Foundations of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology – Kharagpur

Lecture - 14
Curvilinear Coordinate System (Contd.,)

Hello students to the foundation of classical electrodynamics course, today we have lecture number 14 where we will going to continue the understanding of curvilinear coordinate system, which we started in the last class.

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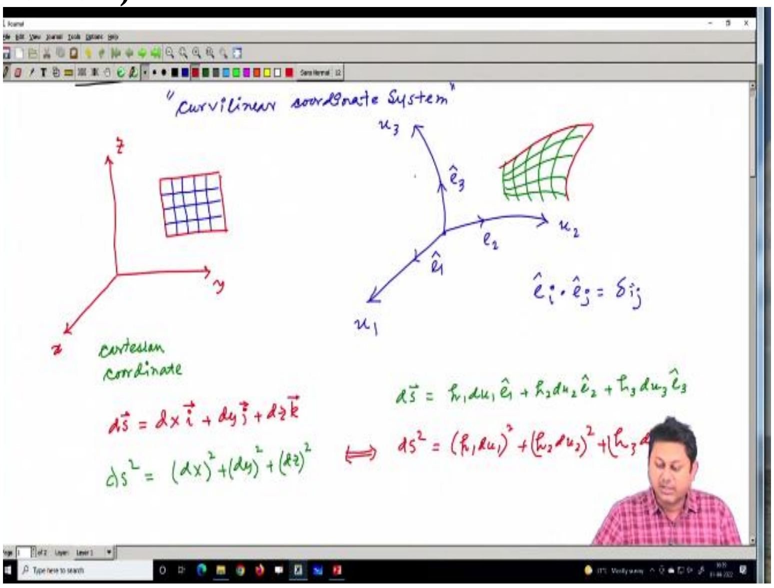

So, today we have class number 14. We are studying curvilinear coordinate system. So, in the last class we already mentioned that curvilinear coordinate system is a special kind of system where coordinates are defined in a curved space not like we have in Cartesian coordinate system in flat space. So, if I compare with the standard Cartesian coordinate system x, y and z. So, you know any plane if I draw any plane it should be a very systematic.

This is a surface element we are having and this is a flat surface where the coordinates are like this, each point is a coordinate point and we can have a flat surface like this. On the other hand, for curvilinear coordinate system, this is Cartesian coordinate system, in a curvilinear coordinate system we can have a surface, which is a curved surface in general. So, the surface should be something like this. So, I have like this and this and then we have coordinate points like this. This is just to show that how two surface differs.

We have a plane surface here and curved surface here. Like x, y, z here we have u_1, u_2, u_3 with the unit vector e_1, e_2 and e_3, a generalized form to write the unit vector because I am taking a generalized coordinate system. However, this condition is valid here, they are orthonormal system. So, in this coordinate system if I now in Cartesian coordinate system if I want to find out what should be my length element it is very straightforward and which is dx i + dy j + dz k.

However, if I want to find out the length element here in this coordinate system it is not like this we have a scale factor last day we mentioned that and it will be something like h_1 du_1 e_1 + h_2 du_2 e_2, e_1, e_2, e_3 are unit vectors and h_3 du_3 e_3. If I find out what is this value say ds^2 here we simply have $(dx)^2 + (dy)^2 + (dz)^2$. On the other hand, here if I calculate ds^2 then it should be $(h_1\, du_1)^2$square + $(h_2\, du_2)^2 + (h_3\, du_3)^2$.

So, this is the way we define the square of the length element and if I compare these 2 because Cartesian coordinate system is a special form of the generalized version of curvilinear coordinate system.

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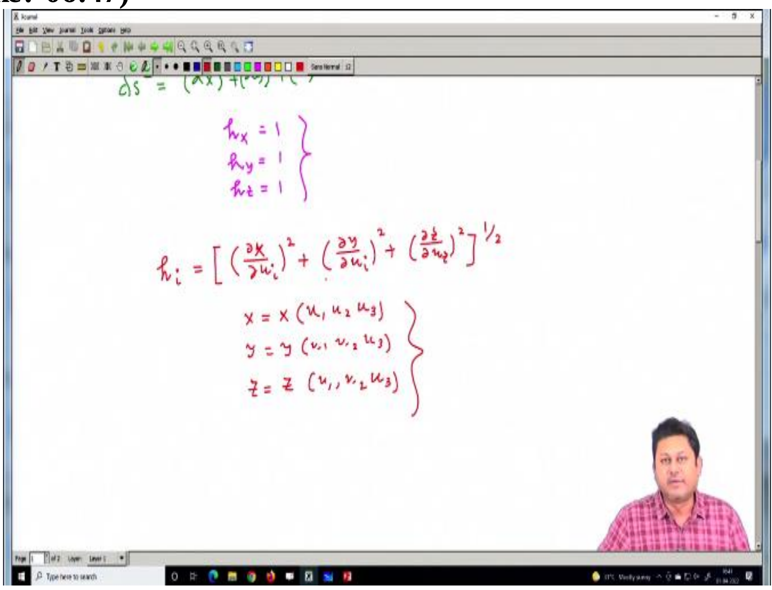

So, if I compare these 2 then I can have readily have the value of h the scale factor h_x, h_y and h_z is equal to 1. So, that is the value of the scale factor for Cartesian coordinate system that we derived last day. Also we have a recipe we used a recipe to derive this scale factor and that recipe also you can use and we calculated that this recipe and that recipe if I write once again that if I want to find out the scale factor h_i then that value should be figured out by using these expressions.

So, the function if you know in terms of u_1, u_2, u_3, y is a function of u_1, u_2, u_3 and z as a function of u_1 then you can figure out what is the value of this scaling factor last day we derived that, but also you can by comparison you can also find out the scale factor. So, for example, I can give you a simple thing like for already I compare here the Cartesian coordinate system with the generalized curvilinear coordinate system and these from this length square expansion I can find out the value of h_1, h_2, h_3 should be this one.

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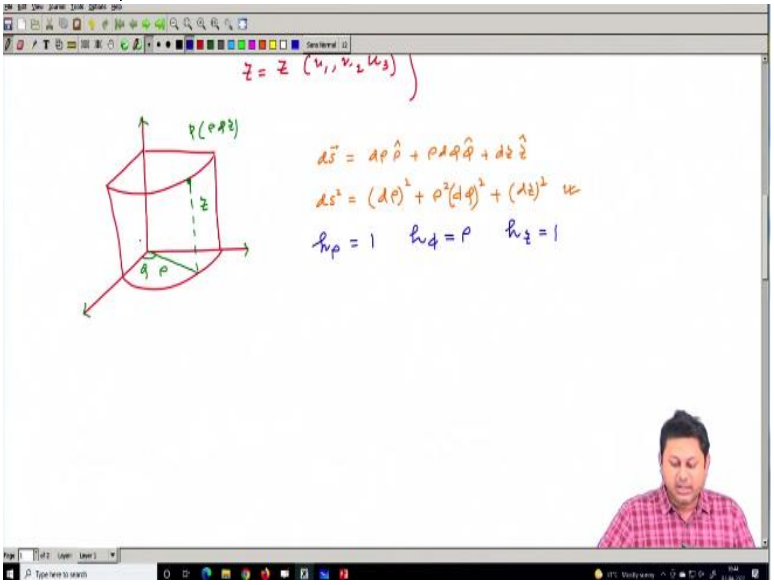

In the similar way, for cylindrical coordinate system, we have a system like this and any point here if this is ρ, this is φ and this is z. So, any point here P can be expressed in terms of ρ φ z and the distance element so, this is the coordinate and the distance element I can write simply ds is dρ $\hat{\rho}$ + ρ dφ $\hat{\varphi}$ + dz $\hat{z}$. Similar way if I calculate the square of this quantity by making a dot product with the same thing.

Then it should be simply the square then you find that it should be $(d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2$ and now, if I compare this equation with whatever the expression we have here that is this one this is the generalized expression we are having this is the generalized expression then if I compare then readily, you can see the value of h_ρ seems to be 1, the value of h_φ is ρ and the value of h_z is 1 exactly the same value we derived in our last class.

But we during that time we use this expression. You can do both the ways and now, this is for cylindrical coordinate system, but for spherical coordinate system we can do the same thing.

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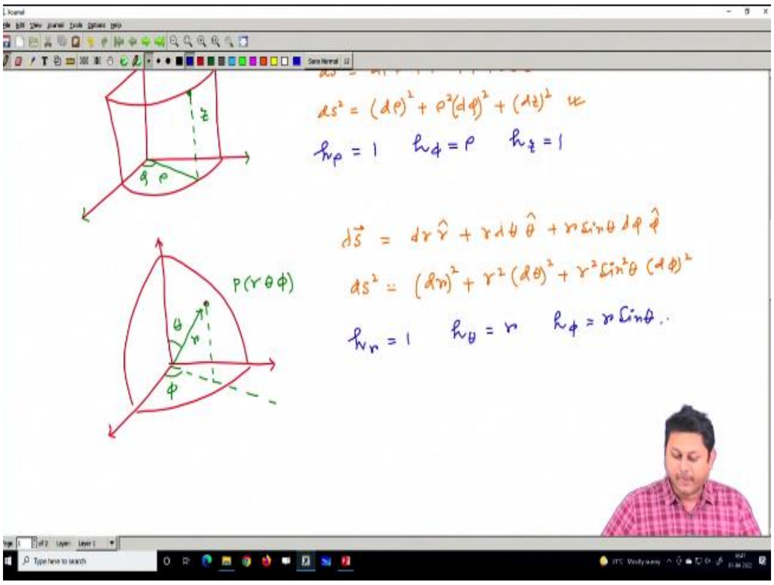

Let us simply draw the spherical coordinate system. So, any point here over some so, I can have r value here so, this is φ and this angle is θ and this is r. So, the value of P here whatever the point we are considering P, which is r θ φ for this coordinate system, if I write the distance element like the way we did we know that how to write the distance element in the different coordinate system. So, ds so, this value if I write it is dr $\hat{r}$ + r dθ $\hat{\theta}$ and r sin θ dφ $\hat{\varphi}$.

Again making this dot product we can have $ds^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\ (d\varphi)^2$. Now, again if we compare with this generalized expression for curvilinear coordinate system that we have here ds^2 so, I can compare and I find that h_1 in this case $h_r = 1$, $h_\theta = r$ and $h_\varphi = r \sin\theta$. So, you can also calculate this by exploiting this expression here say equation 1.

If I write this equation 1 you can also find out the value of these h_1, h_2, h_r, h_θ, h_φ by exploiting this expression 1, that is the thing we did last day. Also in the last class we figured out the expression of the divergence in curvilinear coordinate system.

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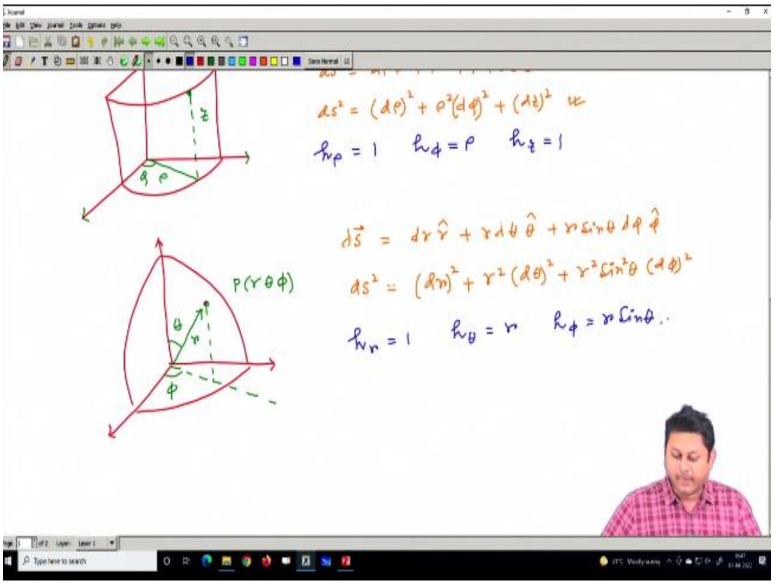

So, the expression of the divergence, not divergence we did the gradient, divergence we will do today, the gradient is in curvilinear coordinate system if I want to find out the gradient, the expression that we figure out is this. This is the operator operating on a scalar field f and if I want to find out the ith element it simply gives us h_i and then $\frac{\partial f}{\partial u_i}$. This is the expression we figure out this is the generalized expression.

Now for these 3 coordinate systems I know the scaling factor starting from the Cartesian coordinate system here I know the scale factor for cylindrical coordinate system and spherical coordinate system. So, using that you can figure out what should be the expression of the gradient in these 2 specialized coordinate systems, cylindrical and spherical. So, this is a most generalized form so you can exploit that.

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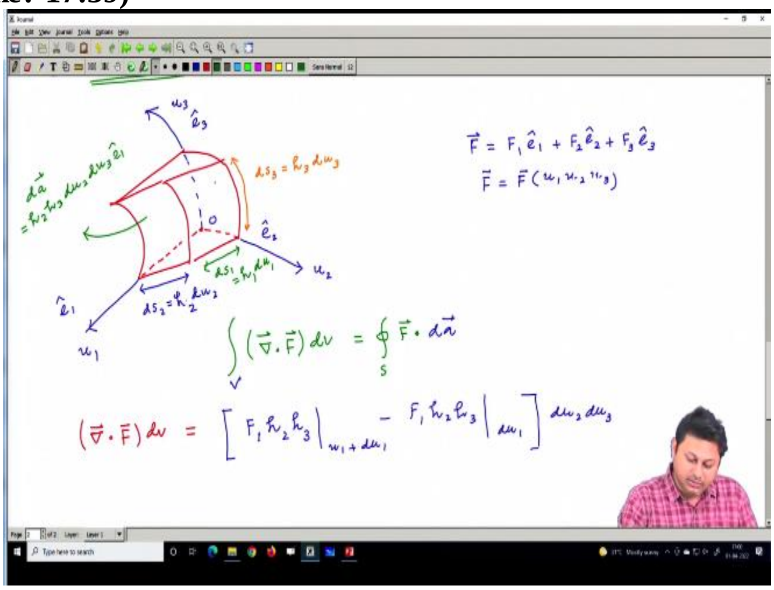

So, today we will extend this thing and try to figure out the expression of so, today we will calculate the expression of divergence. How one can express divergence in you know this curvilinear system? So, let me draw once again this figure for curvilinear coordinate system so, the figure means this the volume element that we draw last thing. So, I am using this so this we are looking system where I should have one line here also. So, the coordinates are like this and here I should so this is my u_1 having unit vector e_i.

That is my u_2 having unit vector e_2, that is my u_3 having unit vector e_3 along this direction and if I know calculate these lengths so from for example here to here this length is ds_2 and that value is h_1 with the scaling factor that is the important du_2. This length from here to here is ds_1, which is h_1 du_1 and this length is ds_3 is h_3 and du_3 that we already defined last class and this is the origin O. Now, we will be going to this is the coordinate system I am having and now I want to find out what is the expression of the divergence.

So, let us go back to the you know the theorem that we had that the divergence theorem and that is if I have a divergence of a vector field say F here which is a vector field and then integrate it over a volume then that value this value should be equivalent to the closed surface integral of $\vec{F} \cdot d\vec{s}$ this is a surface not the length element mind it so better I just removed this to make it surface otherwise there might be some confusion.

So, this is da mind it this is the da, which is the surface element. That is the expression of this is the volume integration and this is a surface integration and that is basically the divergence identity that the divergence rule that we proved. And now, I am going to exploit this to find out this expression. Now, this expression left-hand side if I write and then right-hand side I calculate for this given volume and then we will find what is there.

So, left-hand side if I write simply divergence of F and dv this should be the flux and I am just calculating this flux and now going to integrate and calculate this on the right-hand side so, I have this volume element, whatever the volume element is shown here, so, I need to integrate it over this surface whatever the surface so, again, we have 6 surface here and for this 6 surface if I calculate the right-hand side that I am doing, I am going to calculate this for this surface and calculate the right-hand side that is this integral.

So, the value to first I calculate the value of say F is a vector field and this vector field if I defined here it is F_1 and then unit vector e_i + F_2 then unit vector e_2 and F_3 unit vector e_3. F is a function of the generalized coordinate here whatever u_1, u_2, u_3 it may be x, y, z, it may be ρ, φ, z, it may be r, θ, φ, I am just writing in a generalized way. Now, when you calculate the divergence for each surface you need to calculate.

And if I do then for the first surface, I will have F_1, F_1 is along this direction in this direction. So, I will take account this surface whatever the surface is here. So, for this surface, I have this length element and ds_1 and ds_3. So that we will going to exploit that I will going to put, but the scaling factor is there. So, I need to calculate the scaling with this scaling factor here this should be h_2.

So, F_1 then this h_2 h_3 at which surface? The surface that is here, so, this surface is at $u_1 + du_1$ at this point. Then another surface the opposite one is sitting here in this region, so I can write F_1 the negative sign is because, now the direction of the surface is reversed. So, I should have h_2 this scaling factor h_3 this scaling factor, but I am going to evaluate at du_1 and then I have the surface because du_2 and du_3 is the surface element.

If I just multiply so, what is the surface element here for this particular surface I am just showing once the surface element da and that should be the vector sign and this value is simply h_2 h_3 du_2 du_3. In a similar way rest of the surface you can calculate. Now, this is a unit vector. So, this if I consider this one it should be a unit vector of e_i, this one opposite one this one should have a unit vector with minus e_i so that I am just writing here.

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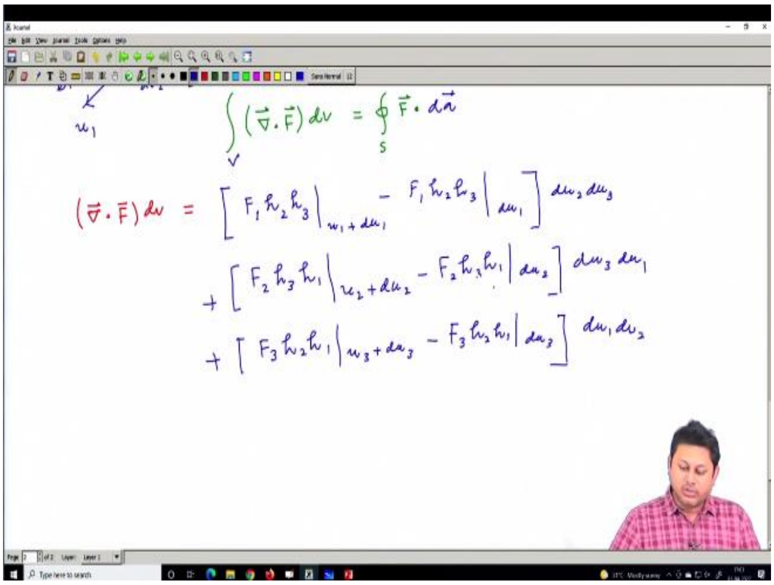

Then, I will add similarly, once you understand that the similarly there is thing is mechanical I should have F_2 3 1 evaluated at $u_2 + du_2$ point $-F_2$ h_3 h_1 evaluated at du_2 and the surface element I am calculating 2 so, it should be 3 and 1. The last one F_3 and h_2 h_1 evaluated at 3 3 and then I have a F_3 h_2 h_1 evaluated at u_3 with 1 and 2. Now, these I simply evaluate and the volume element now, if I understand the surface element the volume element simply let me write here.

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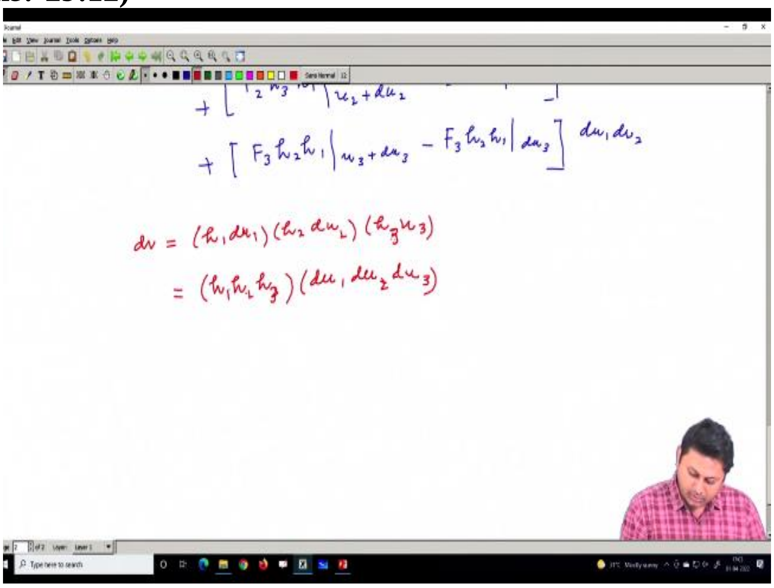

For this curvilinear system is simply h_1 du_1, h_2 du_2 everything is same except the scaling factor you just need to incorporate the scaling factor that is all u_2 and h_3 I am having an issue here with the pen h_3 u_3. So, eventually I should have h_1 h_2 h_3 multiplied by du_1 du_2 du_3. Now, after you know just doing this calculation, it is evaluated u + du_i and it is calculated at du_i. So, I can just expand this is a Taylor series so that if I do I will simply get this one. So, this is a volume element so, simply I will get this one.

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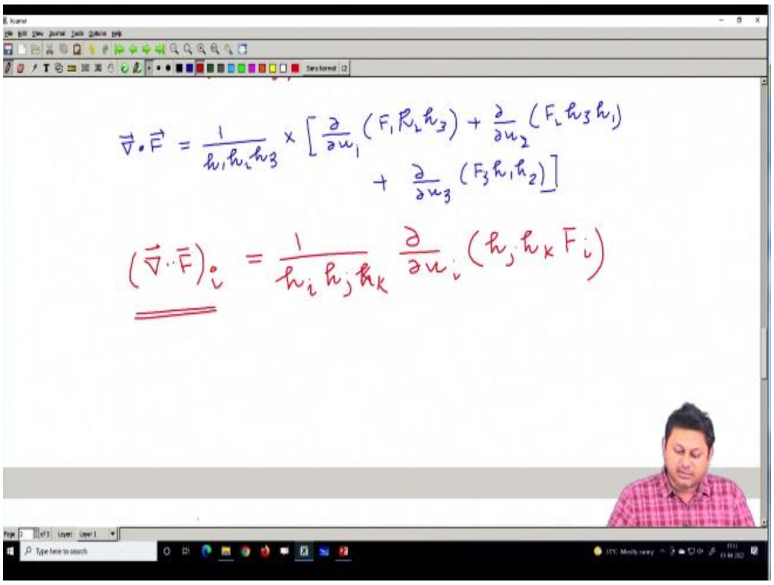

So, left-hand side I will get this dot and that is $\frac{1}{h_1 h_2 h_3}$ and this thing is multiplied by $\frac{\partial}{\partial u_1}$ (F_1 h_2 h_3). I do not know what is going on here and then plus $\frac{\partial}{\partial u_2}$ then (F_2 h_3 h_1) + $\frac{\partial}{\partial u_3}$ (F_3 h_1 h_2). So, I do not know I mean pen is not working properly anyway h_2. So, these things are important I mean once you have this expression then it is clear that in generalized form if you know h_1 h_2 h_3 then you can find out what should be the form of this divergence in any coordinate system.

You can see there is a significant difference between the Cartesian coordinates system to spherical polar coordinate system to this. Now, after that I will include another very important operator let me go to the next page.

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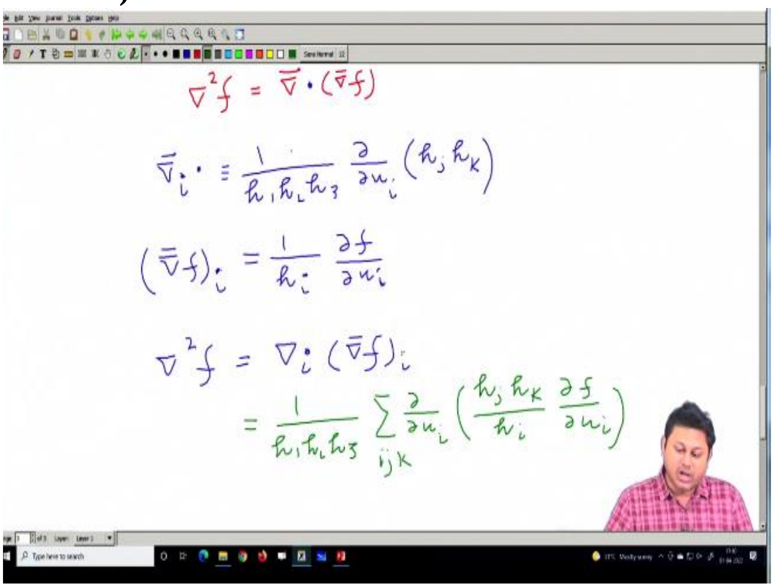

And that is Laplacian. So, the Laplacian operator is defined in this way, dot product here, it is not there is a dot here and then this operator is already calculated it is h_1 h_2, h_3 and then du_i just we calculate du_i the ith one and then h_j h_k if it is i, you can see that once you make a derivative one then the scale factor is 2 and 3 here if it is 2 then scale factor is 3 and 1 that is why if it is i then it will be j and k so that is the formula we are having.

Now, if I calculate if I find out what is the form of the gradient of ith element then it should be $\frac{1}{h_i}$ this is not and then $\frac{\partial f}{\partial u_i}$. So, if I combine these 2 I should simply have it is the ith of this and ith of this because this is the way defined it is i. And that thing is equal to $\frac{1}{h_1 h_2 h_3}$ and then sum over du_1 because they are i j k, so, I should write i here j here and k here. And the rest of the term what we have is $\frac{h_j h_k}{h_i}$, which is here and then $\frac{\partial f}{\partial u_i}$ this is the generalized form.

So, I will show you later in the next class maybe in an explicit way for different coordinate system how you find these values? So, Laplacian, you may remember that you can define the Laplacian by the gradient of a scalar field and then divergence of that. Now, I calculate both the expression here I calculate the divergence and the gradient last day. So, this is the expression of the gradient and this is the expression of the divergence. If I want to find out the ith value of the divergence.

If I calculate what is the ith form of the divergence then, it is simply this is i, j and h_k and then this is $\frac{\partial}{\partial u}$ if it is ith then it is i and it is operating over F so, it should be h then j k and F_i. This is the way we can define the ith component of this divergence operator and we mix up this thing here in calculating the Laplacian, if you mix this thing, then first you have this the divergence and then we have you know the gradient and if you put this since, we have a summation, because for every ith you have i j k that this will be there.

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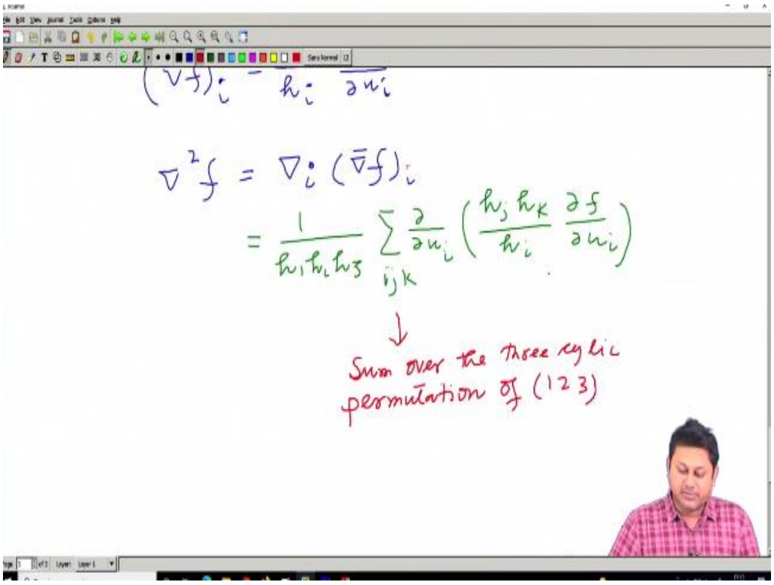

So, this is sum over the cyclic so I should write it here. This is sum over the 3 cyclic permutation of 1, 2, 3 the cyclic permutation should be so once it is 1 2 3 then 2 3 1 then 1 2 3 and then 3 1 2. So, today I do not have that much of time to discuss more about this Laplacian operator. Maybe in the next class I will define again I will discuss again the Laplacian and then curl and then just now finally, what should be the general form of all the coordinate system with these operators.

Gradient operator, divergence operator and curl operator, which we already defined earlier. Again, we will do but in the form of in the generalized way by just understanding the curvilinear coordinate system. So, with this note I would like to conclude today's class. Thank you very much for your attention and see you in the next class.