Foundations of Classical Electrodynamics
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Lecture – 15
Curvilinear Coordinate System (Contd.,)

Hello students to the foundation of classical electrodynamics course. So, today we have lecture 15. And we will continue with the curvilinear coordinate system that we started couple of classes ago. We try to understand the different operation how it looks like in curvilinear coordinate system in generalized way.

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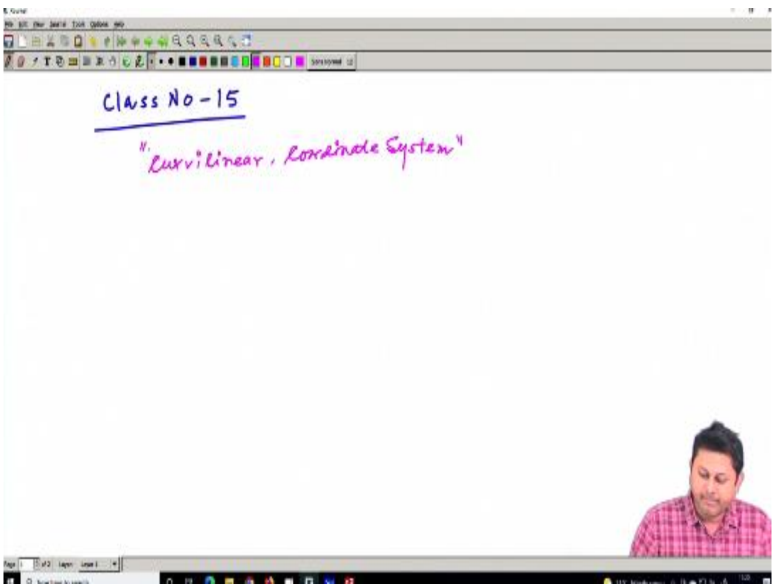

So, today we have class number 15 and doing the same thing curvilinear coordinate system. So, today we will try to understand we already figured out how the Laplacian is defined, how the gradient is defined and you know how the divergence is defined. So, today we will understand how the curl is defined.

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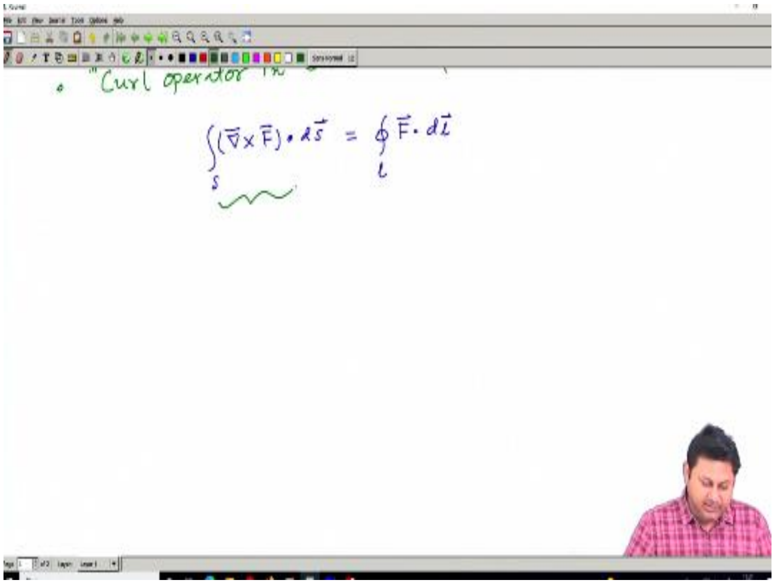

So, the curl operator in curvilinear system curl operator. So, we know again we will go to exploit that expression that you know this integration if F is a vector field and then if I do a surface integral of this quantity, then what we have here the right-hand side I have a closed line integral and the line should be such that it encircling the entire surface. So, that means, if I want to find out the curl of these things then simply what I do I need to find out the line integral for given F, F is same let me draw the coordinate system once again the same old figure.

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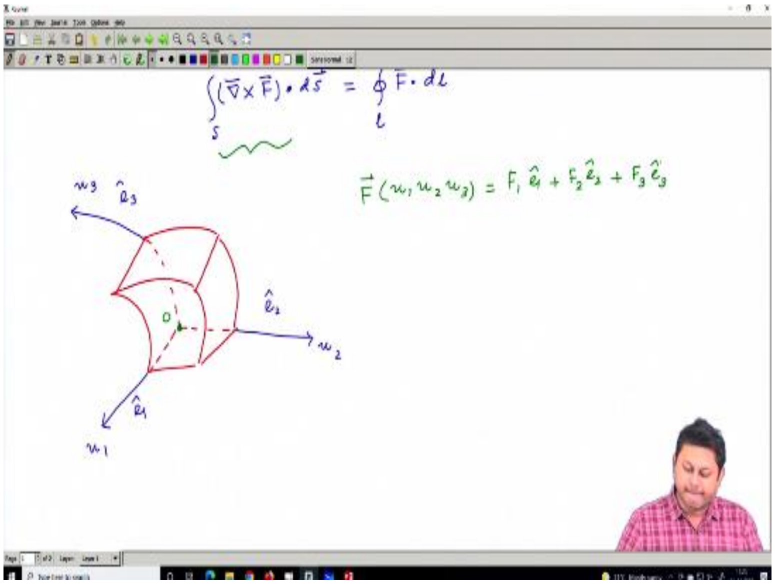

A volume element in curved space. This is u_1, this is u_2, this is u_3, e_1, e_2, e_3 and this is my origin here, this point o. So, this is the element we are having, volume element we are having here. Now, as the way we did in the previous class we will proceed in the same way and first we calculate this quantity that first let me define F the vector field and the vector field F, which is a function of u_1 u_2 u_3 is simply defined as F_1 unit vector, F_2 unit vector and F_3 unit vector this is my vector field well.

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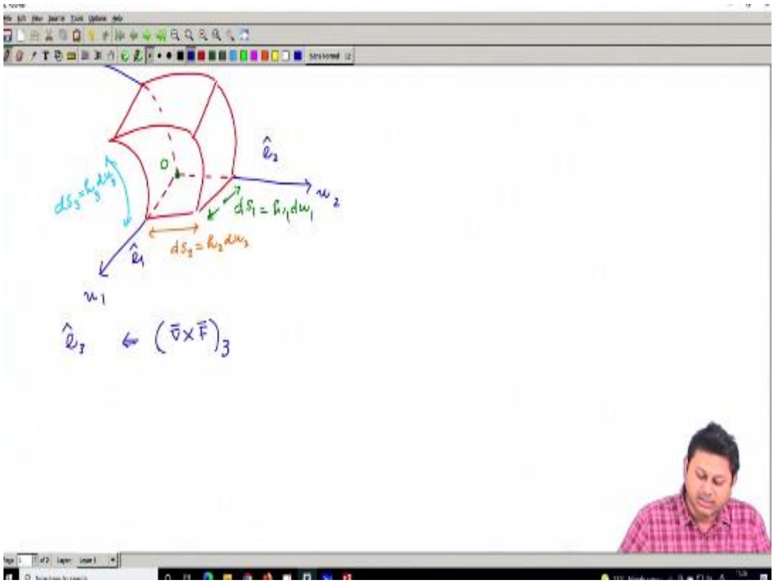

So, the distance element that we also write here I am writing once again. So, this is d this distance is ds_2 and that value is h_2 du_2, this distance from here to here is ds_1, which is equal to h_1 and du_1 and finally, this length should I use another colour, this length here to here is ds_3, which is equal to h_3 du_3 this is the length, length element. Now, I calculate this quantity, I calculate $\vec{\nabla} \times \vec{F}$ and maybe I calculate 3rd component because this is a vector quantity I am just calculating the third component.

So, that I can exploit the, you know these this is a third component so, that means it should be along e_3 direction. So, I can exploit it if I calculate this I can exploit this area, which is associated with u_1 and u_2. So, that means the area we are having here is this. On the left-hand side, I have area

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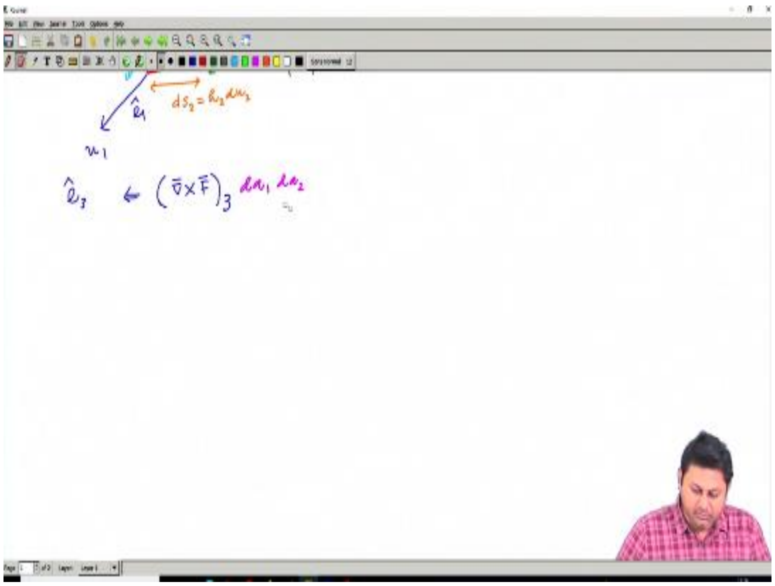

So, I am making a different notation here because area is also normally defined with s but I am making da_1 da_2 sorry, area should be, area I can simply write in terms of distance. So, I do not need to do that actually.

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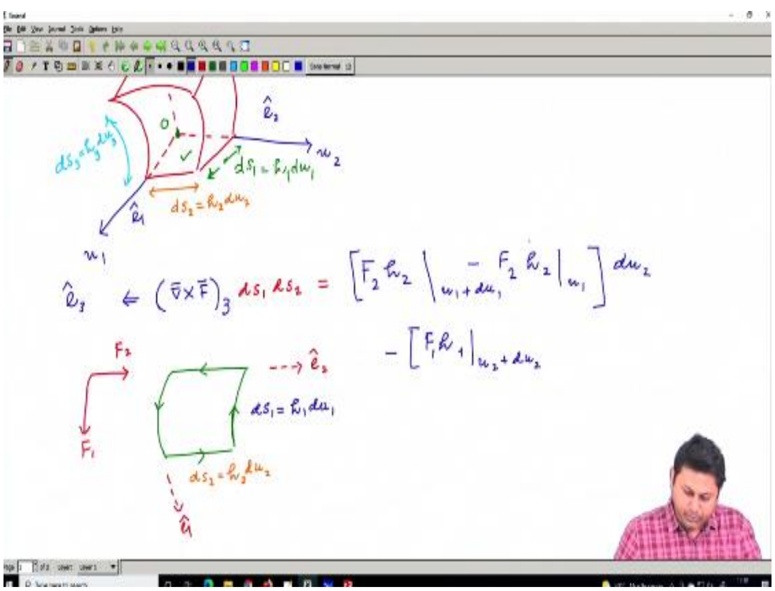

So, simply I write ds_1 distance ds_2 that gives me the area. In the right-hand side, what we have? In the right-hand side, I have this closed line integrals. So, closed line integral means I need to find out first the surface and then line. So, the surface if I draw here it should be something like this, curved surface and this surface is this one I am talking about this surface so, here I need to go with the direction as well because I am calculating the closed line integrals. So, direction is important.

And along this direction, F if I find out along this direction, this is F_2 and along this direction, this is F_1 because this is the direction of e_2 and this is the direction of e_1 unit vector and I have the surface element here where I can calculate the line integral, which is encircling the surface that means the boundary line. So, here already I calculate ds_2 and ds_1, which is this, these 2. So, again I am writing this, so ds_2 is h_2 du_2 and ds_1 I calculate here if you look so, ds_1 is h_1 du_1, I am now set.

Now I calculate this quantity, now this quantity I calculate so, first if I start with this point then definitely my right-hand side I should write, I calculate F_1 then where in this line so, h_2 is there and I need to execute as the point $u_1 + du_1$ and then I calculate this one this upper line, which is in the different direction opposite direction. So, I should have F_1 and then h_2, but I will execute at u_1 point. And then multiplied by the length element length element h_2 I already put here. So the rest part I am going to put and that is du_2 this portion.

In a similar way, I can do this portion and here if I calculate the first this one, so that is in the negative direction, so I write F and then h_1 F_1 because F_1 is this direction sorry, this should be F_2 not F_1 because F_2 is along with this direction, I am making a mistake here. So, F_2 and this one will be F_1 and F_1 and then it should be h_1 executing that, I am talking about this line. So, it is $u_2 + du_2$ let me check once again; this is this line. So, this is u_2.

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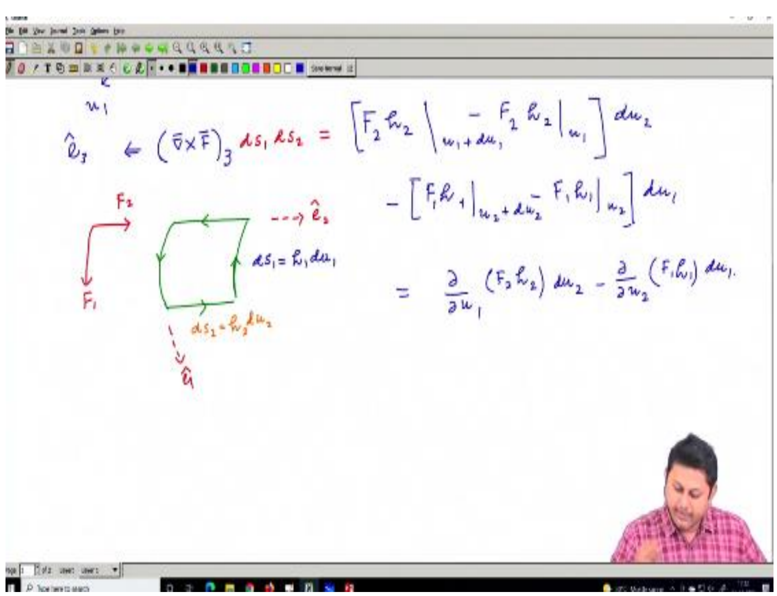

And then - F_1 h_1 executing at u_2 point and then du_1 this is the line. So, this if I calculate then simply it gives me this s_1 s_2 I put it here. So, I can simply get the partial derivative with respect to u_1 of F_2 h_2 and then minus of partial derivative with respect to and then there will be a multiplication of this quantity du_2 and then $-\frac{\partial}{\partial u_2}$ and then F_1 h_1 and that should be in multiplied by du_1.

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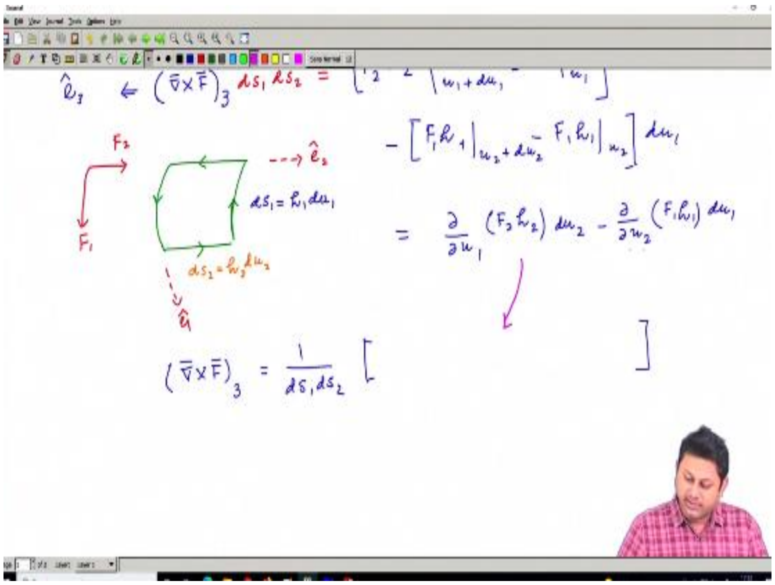

Now, if I calculate the $(\vec{\nabla} \times \vec{F})_3$, the third component, then it should be $\frac{1}{dS_1\, dS_2}$, and whatever I have here, this is this one, I am not writing the entire. So, this will put here and ds_1 and ds_2 is h_1 u_1 and h_2 u_2. So, if I put them this du_2 and du_1 will cancel out, and eventually I will get this one.

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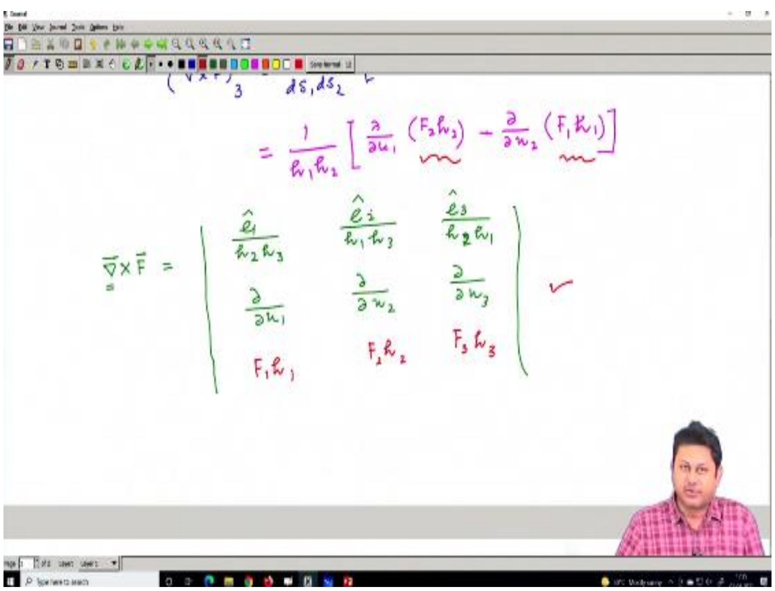

Eventually I will get $\frac{1}{h_1 h_2}$ and then $\frac{\partial}{\partial u_1}$ (F_2 h_2) - $\frac{\partial}{\partial u_2}$ (F_1 h_1), this is the third component. In the similar way, you can come to find out the first component and second component in general, if you do all the components and find out it should look like this. $\vec{\nabla} \times \vec{F}$ in curvilinear coordinate system or in generalized way one can writing this way e_1 unit vector divided by h I think it should be h_2, h_3. This division what we have h_2 h_3, e_2 unit vector divided by the scaling parameter 1 3, e_3 unit vector divided by the scaling parameter 3 sorry 2, 1.

And then partial derivative with respect to u_1 for this curl operator for form this one and this one and finally, whatever we have in here. So, here we have the field whatever the field associated with the scaling factor, field component associated with the scaling factor and field component associated with this scaling factor. So, this is the generalized way to write the curl of any coordinate system you can now put h_1 h_2 h_3 in different coordinate system because you know what is the value of h_1 h_2 h_3 for Cartesian coordinate system.

What is the value of h_1 h_2 h_3 for cylindrical coordinate system and spherical coordinate systems you need to use this value to find out what is the curl expression. Now I am going to write with this note I am going to write for 3 coordinate systems what are the values just list it.

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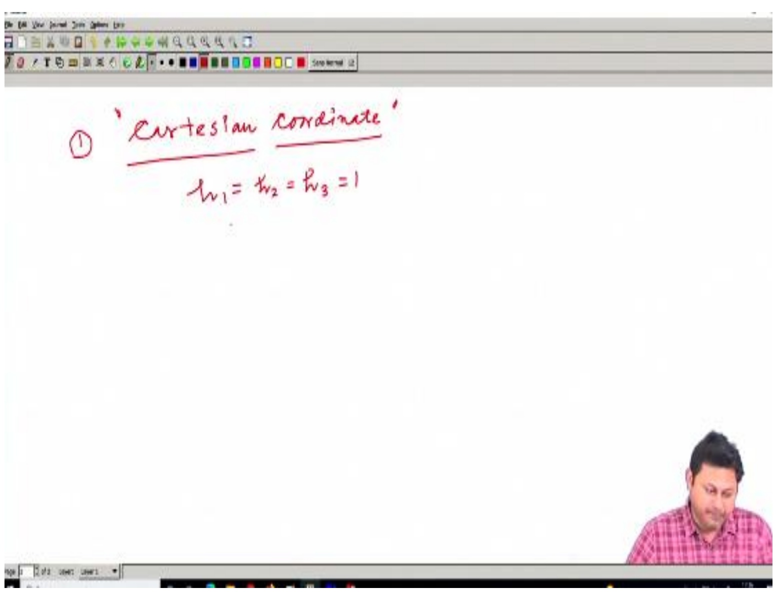

So, number 1 is for Cartesian coordinate system, which is very easy, for Cartesian coordinate, for Cartesian coordinate what we have is this, for Cartesian coordinate system we have $h_1 = h_2 = h_3 = 1$ because h_1 h_2 h_3 is 1 here.

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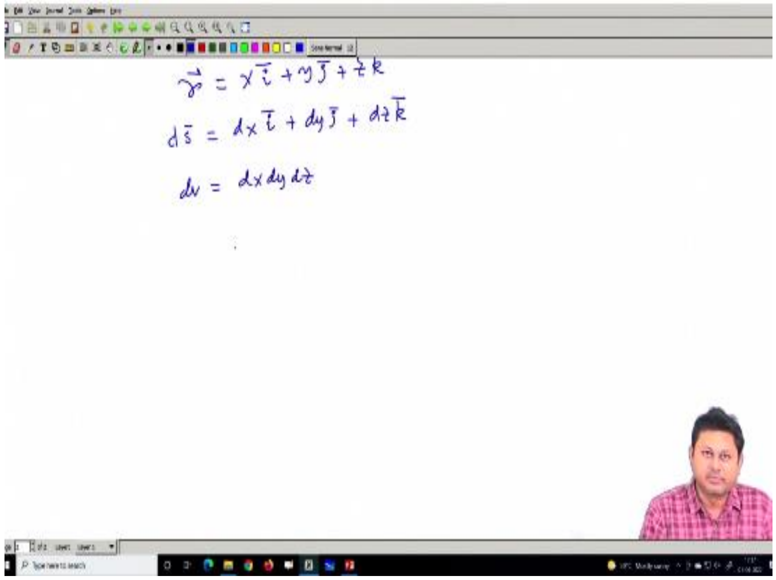

What is the r vector, which is x $\hat{\imath}$ + y $\hat{\jmath}$ + z $\hat{k}$ what is d distance vector important and that is dx $\hat{\imath}$ + dy $\hat{\jmath}$ + dz $\hat{k}$, each h factor is 1 that is why life become very simple here. What is dv, the volume element? The volume element is simply dx dy and dz this is the volume element.

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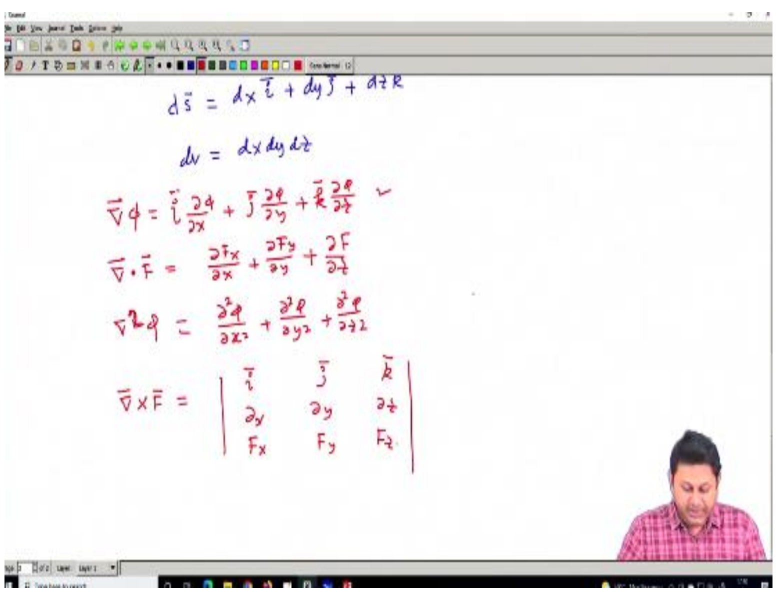

Now, what is the operator? Simply if I have a gradient operator, which is simply i $\frac{\partial\varphi}{\partial x}$ + j $\frac{\partial\varphi}{\partial y}$ + k $\frac{\partial\varphi}{\partial z}$. What is the gradient of a scalar field F, which is also very simple. No this is a scalar quantity so I should not have i here it is $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$. What value of Laplacian we have, what is the expression of Laplacian? The expression of the Laplacian this is 2 is this it is $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ and curl simply i j k ∂_x ∂_y ∂_z F_x F_y F_z.

So, this is all these things are well known, so, nothing much to you know nothing much to remember here. But once you now go for a different coordinate system that is for cylindrical coordinate system.

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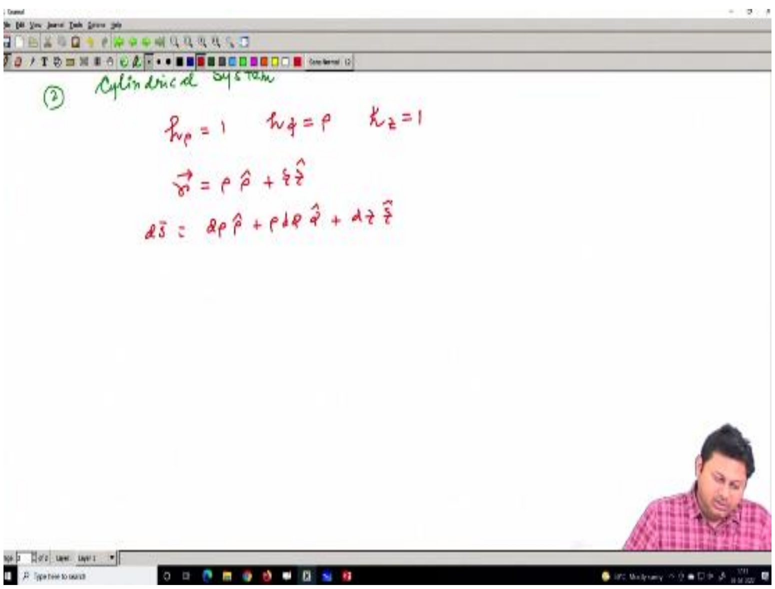

The next one 2 we have cylindrical system, in cylindrical system what we have is h_ρ for me maybe I can use a different colour to distinguish this. So, h_ρ is 1, h_φ is ρ that is a difference here and h_z is 1, r any position can be presented by ρ $\hat{\rho}$ + z $\hat{z}$. What is the distance element here which is dρ $\hat{\rho}$ + ρ dφ $\hat{\varphi}$ + dz $\hat{z}$ we derived it.

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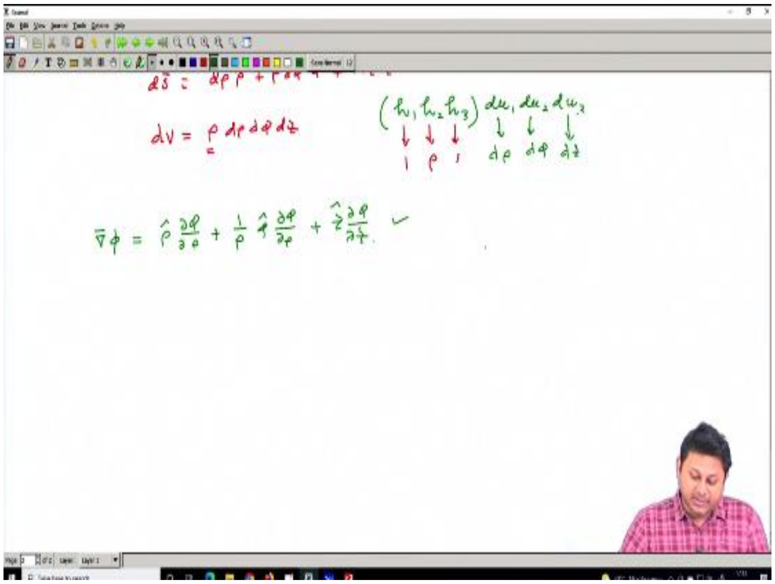

What is the volume element here? The volume element is ρ dρ dφ dz this is not dρ dφ dz, ρ is here because we have a scaling factor ρ seating here and in generalized way the volume element if I write the generalized volume element then if you may remember this is h_1, h_2, h_3 and then du_1, du_2 and du_3 here du_1 is dρ, du_2 is dφ and this is dz, h_1 is 1, h_2 is ρ and this is 1. Now, if I put all these things here you will find the volume element, which is this one.

Now, what is the expression of this gradient? The gradient expression is this $\hat{\rho}\,\frac{\partial\varphi}{\partial\rho} + \frac{1}{\rho}\,\hat{\varphi}\,\frac{\partial\varphi}{\partial\varphi}$ and then I have $\hat{z}\,\frac{\partial\varphi}{\partial z}$. So that we also derived and also you can check it with the standard by just putting the scaling factor and you will get the same result here.

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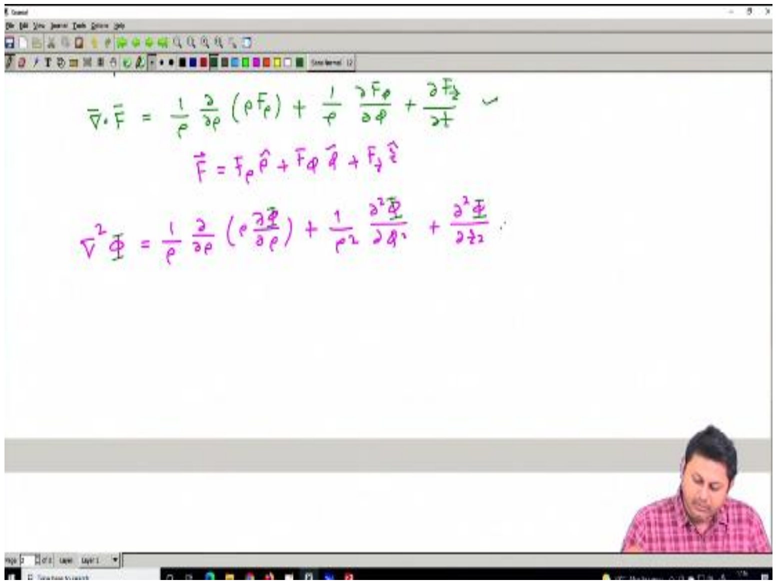

Next what is the expression of the divergence? Expression of the divergence is this $\frac{1}{\rho}$ and then $\frac{\partial}{\partial\rho}$ (ρ F_ρ) where F_ρ is the ρ^{th} component of the vector field F, then $\frac{1}{\rho}$ and then $\frac{\partial F_\varphi}{\partial\varphi}$ and then I have $\frac{\partial F_z}{\partial z}$. So, this is the way you can define the divergence in cylindrical coordinate system and finally, the Laplacian F mind it I should write it here F is defined in this way this is $F_\rho\,\hat{\rho} + F_\varphi\,\hat{\varphi} + F_z\,\hat{z}$.

So, the next thing that I like to show is the very important Laplacian over a scalar field Φ and it is $\frac{1}{\rho}\frac{\partial}{\partial\rho}$ and then ρ $\frac{\partial\Phi}{\partial\rho}$, a little bit complicated because 2 operational I am simultaneously doing here and if you go to the generalized form and try to understand you will find the result $\frac{1}{\rho^2}$ next term $\frac{\partial^2\Phi}{\partial\varphi^2}$ this I am now in order to distinguish let us make it Φ here, let us make a Φ here and Φ here because already we have a φ and then I have $\frac{\partial^2\Phi}{\partial z^2}$ here.

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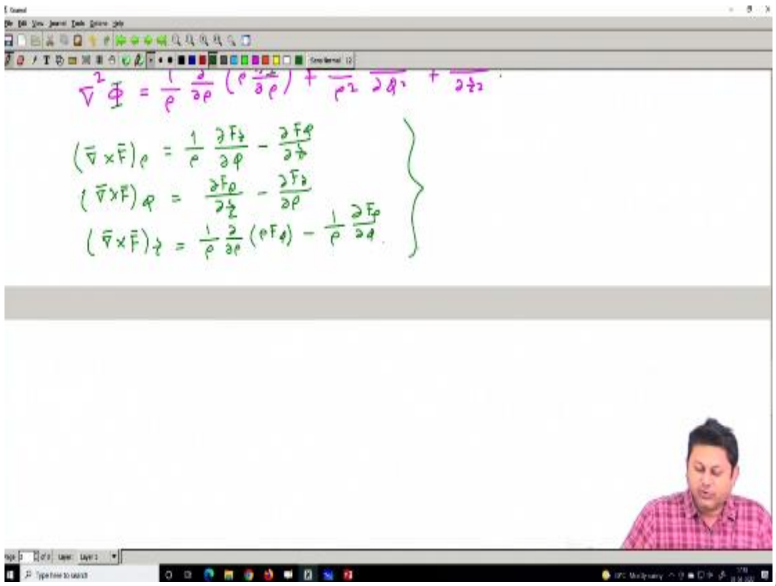 ρ

What are the curl elements that also you can find let me quickly write this. So, the $\vec{\nabla} \times \vec{F}$ in cylindrical coordinate system mind it ρ^{th} component it should be $\frac{1}{\rho}$ and then $\frac{\partial F_z}{\partial \varphi}$ and then $-\frac{\partial F_\varphi}{\partial z}$. What is the component of φ here? This is $\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}$. And finally this z this is a $\frac{1}{\rho}\frac{\partial}{\partial \rho}$ and then $\rho\ F_\varphi$ - $\frac{1}{\rho}$ and $\frac{\partial F_\rho}{\partial \varphi}$.

A complicated kind of expression you find because you are doing the curl but at least once you need to know that how it is coming maybe it is not that much useful. Because there are many terms you need to remember but at least you know that how it is coming that is the point. And then finally I will go for this cylindrical after cylindrical coordinate system.

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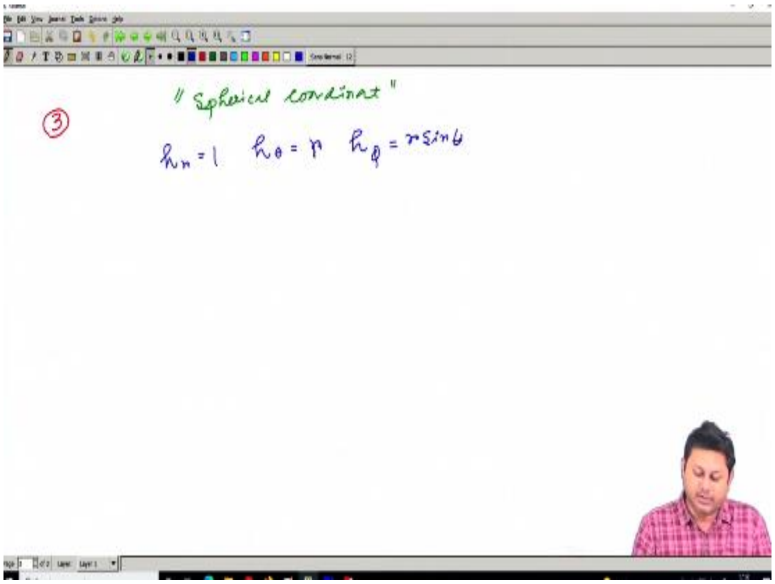

Finally, I am going in the spherical coordinate system how these things look like more complicated thing will be there because the scaling factors are now, here you can see once you have here the scaling factor is ρ because of these you are having all the complicated terms, it is coming in a complicated way because just 1 scaling factor is not equal to 0, but for cylindrical coordinate system there are 2 scaling factors, which are not equal to 1.

And that is why it creates not problem but things will be a little bit problematic complicated in terms of problem if I say it is problem that means, it is complicated not so, spherical coordinate. In spherical coordinate, what value we are having? So, we are having $h_r = 1$, $h_\theta = r$ and $h_\varphi = r \sin \theta$ that we are having.

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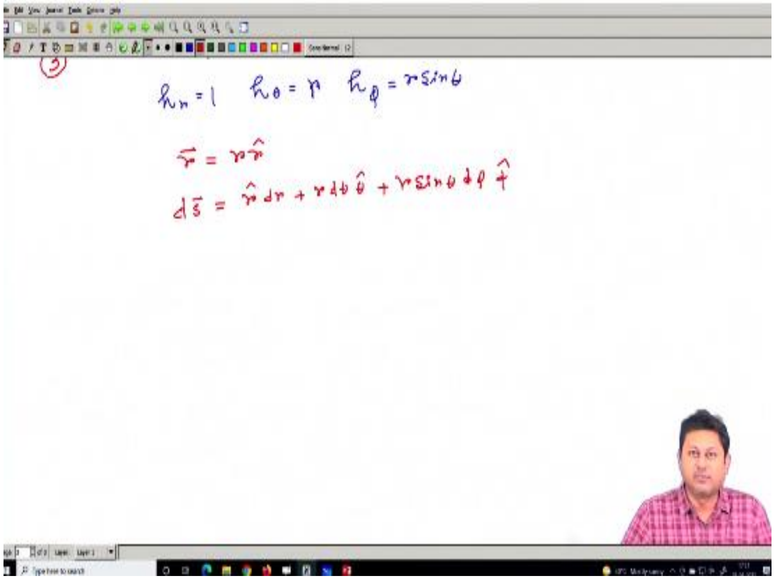

So, the position vector if I want to express it should be simply $r\hat{r}$, displacement or distance vector. So, this is $\hat{r}$ dr + r dθ $\hat{\theta}$ + r sin θ dφ $\hat{\varphi}$. This is the way we define the distance these things.

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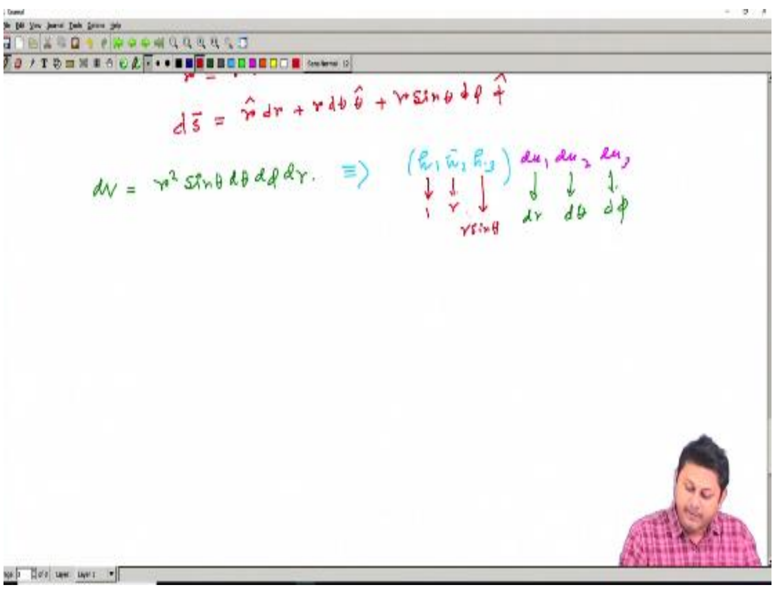

What is the volume element dv? dv is r^2 sin θ dθ dφ dr. Again I like to make you note that in spherical, in Cartesian coordinate, in cylindrical coordinate system if you have the volume element it was h_1 h_2 h_3 and then you had du_1 du_2 and du_3. Now, du_1 here is dr, du_2 here is dθ and this is dφ. On the other hand, h_1 here is 1, h_2 here is r and h_3 is r sin θ. Now, if you put all this together you will find that this is nothing but this one r^2 sin θ dr dφ dθ. So, now what about the gradient, how the gradient going to operate?

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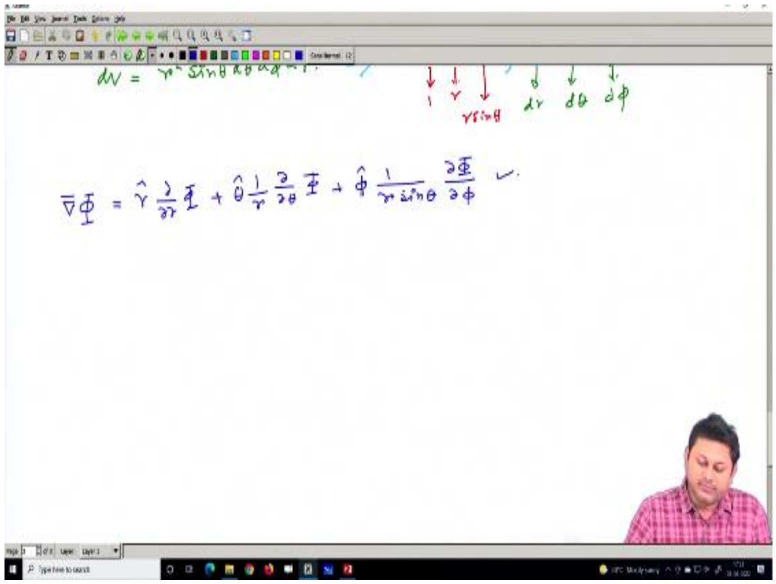

In this coordinate system, so, it is simply r then $\frac{\partial}{\partial r}$ Φ. Let us put Φ again otherwise I will have problems because here we have a φ variable and then $\hat{\theta}\ \frac{1}{r}$ and then $\frac{\partial}{\partial \theta}$ Φ and then I have $\hat{\varphi}\ \frac{1}{r \sin\theta}$ and then $\frac{\partial}{\partial \varphi}$ Φ. So, this is the way one can define the gradient.

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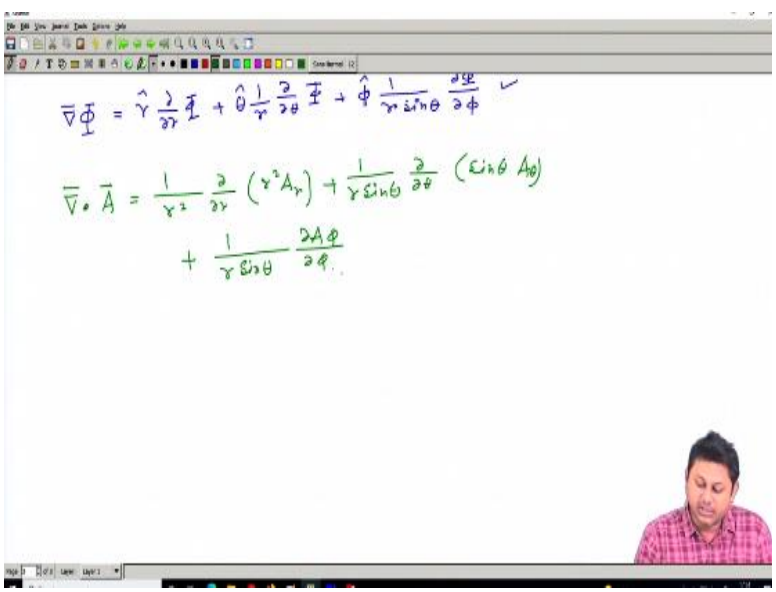

What about the divergence? A vector field say, A so, it is $\frac{1}{r^2}\ \frac{\partial}{\partial r}$, a complicated kind of things because of the scaling factor we are having here 2 scaling factor is not equal to 1, A_r + $\frac{1}{r \sin\theta}$ and then $\frac{\partial}{\partial \theta}$ and then sin θ A_θ. And then finally, $\frac{1}{r \sin\theta}$ this is $\frac{\partial A_\varphi}{\partial \varphi}$. So, I also request you to please kindly check whatever I am writing maybe there is some error because I am writing so, please check it from the standard book.

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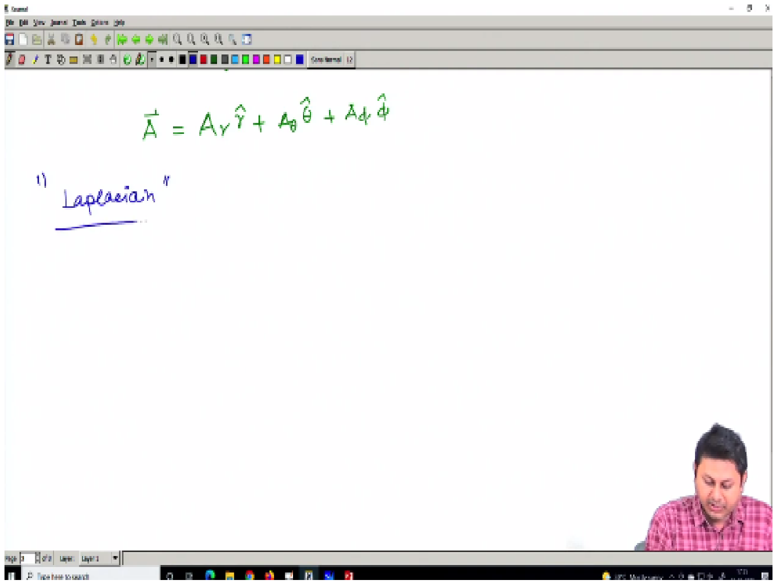

A is defined like $A_r\ \hat{r} + A_\theta\ \hat{\theta} + A_\varphi\ \hat{\varphi}$. Now, the most important Laplacian, how the Laplacian we are going to define here.

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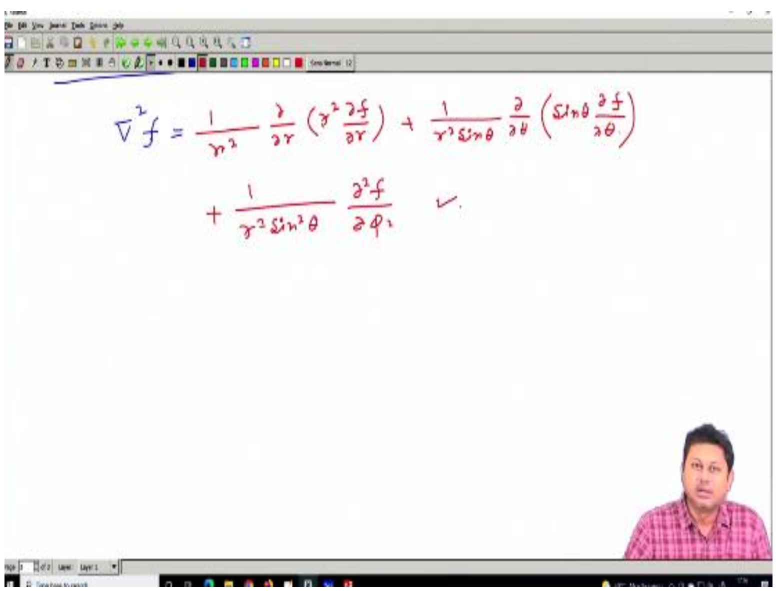

Laplacian is ∇^2, it is operating on a scalar field f, and the expression is this is a very important and very useful expression. So, I request you to please remember it at least $r^2\ \frac{\partial}{\partial r}$ then r^2 df dr then I have $\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}$ in the argument we have sin θ $\frac{\partial f}{\partial\theta}$. And finally, we have $\frac{1}{r^2 sin^2\theta}$ and then $\frac{\partial^2 f}{\partial\varphi^2}$. So, this is a very important expression in many cases we will find later and we may need to use that.

So, better you should try to remember this expression or try to understand at least in the context of this curvilinear coordinate system how this coming?

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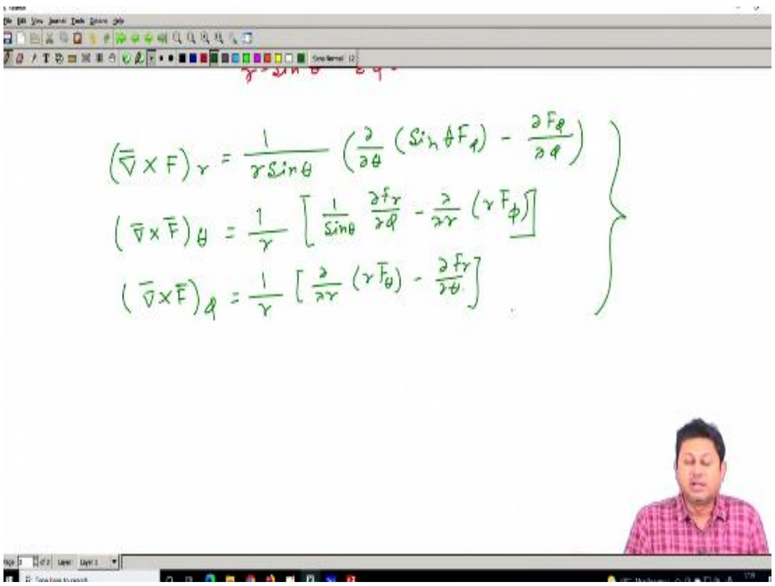

Finally, I like to you know write the curl expression and $\vec{\nabla} \times \vec{F}$ in cylindrical coordinate system the r component will be $\frac{1}{r\sin\theta}$ it is not very useful because you need to remember but at least I am writing so, that you can understand that how these expressions are there this is F_φ - ∂F_φ the generalized calculation is already done in curvilinear coordinate system in terms of the scaling factor, what is here I am putting the specific scaling factor based on the coordinate system I am using θ.

This is $\frac{1}{r}$ and then I have $\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}$ - $\frac{\partial}{\partial r}$ then I have r and F_φ. And finally, I have $(\vec{\nabla} \times \vec{F})_\varphi$ $\frac{1}{r}\frac{\partial}{\partial r}$ then r $F_\theta - \frac{\partial F_r}{\partial\theta}$. So, these are the complicated kind of expressions for the curl of when we are using the cylindrical, spherical polar coordinate system. So, I am just write down all the expression for different coordinate system, how the volume element are there, what is the scaling factor etcetera?

And in general how one can calculate that is also done in the previous class and this class also. So, I am not going to extend any more you please practice and you please check the expression from the standard books maybe I can make a mistake there is a possibility because there are a huge number of expressions I am writing right now, there is the possibility that I make a mistake. So, please check yourself by using some standard book or and then practice few problems where you can calculate the curl.

When the vector field is given in the spherical polar coordinate system and you are asked to calculate the curl of that vector field. So, it will be not that easy like you do for Cartesian coordinate system, you need to use these expressions, whatever just written here, and whatever is showing is shown now in this computer screen. So, today I am going to stop here because I do not have much time. The next class we have the last class probably, I do not know maybe the last class probably for the module 1 where we discuss about the delta function.

Which is very important and we need to understand how the functions, what is the properties of the functions and how one can use this function in different problems. So, with this note, I like to conclude my class here today. Thank you very much for attention and see you in the next class.