Foundation of Classical Electrodynamics
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Lecture – 16
Delta Function

Hello students to the foundation of classical electrodynamics course. So, today we will be going to learn the delta function, which is under module 1.

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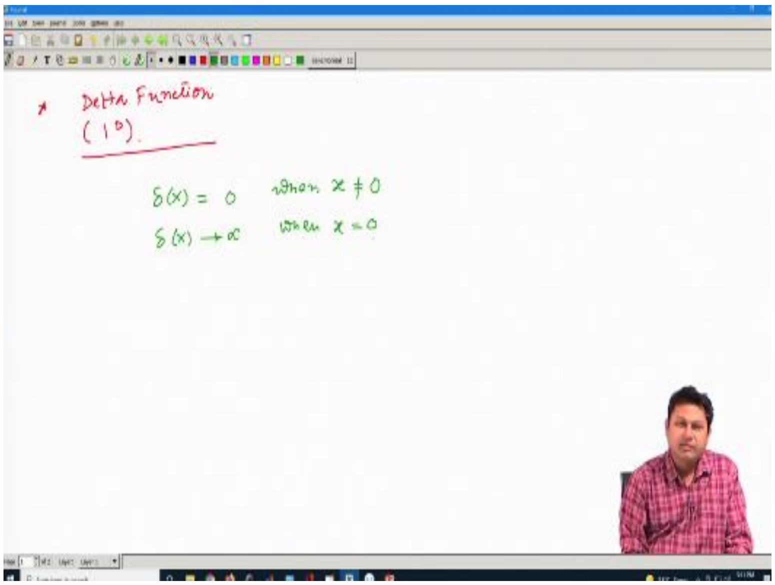

So, today we have class number 16. So, we will start a very important topic today which is delta function. So, in 1D, say, 1D, in 1 dimension the delta function is mathematically defined in this way, a function, which is 0 when x = 0 and this function goes to infinity when x tends to 0 or sorry here when x is not equal to 0 and here in other cases when x = 0 this function goes to infinity.

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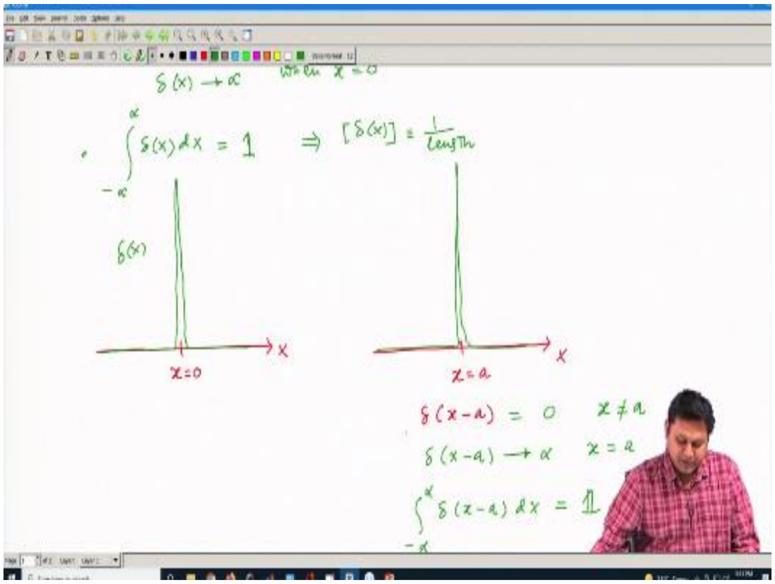

Not only that, we have another very important property that if I integrate this function over the entire x axis because we are taking 1 dimension then this value is 1, this value is 1 so, this is the property of the delta function, this is the way delta function is defined that this function $\delta(x)$ should be 0 when x is not equal to 0, when x equal to 0 this delta function $\delta(x)$ should tends to infinity, but if I integrate this function over entire space or over entire x axis then the resultant value should be 1.

So, here also you should from here we clearly understand one thing the delta function should have a dimension 1 divided by length at least in 1 dimension, then only it should be dimensionless by multiplying dx and at the end of the day whatever we get is simply a number. So, that you just try to understand from this expression this is a very very important expression. Now, if I visualize this function how it look like suppose, this is my x axis and this is at x = 0 point and now, if I plot my delta function here it is 0 here, it is 0 over here.

And suddenly it shoot up going to a very high value and this, this is roughly the structure of the delta function it goes in principle it should go to infinity. So, this is the delta function that we are talking about, and this should be the distribution. Well, if I make these things in a general way, it is not necessarily that always we have the delta function x = 0, we can shift our coordinate and if I shift our coordinate to some other point, suppose this is my x, and this point I am talking about this point, which is x = a.

So, now, my delta function is defined like δ(x - a), a coordinate shift. So, same thing happened here all the values will be 0 and suddenly we have a huge steep here and then it goes to 0 for other values. In this case, this quantity should be 0 when x is not equal to a. And this function δ(x - a) tends to infinity when x = a and if I integrate this quantity $\int_{-\infty}^{\infty} \delta(x-a)$ dx in 1 dimension, then this value gives me a value 1 like before, but this is a general form where we are making a coordinate shift. Another very important property so, these properties are very important.

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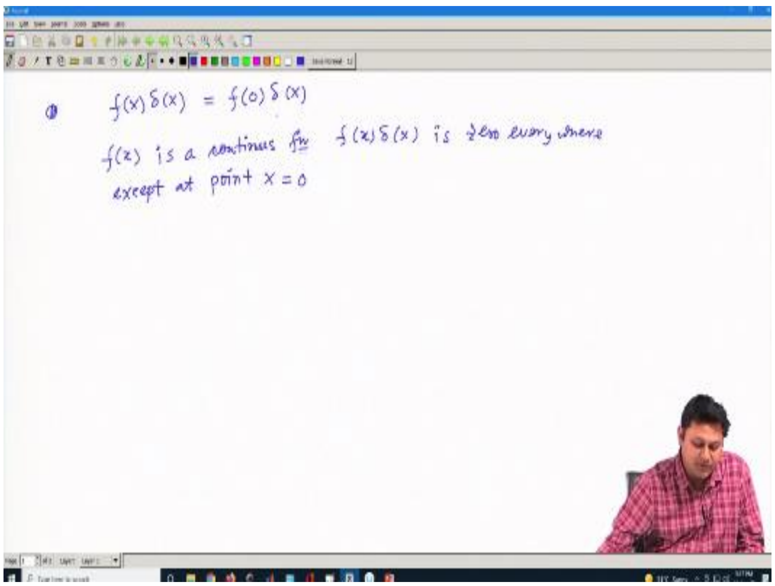

Another very important property is this, if I multiply a function with the delta function, then obviously, this function when x is not equal to 0, so for this case, this quantity is always 0 and this these things will be meaningful only when x = 0, so, the function not x = 0 δ(x) that should be in the right-hand side, try to understand that δ(x) is all the point δ(x) is 0 except x = 0. So, only this expression is meaningful at the point when x = 0.

Because in that case, delta function has some meaning a nonzero value and that is why this multiplication should be meaningful when the function is evaluated at 0, x equal to 0 point multiplied by δ(x). So, that you should understand or you should realize this. So, what I am trying to say is this f(x) is a continuous function and f(x) δ(x) is 0 everywhere except at point x = 0. So, that is why this function is important this function can be evaluated at x = 0 then multiplied by δ (x).

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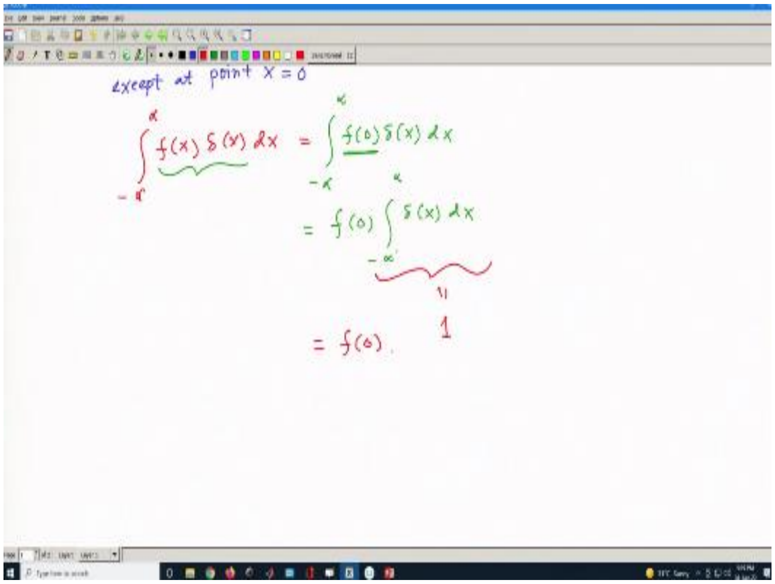

Now, if this is the case, then we can have an interesting property and that is this minus infinity to infinity if I integrate this quantity f(x) then δ(x) dx then what we write in the right-hand side because this quantity I mentioned is meaningful at x = 0 when the function meaningful when we evaluate the function at x = 0 since the right-hand side I should simply write $\int_{-\infty}^{\infty} f(0)\, \delta(x-a)$ dx. Now, you note that f (0) is a fixed value, it is a constant it does not depend on x.

So, I can take this term outside the integral. So, I should write this and $\int_{-\infty}^{\infty} \delta(x)$ dx. Now, you can see that this quantity again by definition is equal to 1 because I just mentioned that when we integrate this delta function minus infinity to infinity over x axis in 1 dimensional case this value gives me 1 so, this is true so, as this one. So, exploiting that expression here we can find that this integral value is eventually f (0).

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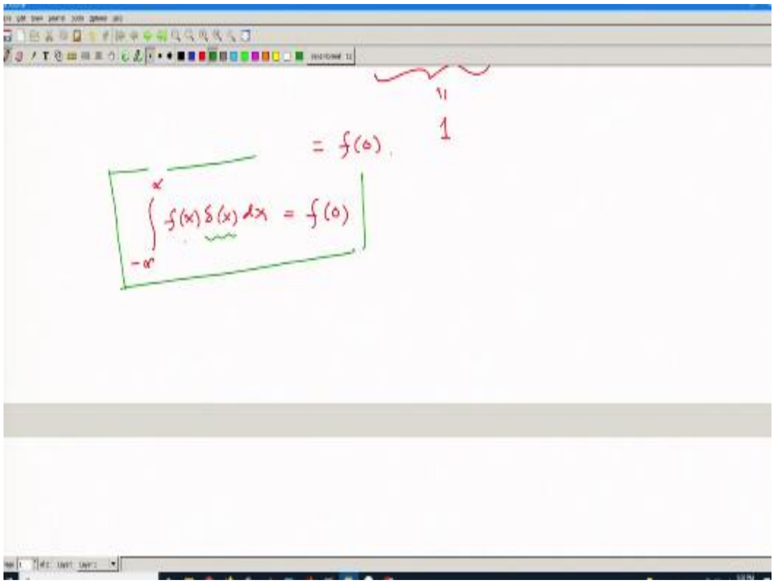

So, we have a very interesting expression and that suggests that if I integrate minus infinity to infinity of any given continuous function 1 dimensional function multiplied by delta function dx, then this integration is simplified and we simply have the result this the value of the function at x = 0 point. So, this is a very interesting and important result and we should note it. So, this is the result where we can find out the integration involving the delta function for any function. Now, we can generalize this because this is also for the coordinate shifting.

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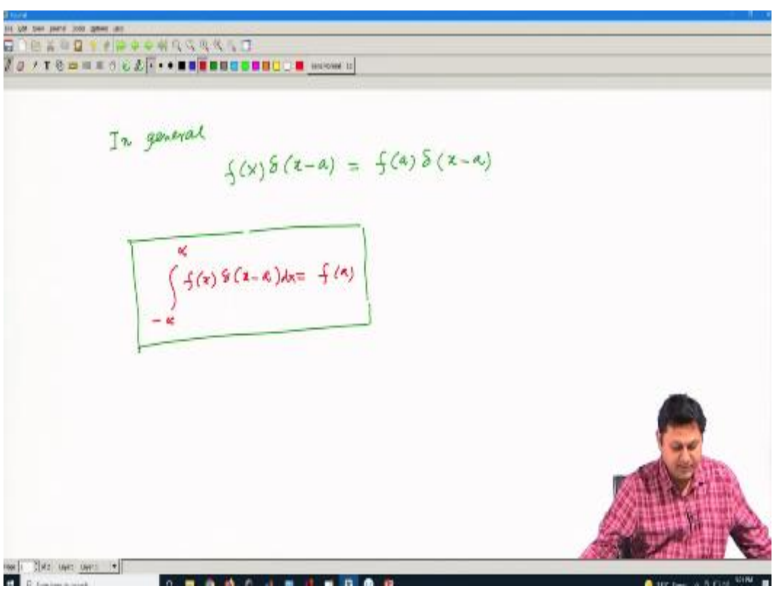

So, in general if I write in general we can have a function of x $\delta(x-a)$ now, I am making a coordinate shifting delta is now defined at x = a point then that quantity should be simply f (a)

$\delta(x-a)$ like before only I put a in place of 0. Now, if I integrate it $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)$ dx = f (a). So, $\delta(x)$ is a very important function as far as the integration is concerned. So, any function multiplied by delta function if I want to integrate over the x axis in this case.

I should put dx here. So, it should be dx so, this is a very important result. So, now, we would like to understand quickly that in reality how the delta function is realized.

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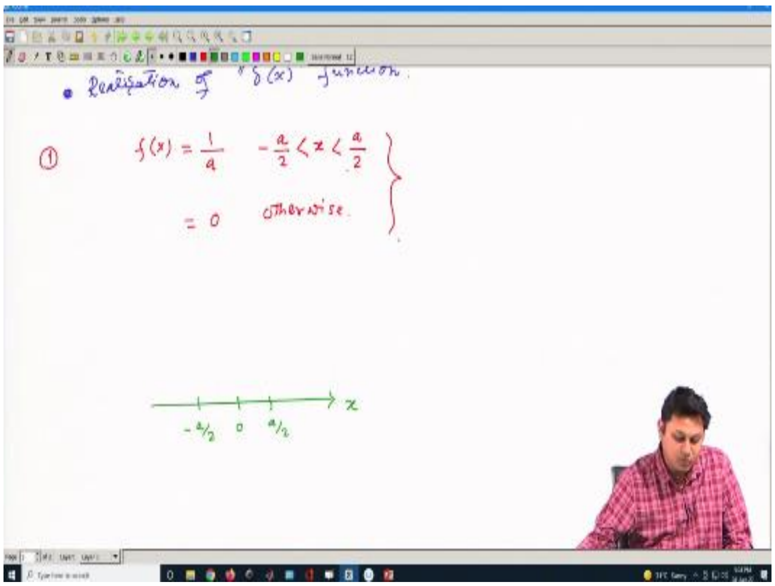

So, realization of delta function so, delta function one can realize under with putting a certain limit of the well-known functions and then one can understand that how the delta function is physically possible. So, let me give the first example quickly I will give few examples. So, the first example is this function suppose, we consider a function, function of x these are few standard example in any book you will be going to find this function of x suppose, given as $\frac{1}{a}$.

And this function is distributed in this way, it is $\frac{1}{a}$ when the x value is in between this limit and 0 otherwise this is the way the function is defined mathematically. Now, if I plot this function the plot will be very simple that this is my x axis I have this point is 0 I have a point here $\frac{a}{2}$ I have a point here $-\frac{a}{2}$ and when the x value is in between $\frac{a}{2}$ and $-\frac{a}{2}$ we have a value constant value $\frac{1}{a}$.

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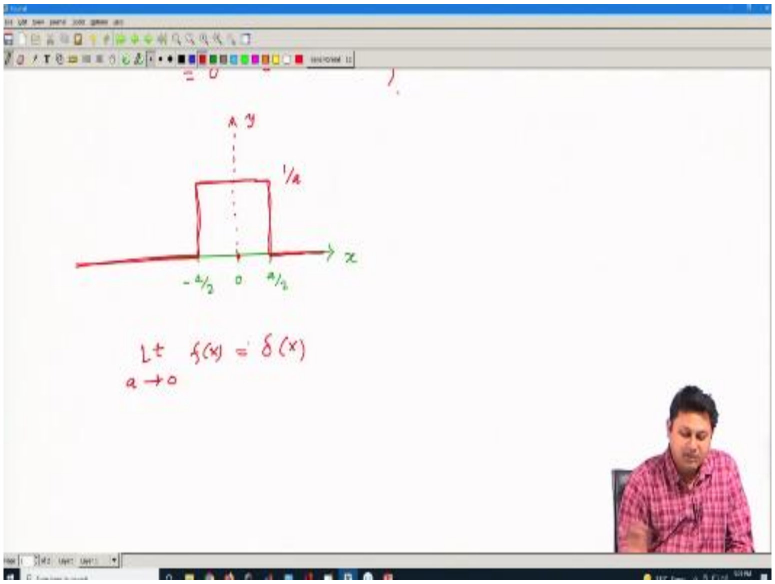

Suppose, we have this value $\frac{1}{a}$ here and that means, eventually I am having this box kind of function. And it is 0 all others for all the other values, so, my function is this from here to here it is 0. Suddenly there is a jump. We have fixed value $\frac{1}{a}$ here and again there is a sudden drop and then it goes to 0. So, this value is $\frac{1}{a}$ and this is my y axis and this is x = 0 point. Now, one can understand this limit. So, how the delta function is defined for this case, the delta function can be defined for this case in this way.

If I take limit a tends to 0 for this function f(x) that eventually gives me delta function and you can understand that, so, a tends to 0 means, I will go I will reduce this length from because a tends to 0 if you reduce this length, this height will gradually increase. So, what kind of figure I will be going to get under this limit.

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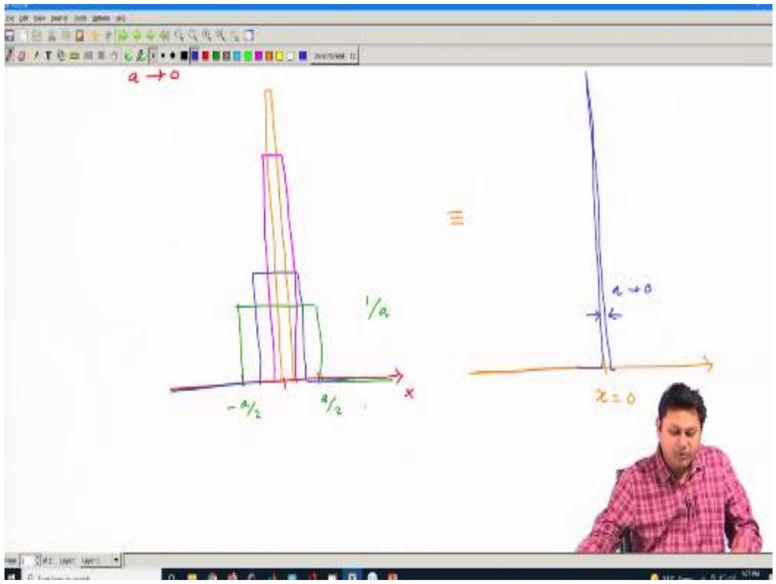

I can draw here it is something like this suppose this is my x axis and I start with this a value this is 0 and my initial function is this form this value is $-\frac{a}{2}$ and this value is $\frac{a}{2}$ and this value is $\frac{1}{a}$ that was the initial function. Now, I put the limit I gradually decrease the a in such a way that these things will shrink as a next point we will get the next if we reduce that, I will get a function like this because I am reducing a. In a similar way, if I reduce a more I will get a function like this. If you reduce a more I will get a function very sharp function like this.

So, gradually this function will going to shrink and eventually we will going to get the ideal delta function an ideal delta function I already mentioned that it goes to 0 at x = 0 point, it goes to infinity at x = 0 point and 0 otherwise. So, the ideal delta function is simply this goes to 0 very sharp increments and this is the condition. So, this limit, which is the width of this function, which is a tends to 0 here and this is precisely the case for this function.

So, this is the way one can realize the delta function and this is not the only way one can realize delta function there are other way also for example, I am just giving one function and that is this one.

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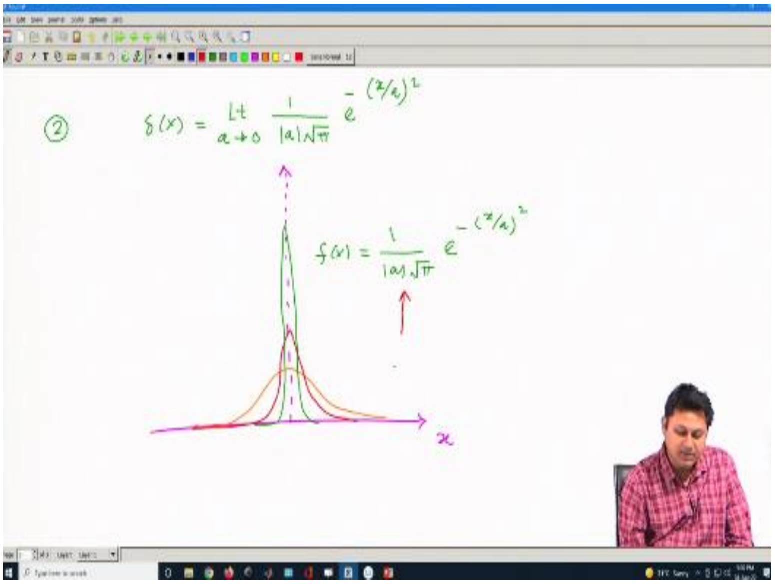

So, delta function can also be considered as limit a tends to 0 and then $\frac{1}{|a|\sqrt{\pi}}$ and then $e^{-(\frac{x}{a})^2}$. So, this is typically a Gaussian function having a width $\frac{1}{|a|}$. So, again if I want to plot this it looks similar, whatever is drawn here, it looks very similar, when we make our limit a tends to 0, then the function will be something like this evolution of the function will be this, so, this is the x = 0 point so, let me draw it. So, this is x axis.

So, the initially we have a Gaussian like this for large value of a now, if you gradually reduce the value of a we will going to get a function like this. If you reduce this the value of a more I will eventually get a function Gaussian distribution like this. So, again you can see that we are gradually terms a delta function kind of structure under the limit. So, this is the function I am plotting which is this one $\frac{1}{|a|\sqrt{\pi}} e^{-(\frac{x}{a})^2}$ so, this is a typical Gaussian function.

And now I change this limit, I change the value of a, so, under the limit a tends to 0, I am going to get a delta function here.

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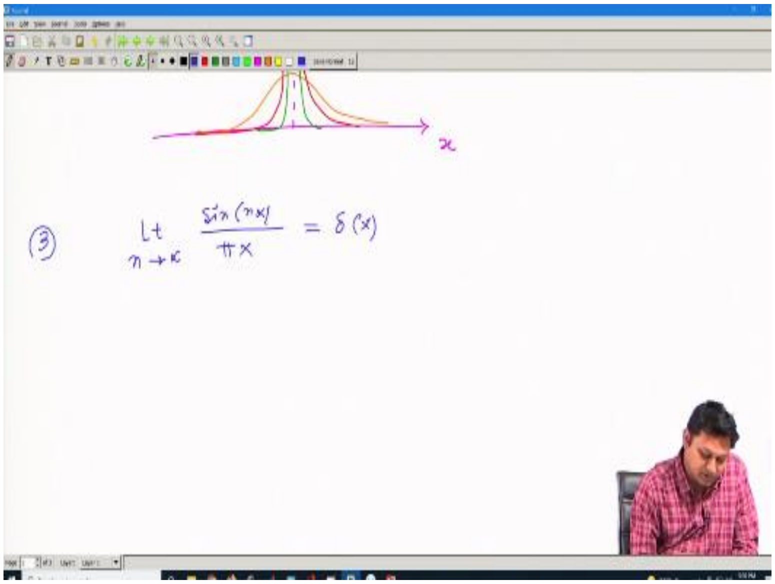

Another example is this and that is $\lim_{n\to\infty} \frac{\sin(nx)}{\pi x}$, this is also considered to be a potential delta function, and this is a sinc function this is related to a sinc function this function is called sinc function. So, we know that how the sinc function evolves.

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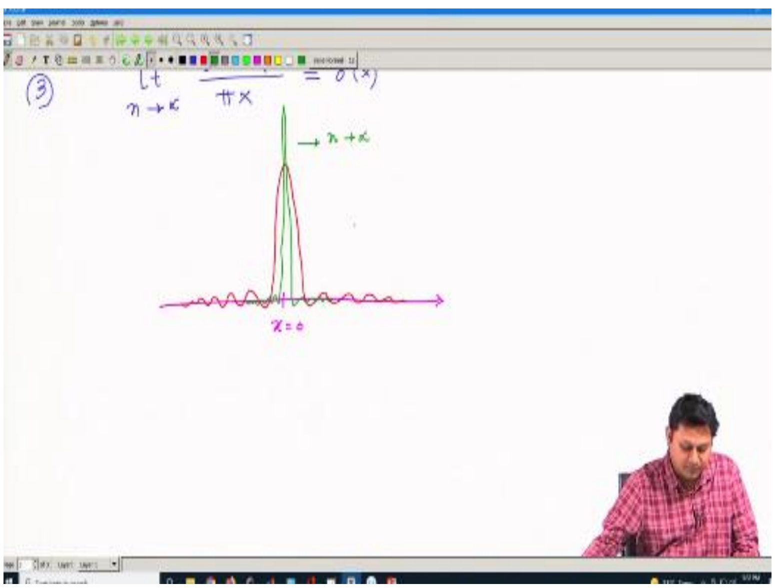

And again if I draw this sinc function typically, if this is x = 0, the typically sinc function is this, this is roughly the figure. So, here we have a peak. Now, if I gradually change, so, this is if I now make this n tends to infinity, then what happened that I will get more and more sharper this sinc function and eventually which looks like a delta function. So, I am just roughly drawing the condition when this is for when n tends to very high value tends to infinity. So, it basically reaches it basically arrange itself like a delta function.

So, this is the way one can realize. So, these are the few examples where one can realize the how delta function can be possible with the usual functional form by exploiting the usual function under certain limit, but the properties are very important for the delta function that we mentioned. Now, we will try to understand few more properties of the delta function, which is very important and few identities rather.

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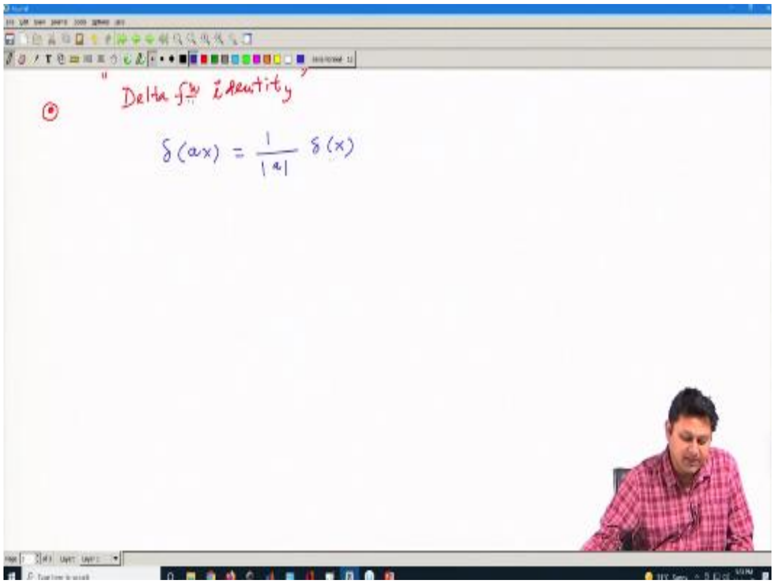

So, the next thing we will understand the delta function identity. So, the first identity is this, if I have a delta function and in the argument I have some constant a and x, that value is eventually $\frac{1}{|a|}$ and then δ(x) a quite a very interesting looking identity we are having, so instead of having an argument x, if I have a constant a then in the right-hand side, I can write the delta function is in its usual form with delta x, but then it should be $\frac{1}{|a|}$. So, we can prove this there is a way to prove this.

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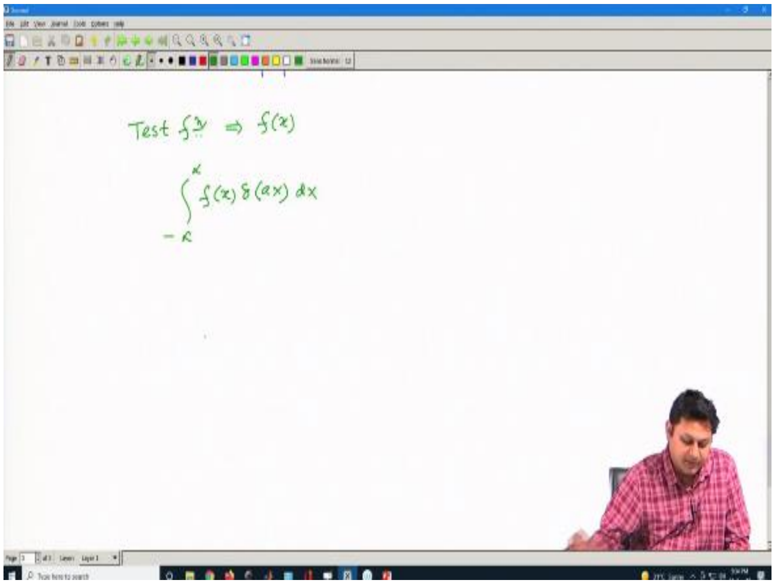

And in order to prove we need to first consider a test function. So, let us take this test function and this test function is f(x) over which we will operate this operator or this function. So, now I want to find out this quantity minus infinity to infinity our favorite integral f(x) instead of delta you will want to find integration for this given function $\delta(ax)$ dx.

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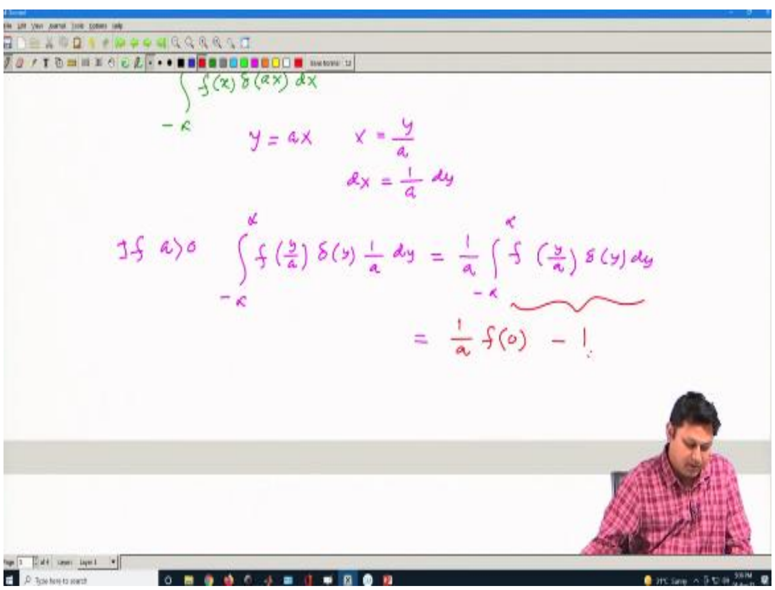

Now, after that, we will make some transformation like y = ax then x should be equal to $\frac{y}{a}$ so that dx will be $\frac{1}{a}$ dy. Now, if a is greater than 0, it can be greater than 0 or less than 0, then minus infinity to infinity f(x) has now become $\frac{y}{a}$ and δ(y) and then we have $\frac{1}{a}$ dy after making this, this rescaling, so, we will eventually have $\frac{1}{a}$, then integration minus infinity to infinity f($\frac{y}{a}$) and δ(y) dy.

So, that value, I know this condition, I know, because this integral is a very famous integral I have a delta function and function, so, it should be evolve at 0 point. So, eventually I will get the result like $\frac{1}{a}$ and then f(0), this is my equation 1.

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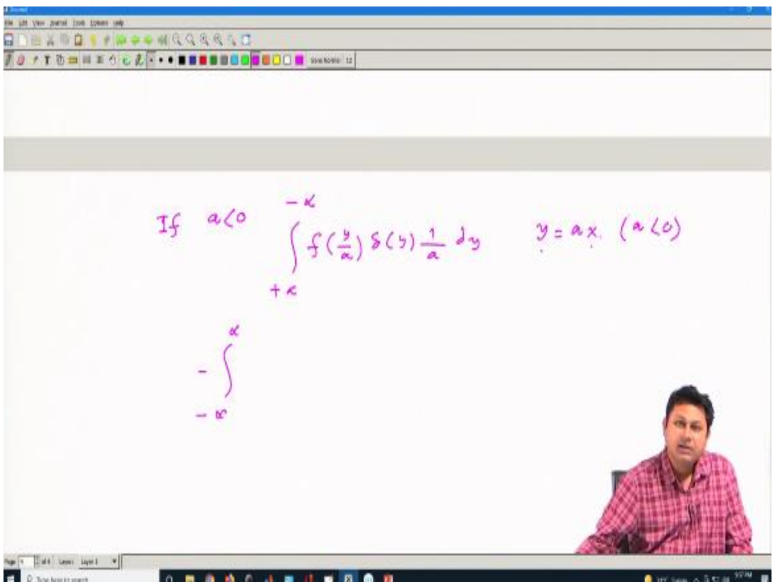

In the similar way, if a is less than 0, that is another condition, we can think of that now, my if a is less than 0, what we have. We have $\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\ \delta(y)\ \frac{1}{a}$ dy. Now, since we are having a negative so the limit should be, since a is going to the negative side, the limit should be this one. And in order to make this limit correct, we will simply invent a minus sign, and we should have minus infinity to infinity. Because here a is negative means when you when we make the transformation y = ax and a is negative.

That means, when x tends to infinity, y goes to minus infinity and vice versa. That is why there is a change in the limit under the condition that we are having y = ax with the condition that a is negative. So, when x tends to infinity, y tends to minus infinity, y goes to minus infinity. And when x tends to + infinity, y goes to minus infinity because there is a negative sign carried by this value a, I can fix it by putting another negative sign.

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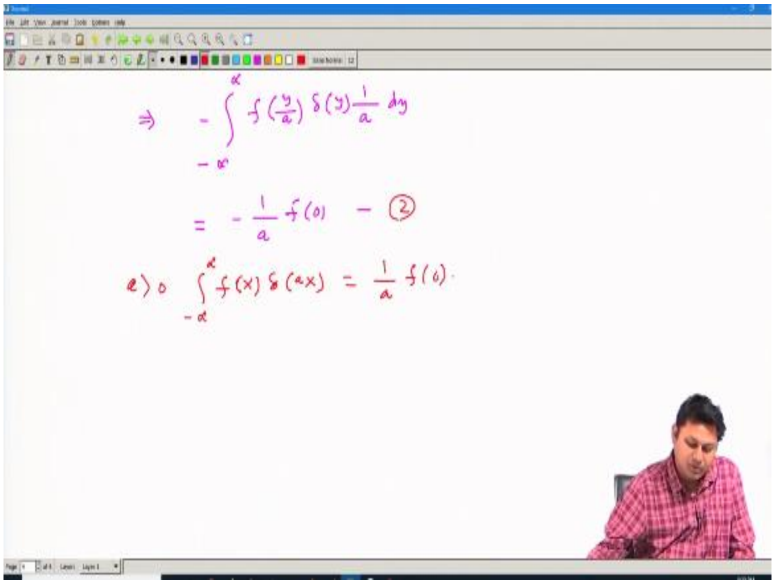

And then it simply becomes f($\frac{y}{a}$) δ(y) $\frac{1}{a}$ dy. So, I will get simply I will get $-\frac{1}{a}$ f(0), that is the value I am getting and that is my equation 2. So, now, if I tally equation 1 and equation 2, then you can see that for a greater than 0, I have the integral integration whatever I have $\int_{-\infty}^{\infty} f(x)\ \delta(ax)$ dx = $\frac{1}{a}$ f (0). That value we have.

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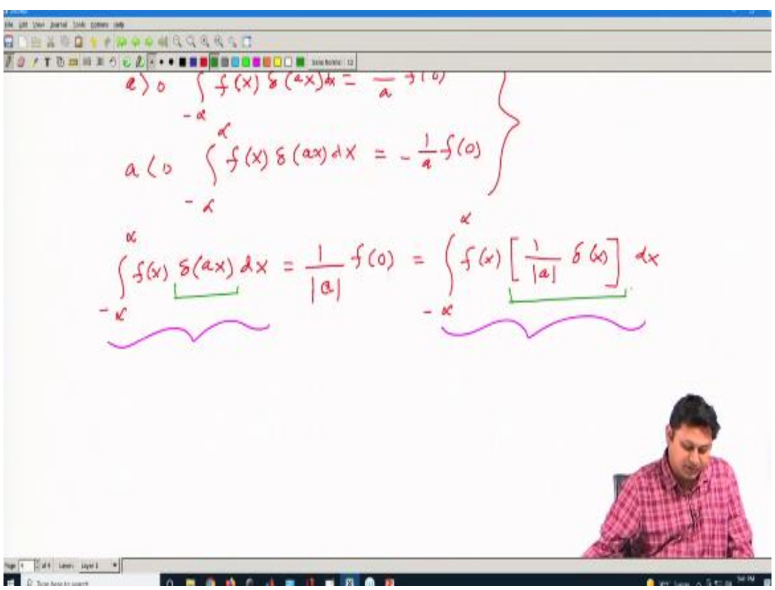

On the other hand, when a is less than 0, I am having minus infinity to infinity f(x) the same integral I am getting. Here we have dx, ax and dx. That value I get with a negative sign so, combining these 2 because for a positive we have positive for a negative a negative sign is already there. So, with this note I can write, we can simply I can simply write this integration $\int_{-\infty}^{\infty} f(x)\ \delta(ax)$ dx is simply $\frac{1}{|a|}$, because whatever the a you choose positive or negative at the end of the day you are getting this result is 0.

Now, this quantity these things I can rewrite in this way, this I can rewrite $\int_{-\infty}^{\infty} f(x) \frac{1}{|a|}\ \delta(x)$ dx, I just rewrite this expression in this way. Now, if I tally this side and this side is simply gives me that this quantity $\delta(a\text{x})$ is simply $\frac{1}{|a|}\ \delta(\text{x})$.

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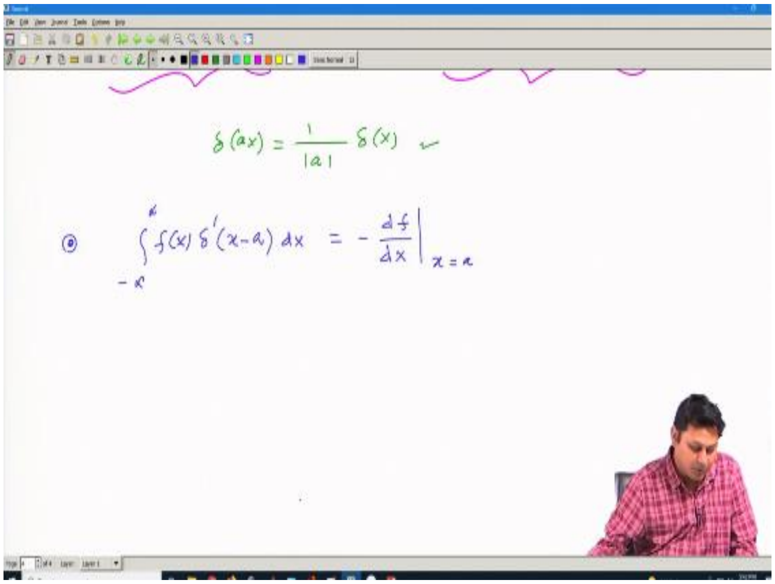

So, that proves that $\delta(a\text{x})$ = $\frac{1}{|a|}\delta(\text{x})$ this is a very important proof. Next, I want to extend this part another identity, but I am not going to prove this identity. So, that identity is this one. So, minus infinity to infinity if a function we have a f(x) and instead of having delta function if I have delta prime, which is a derivative of the delta function in interesting looking function, it is not a delta function, but derivative of that this prime suggests it is a derivative dx is equal to $-\frac{df}{dx}$ that will be evaluated at x = a point.

So, the value of this quantity should be this one and the process is straightforward so, I am just giving you the hint and the hint is you just calculate this left-hand side integral you just calculate the left-hand side integral with by parts.

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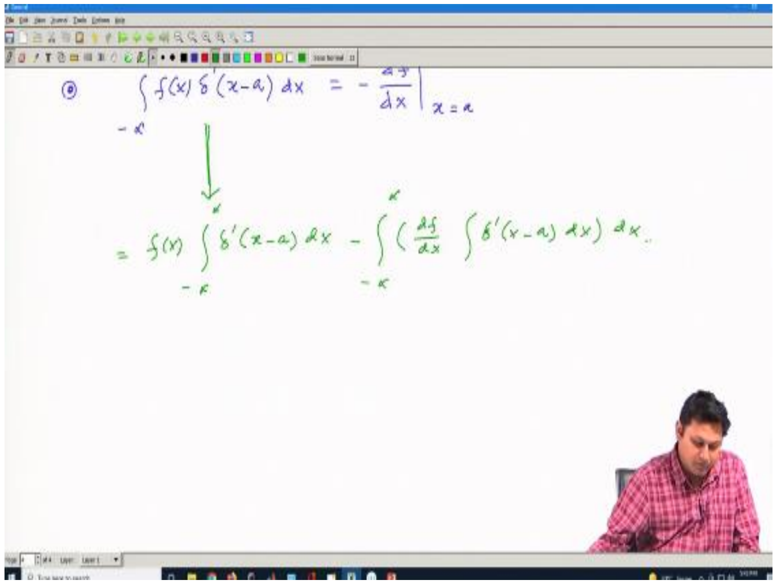

Because this integral left-hand side integral if I want to evaluate it should be first function f(x) make these 2 function f(x), first function f(x) and then integration of the second function minus derivative of the first function multiplied by integration of the second function and then whole dx. Now, if I integrate this function integrate the derivative of the delta function what we get?

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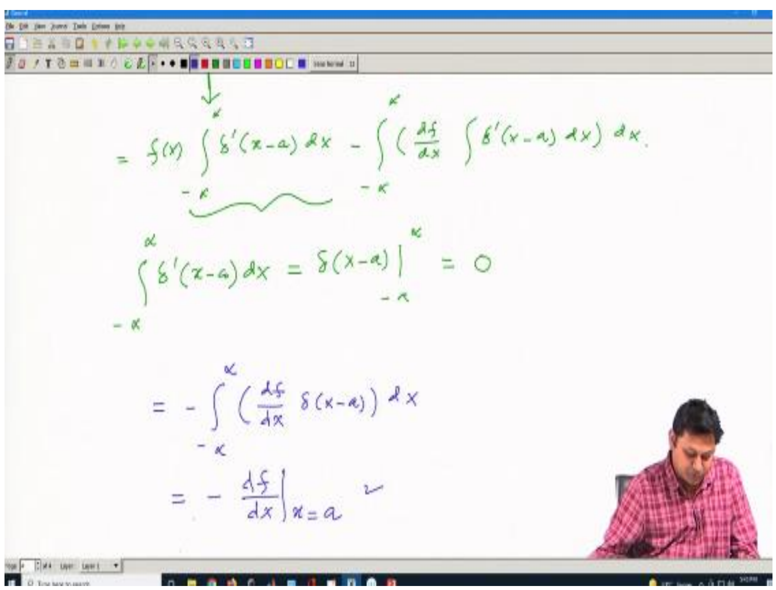

$\int_{-\infty}^{\infty} \delta'(x-a)$ dx if I integrate the derivative of this function, then we simply get back to our delta function with the limit minus infinity to infinity. Now, delta function is meaningful only at x = a point by definition set minus infinity and + infinity in both the point delta function is 0. So, this quantity is simply 0. So, eventually I have the first term 0 what about the next term the next term is this one, it is minus of infinity to infinity.

And then I thought that I should not do it, but I am doing it I can give you other problems no issue and then integration of this quantity is simply a delta function and we have dx. So, this quantity is nothing but again a function associated with the delta function at x = a. So, if I now extract this integration if you do this integration it should be simply $\frac{df}{dx}$ evaluated at x = a point that should be our result and that is the thing I just write here so I prove it. Even though I thought that I should not, but anyway I will give you other problems also here. So, there are ample amount of problems.

Now, before concluding I like to you know, I like to show a very important identity and that identity is this one.

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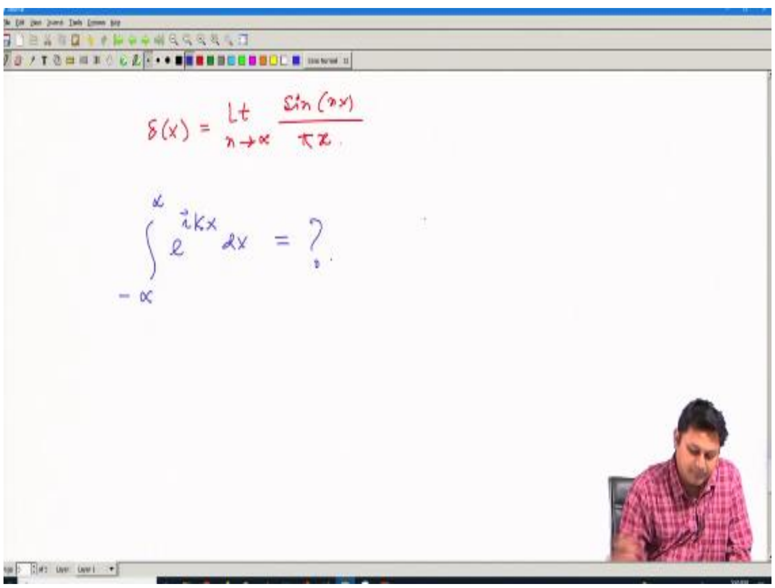

So, I already mentioned that δ(x) let me write it is equal to $\lim_{n\to\infty} \frac{\sin{(nx)}}{\pi x}$ so, that is the way one can define delta function. Now, the question is if I want to find out this quantity this integral what should be the value. The integral is $\int_{-\infty}^{\infty} e^{ikx}$ dx a very important integration and the question is what should be the value of this integration. So, this integration we will do in this way.

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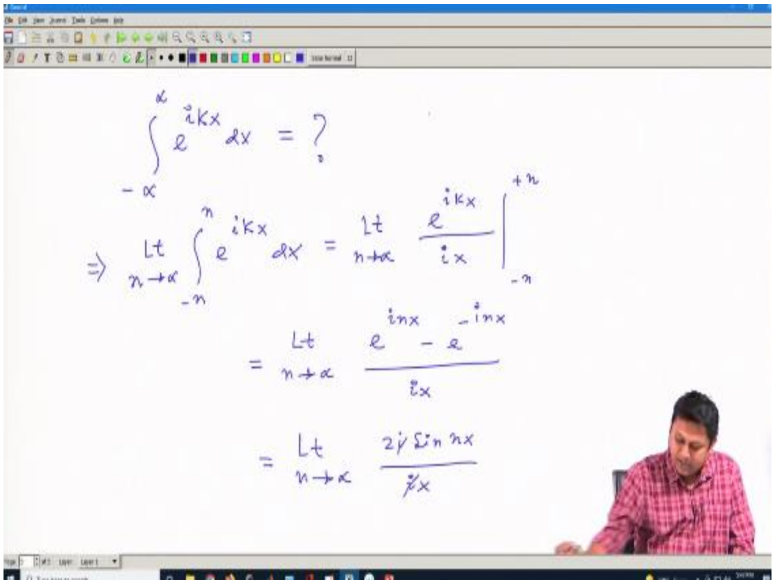

So, let define the function let define n in such a way $\lim_{n\to\infty} \int_{-n}^{n} e^{ikx}$ dx. So, this quantity is how much $\lim_{n\to\infty} \frac{e^{ikx}}{ix}$ evaluated at - n and + n. So, this quantity is $\lim_{n\to\infty} \frac{e^{inx} - e^{-inx}}{ix}$, then limit n tends to infinity this quantity is $\frac{2i \sin nx}{ix}$ this i will cancel out.

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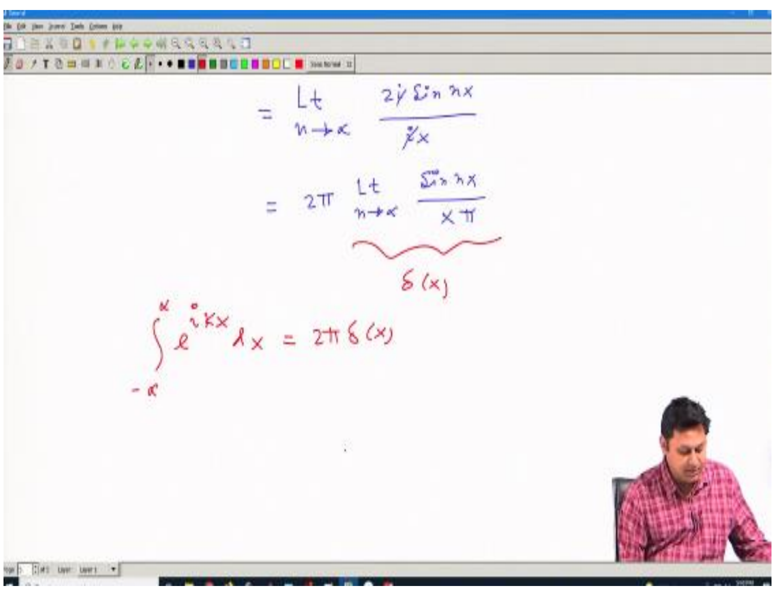

And eventually we have term like 2π, I put another π $\lim_{n\to\infty} \frac{\sin nx}{x\pi}$. So, this quantity πx because I put another π. So, you can see that this quantity is nothing but my delta function x. So, eventually I get $\int_{-\infty}^{\infty} e^{ikx}$ dx = 2π δ(x) this is a very important result.

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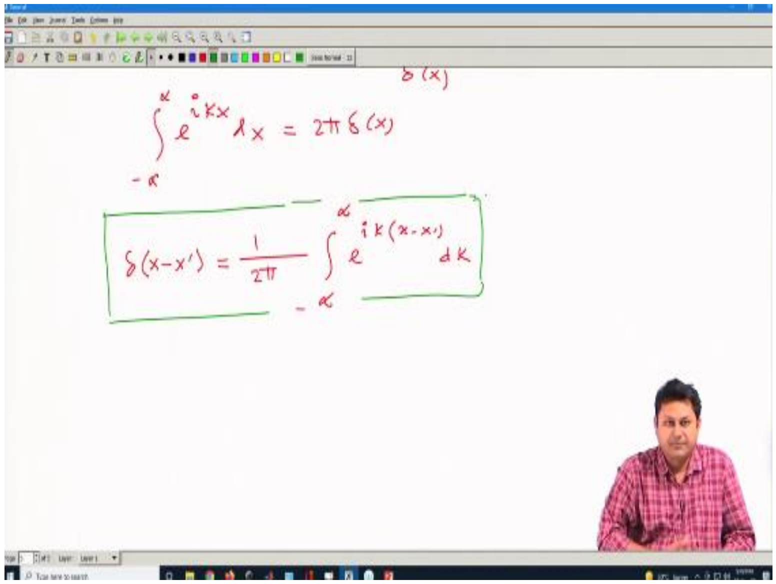

In general, I can write it $\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$. So, this is a very important expression and maybe in future we will be going to use this. So, I just want to show that the delta function can also be represented in different ways. So, this is also an identity of delta function that we will be going to see, a useful relation and that we are going to use in the future class. So, with that note I do not have much time today. So, with that note, let me conclude. So, thank you very much for attention see you in the next class.