Class number 13 and today the topic is curvilinear coordinate system. So, this is a general kind of coordinate system today which I want to discuss and please be careful because there are lengthy calculations and conceptually this is very enriched. So, I want you to concentrate more on this today's class. So, the general notation for this coordinate system is this. Let us first define the general notation.

So, I am not using Cartesian coordinate system, I am not using cylindrical coordinate system or I am not using the spherical polar coordinate system. So, this is a in general coordinate system and the general coordinate system demands some general notation and that general notation now I am, so, the 3 coordinates we write with this u_1, u_2, u_3 like x, y, z. So, in Cartesian coordinate system, u_1, u_2, u_3 in Cartesian coordinate system it is simply x, y, z.

In cylindrical coordinate system, it is ρ φ z and in spherical polar coordinate system, it is r θ φ. So, in general now I am writing it is u_1, u_2, u_3 this is what? This is the 3 coordinates, denotes 3 coordinates. Since we are denoting the 3 coordinate system, 3 coordinate the notations. So, now, the next thing I like to note I like to mention is the unit vectors along these 3 coordinates points and these 3 unit vectors are e_1, e_2 and e_3 this is a generalized way to write the unit vector.

So, these are the corresponding unit vectors. So, similarly I can write here that e_1, e_2, e_3 for our known 3 coordinate system, this is simply i j k for Cartesian coordinate system, then $\hat{\rho}$ $\hat{\varphi}$ $\hat{z}$ in cylindrical coordinate system and finally, $\hat{r}$ $\hat{\theta}$ and $\hat{\varphi}$ in spherical coordinate system. So, this is the way I am writing here, so, e_1, e_2, e_3 and u_1, u_2, u_3 the coordinates and the unit vectors and this is a general notation.

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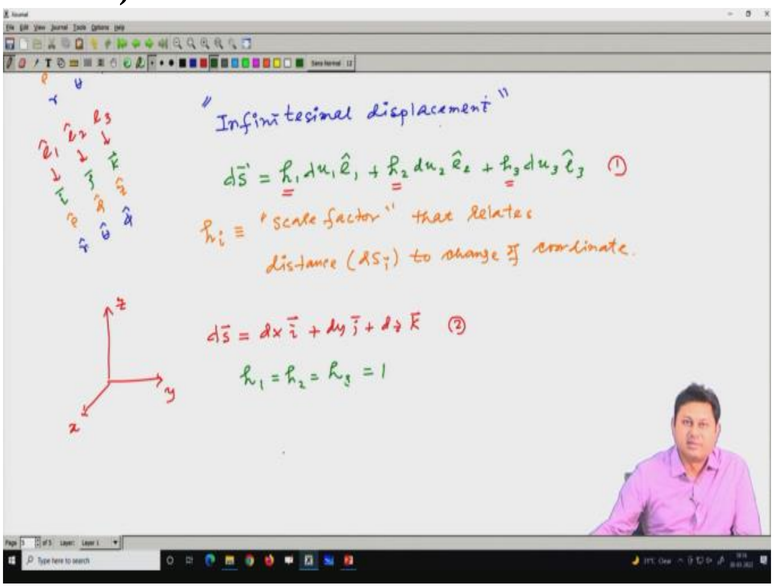

Now, in for this general coordinate system now, the fact is if I want to find out the infinitesimal displacement how to write it so, the length is the distance element displacement so, this infinitesimal displacement in general one should write like ds let us put a vector here is h_1 du_1 and then you want unit vector + h_2 du_2 $\hat{e}_2$ and h_3 du_3 $\hat{e}_3$. So, now, this h_1 h_2 h_3 are important here let me first define then I will discuss it.

This h_i in general i goes to 1, 2, 3 is simply the scale factor, is called the scale factor and for different coordinate system the scale factor differs how it differs and how to calculate we will discuss this, this is a scale factor that relates distance ds_i to change of coordinate. You may remember that for Cartesian coordinate system the distance if this is a Cartesian coordinate system say x y and z, the distance ds this is the vector we use, but if I want to find out the distance in cylindrical coordinate system it is not that simple that is why the scale factor is here.

So, if I compare this general distance, the way we define the distance vector here and the distance vector that we have the simplest case for Cartesian coordinate system then if I compare, say this is my equation 1 and this is my equation 2, equation 1 is a generalized version of equation 2 then I can readily find one very important thing and that is the scale factor h_1 h_2 and h_3 is 1 for Cartesian coordinate system.

If you simply put h_1 h_2 h_3 1 then from this equation 1 you will get equation 2 because in equation 1 you have du_i what is u_i? u_i is the coordinate. So, here we also have the coordinate x, y and z. So, here you have du_i du_2, du_1 du_2 and du_3 and also you have a unit vector e_1 e_2 e_3 and this unit vector is representing here in terms of i, j, k because we are dealing with Cartesian coordinate system. So, that is the information we have the first time.

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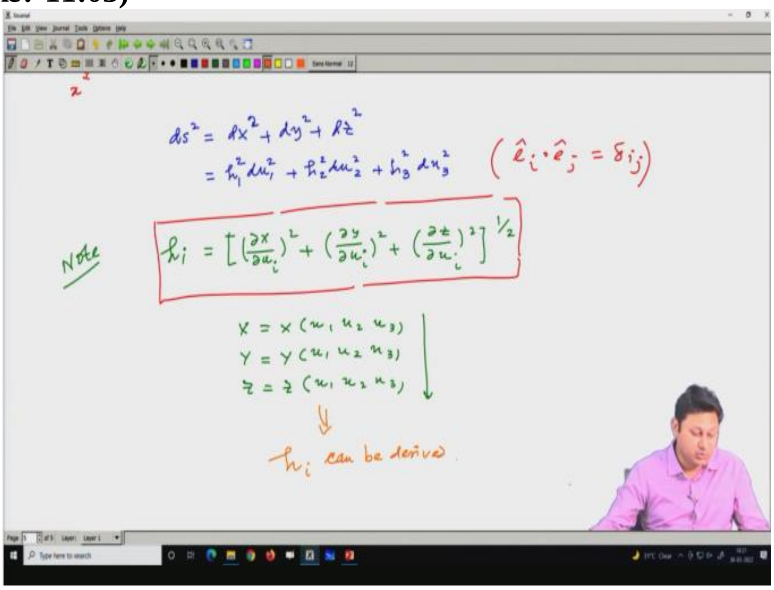

Now, another thing that how to calculate the you know this h and that is a very, very important thing now, I am going to say that ds^2 so, I am giving you a recipe here ds^2 this is the square of a distance is simply dx^2 in Cartesian coordinate it is $dx^2 + dy^2 + dz^2$. Now, these things is in Cartesian coordinate system in general what is this value? This is $h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$.

Because ds is defined in this way and if you make a square of that, then I will get this result under the condition that $\hat{e}_i \cdot \hat{e}_j$ is δ_{ij}. So, that means, I am still in orthogonal orthonormal coordinate even though this is a generalized coordinate but this coordinate is orthonormal in nature. So, I am having these in my hand that is why you can write it. So, from that I like to note here make you note h_1^2 I can have simply this.

Or h_i rather I can simply have this you just correlate this $(\frac{\partial x}{\partial u_i})$ if I want to find out $((\frac{\partial x}{\partial u_i})^2 + (\frac{\partial y}{\partial u_i})^2 + (\frac{\partial z}{\partial u_i})^2)$ that whole to the power $1/2$. So, this is a very, very important expression for the time being let us consider this is true and then we will show the later case that how it is coming. But, intuitively you can write it the h factor in terms of x and u in this form where x y z in the Cartesian coordinate system.

And u_1, u_2, u_i are in different coordinate system the coordinate system if I want. So, if I know the relationship, if you know x as a function of u_1 u_2 u_3, y as a function of u_1, u_2, u_3 and z as a function of u_1, u_2 u_3, then eventually you can find out the scale factor. So, from here from this information we can find out the scale factor. So, that is information we will want to use later.

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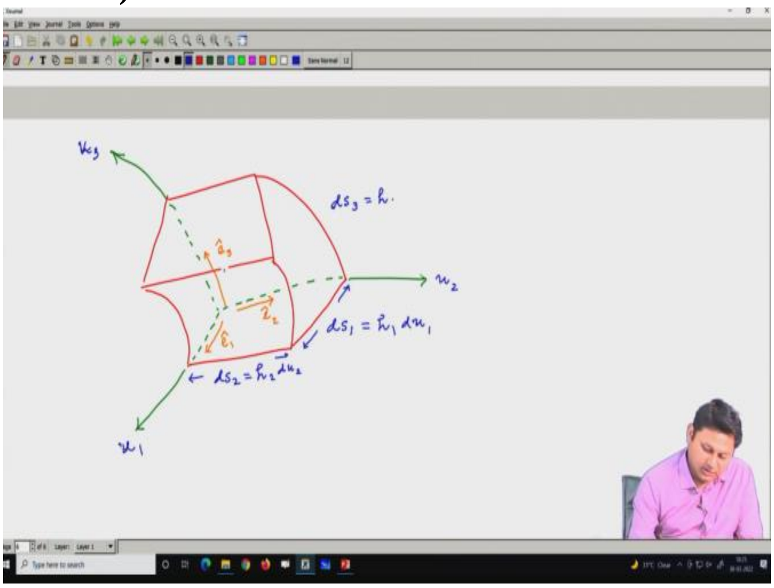

Now let us try to you know understand this curvilinear coordinates as I mentioned this is not any this is a so, I can have a simply a volume element here and that volume element is not a regular volume element so, it can be something like this. So, let me draw it so, maybe I can have an arbitrary volume element it looks like this and here I am having say my coordinates so, these are the coordinates.

And this is the volume element it is a very ugly looking volume element but this is a generalized version I do not have any spatial geometry here just the arbitrary thing and try to correlate with the so, what is e_1 then? From here to here the unit vector I need to do it in different colours so, this is the colour so, along this direction this is say, let us make this along this direction we have unit vector.

And along this direction we have unit vector. Suppose this is my e_1, this is say e_2 and this is e_3 so, this is the unit vector that is distributed along these coordinates and this is my you know u_1, the coordinate wise this is u_1 and this is u_2 and this is u_3. So, what are the surface element that it is better that we can find what is the surface element here. So, this is so, along this direction we have i.

So, the distance first we try to find out the distance from here to here, from here to here, from here to here because the scaling factor is there. So, the distance if I calculate so, from here to here the distance is this ds_1 this is the distance and this distance is scaling factor multiplied by du_1. Every time you need to multiply the scaling factor when you calculate the distance this one is ds_2, which is scaling factor du_2. I am talking about this distance.

And finally here or this one maybe it is ds_3 is equal to scaling factor h_3 and then du_3 from this reduction it is u_3. Now with this coordinate system if I define the gradient of over a function how it look like that is the first thing we will do today.

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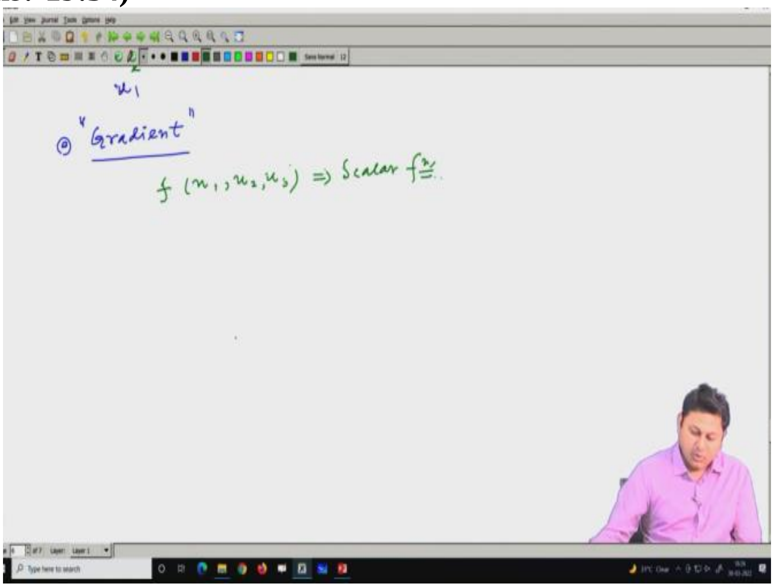

So, I will try to find out the gradient the operation and we know the gradient operator how it operates, but for curvilinear coordinate system what is the form that we want to check. So, let us start with the scalar function, f (u_1 u_2 u_3) so, this is my scalar function or scalar field.

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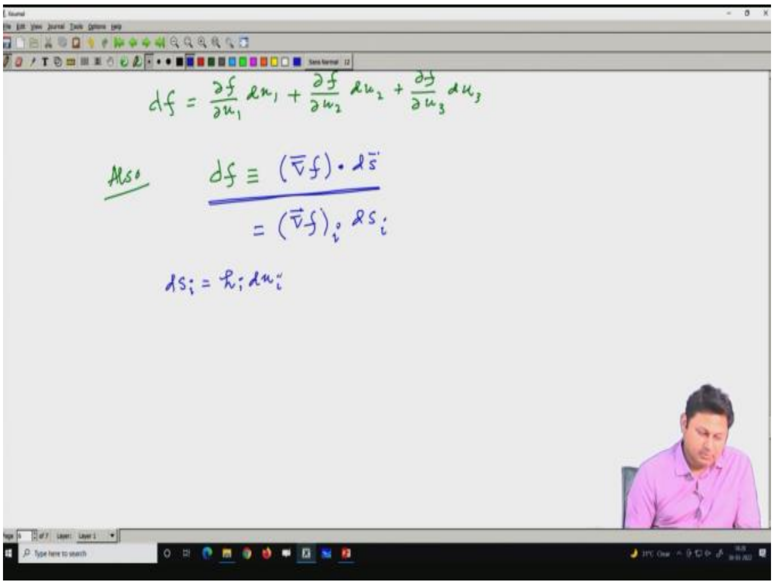

Now, what is df here? We know it. This and also we know that df is equivalent to $(\vec{\nabla} f) \cdot d\vec{s}$ that we know this is the identity, this is the fundamental identity we have. So, keeping that in mind I can have this if I expand what I get? I get I have the gradient and the dot product. So, in Einstein Notation it is i and ds_i what is ds_i? It should be h_i and then du_i.

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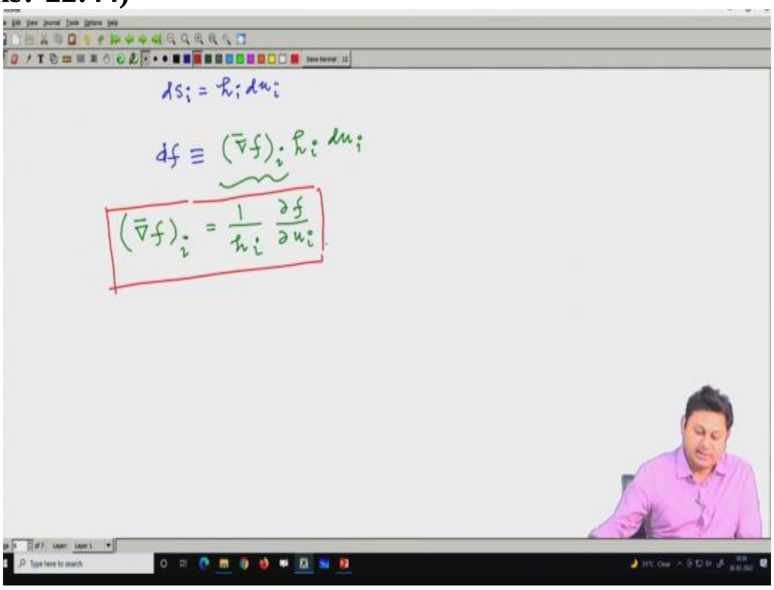

So, df if I find whatever is written here df is ith component and ds I write it as my h_i du_i. So, what is the ith component? What is this then? If I write this the gradient of the ith component is simply $\frac{1}{h_i} \frac{\partial f}{\partial u_i}$ so, this is the general way to define the gradient in curvilinear coordinate system this is the way one can define. Now, you can appreciate that you should appreciate that this h_1 if for Cartesian coordinate system what happened?

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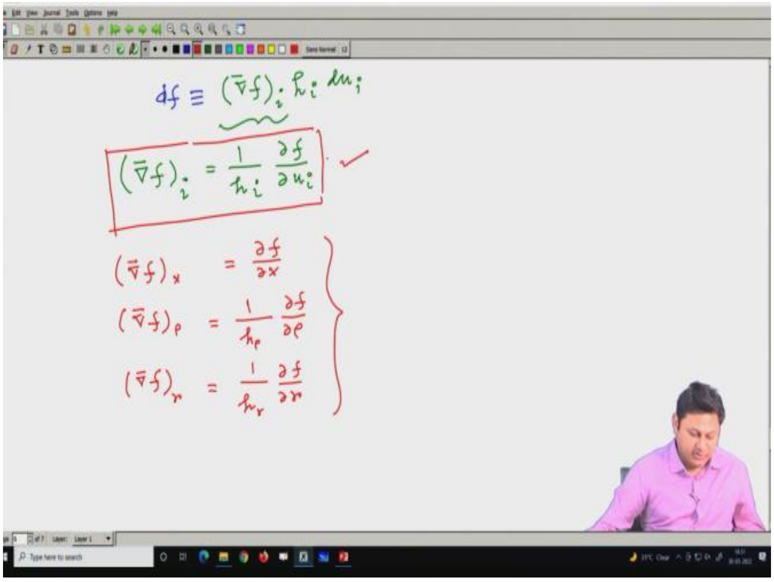

So, if I want to find out this quantity in x y z this system then for x it is simply $\frac{\partial f}{\partial x}$. I am just writing the component not the vectors here and whatever here y if it is y then it should be y. So, let us put otherwise it will be confusing, so, I am just taking the x component but it is the Cartesian coordinate system, this is the Cartesian coordinate system. Now, similarly, if I want to find out these in other coordinate system like what is this in ρ φ z.

So, I need to find out. So, the general form is I need to find out the h factor of ρ and then the function derived by ρ. In the similar way, if I want to find out in polar coordinates, in spherical coordinate then I need to find out the scaling factor here and the scaling factor is h_r and it should be $\frac{\partial f}{\partial r}$ so, this is the way we should proceed, but you should remember that the general form is this one.

Now, next after that I like to extend this idea for divergence how to calculate the divergence? It is a little bit lengthy calculation but, straightforward. So, let me check how much time I am having. So, let me quickly show before that let me quickly because we are here quickly show how to convert how to find this h_ρ we are talking about and we already had these information in our hand. So, how h factor one can calculate.

So, I mentioned that if we know the relationship with x y and z with u_1 u_2 u_3 then you can calculate from this you can calculate h_1 h_2 h_3 for different coordinate system. So, for let us do that for cylindrical coordinate system. So, that let us do this exercise whatever the time we have.

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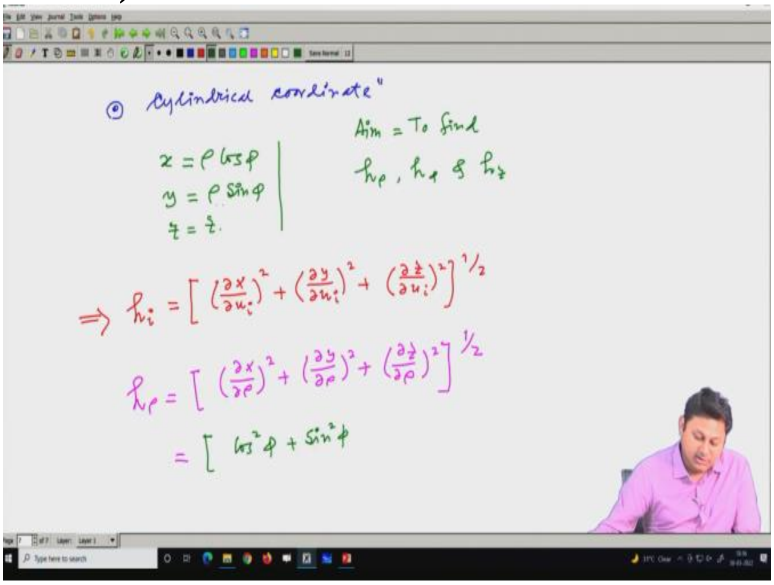

So, for cylindrical coordinate system we have the relation and that relation I am going to exploit. x = ρ cos φ, y = ρ sin φ and z = z. What is my aim? To find h_ρ h_φ and h_z these are the scaling factor of the cylindrical coordinate system these scale factor I want to find, this is my aim and how we find that? Because I know the general I mean I know how to find h_i in general h_i is this one $(\frac{\partial x}{\partial u_i})^2 + (\frac{\partial y}{\partial u_i})^2 + (\frac{\partial z}{\partial u_i})^2$ whole to the power $\frac{1}{2}$ this is the recipe I am having.

So, this recipe going to exploit here. So, what is now the first thing I calculate is h_ρ. So, h_ρ will be how much? Let us calculate. So, the first thing is this. So, it will be $(\frac{\partial x}{\partial \rho})^2 + (\frac{\partial y}{\partial \rho})^2 + (\frac{\partial z}{\partial \rho})^2$ whole to the power $\frac{1}{2}$ according to recipe I should write it now, what is this quantity? This is known because I know the explicit dependence.

So, if I do if I make a derivative, so, x is ρ cos φ, so, if I make a derivative with respect to partial derivative, so, this value is $\cos^2\varphi$ then I have $\sin^2\varphi$ because I know the relationship with y and ρ, x and ρ φ y with ρ and z and finally z, but you can see that there is no dependency over ρ in this.

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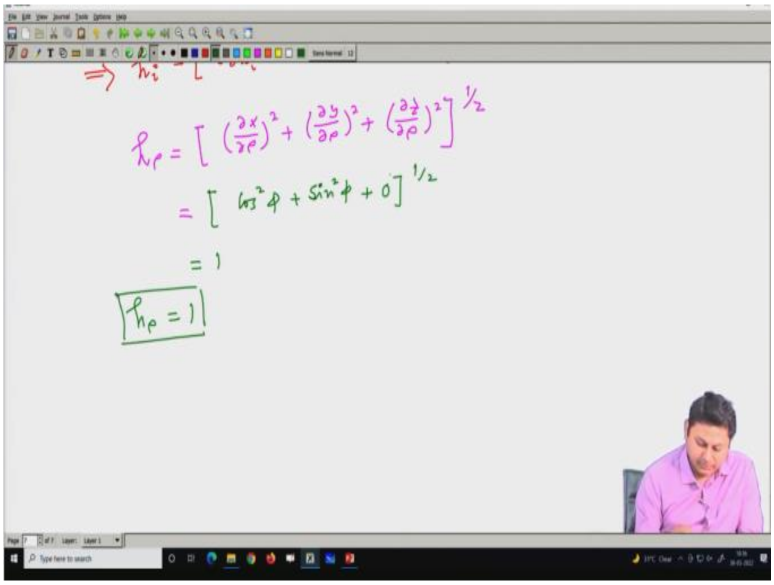

So, I can have this plus 0 whole to the power $\frac{1}{2}$. Now, this quantity is simply 1. So, the first thing I find here is $h_\rho = 1$ that is the first thing I find.

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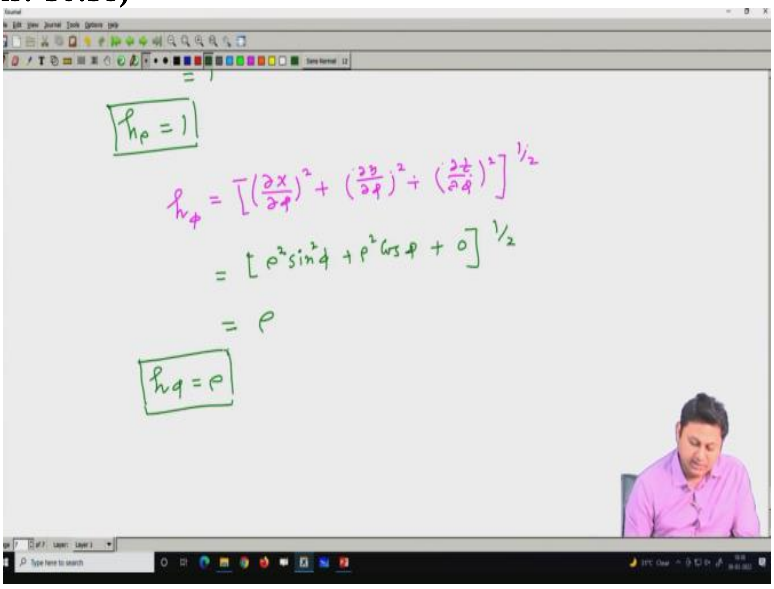

Next I need to calculate what is h_φ? h_φ is simply $(\frac{\partial x}{\partial \varphi})^2 + (\frac{\partial y}{\partial \varphi})^2 + (\frac{\partial z}{\partial \varphi})^2$ that you just go with the recipe that is shown there then hold to the power $\frac{1}{2}$. By doing this what do we have let us check so, I need to make a φ derivative with respect to φ derivative here. So, x is there, so, I am getting the value this. So, it should be ρ^2.

And when you make a derivative with respect to partially derivative φ cos becomes sine, so, it should be sine square with a negative sign but square is there, so, I will get simply $\sin^2\varphi$. This value will be $\rho^2 \cos^2\varphi$ and again the last value is 0. So, to the power half so, we are having now interesting value here and that is $h_\varphi = \rho$. So, the next thing I am getting that $h_\varphi = \rho$.

The scaling factor is no longer become 1 here previously for h_x h_y h_z or h 1 2 was 1 for Cartesian coordinate system, but here this is not the case.

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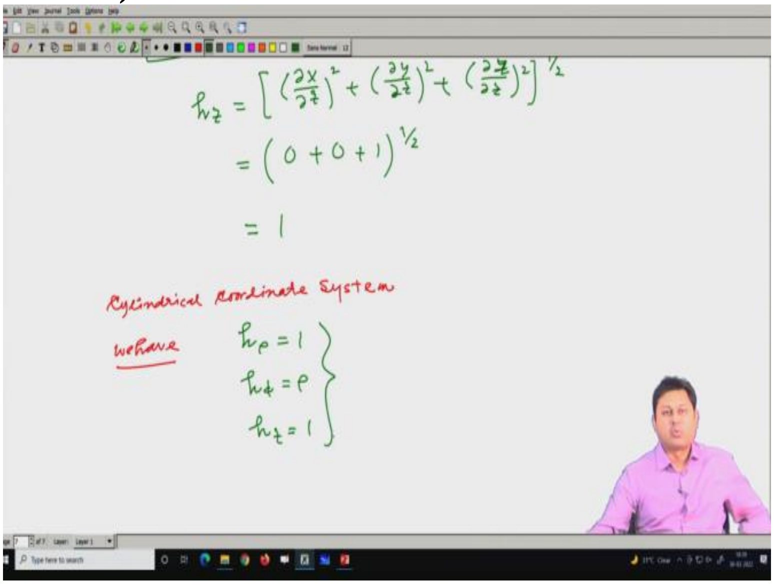

And finally, what is h_z very straightforward because z = z. So, I mean even if you do rigorously will get 1 but from just looking the expression you can get the value 1 square, square sorry this is z^2 whole power $\frac{1}{2}$. So, this value is 0, this value is 0 and this value is 1 whole to the power $\frac{1}{2}$. So, you will get simply 1. So, for cylindrical coordinate system I evaluate we have what we have this $h_\rho = 1$ these are the scaling factor $h_\varphi = \rho$ and $h_z = 1$.

So, this is the scaling factor we are having for cylindrical coordinate system. I do not have much time to you know elaborate that. So, in the next class maybe we can find out the scaling factor for the spherical polar coordinate system the process will be exactly identical and you find some values we will find some values and exploiting that values now, the next thing is to how to understand what should be the form of this operator the divergence operator.

And then the curl operator and the Laplacian how these operators Laplacian operator, divergence operator or curl operator will look like in different coordinate system from this general expression that we are going to exploit. So, already I figured out the h_i for cylindrical coordinate system. So, you know that if I want to find out here so, these values if I want to use, so what value you will get?

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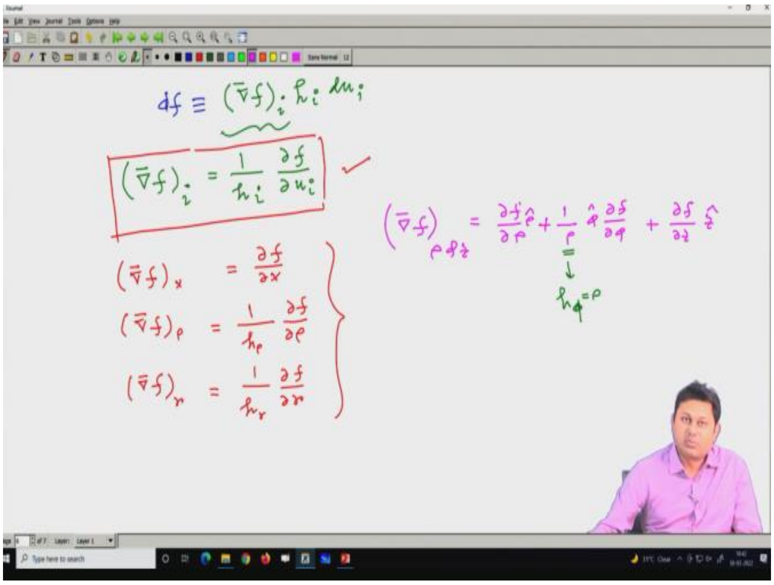

This I now calculate in cylindrical coordinate system ρ φ z the scaling factor is now known. So, in the first value, it should be simply $\frac{\partial f}{\partial \rho}$. What is the second value? Because h_ρ is 1 what is the second value? Second value with the unit vector $\hat{\rho}$ and then what is the second value? Second value is $\frac{1}{\rho}$ then $\hat{\varphi}\,\frac{\partial f}{\partial \varphi}$ this $\frac{1}{\rho}$ term is coming.

Because of the fact that h_ρ h_φ sorry is equal to ρ that we just calculate and finally, again the last term which is 1 again h factor 1 so, I have $\frac{\partial f}{\partial z}$ with unit vector z. So, this is the form of the gradient operator in cylindrical coordinate system. You can do that also for spherical coordinate system and you will see that the form is different, but in that case we need to find out the h_1 h_2 h_3 the scaling factor for this spherical polar coordinate system that we do in the next class.

So, I do not have much time today. So, with that note I like to conclude. So, thank you very much for your attention. So, see you in the next class, where we are going to expand more about this curvilinear coordinate system and trying to find out the scaling factor and eventually we will get the expressions of the curl operator and then divergence operator and the Laplacian operator. Thank you very much. See you in the next class.