Foundations of Classical Electrodynamics
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Lecture - 17
Delta Function (Contd.,)

Hello students to the foundation of classical electrodynamics course. So, today we will have lecture 17, where we discuss delta function, which we already introduced in the last class.

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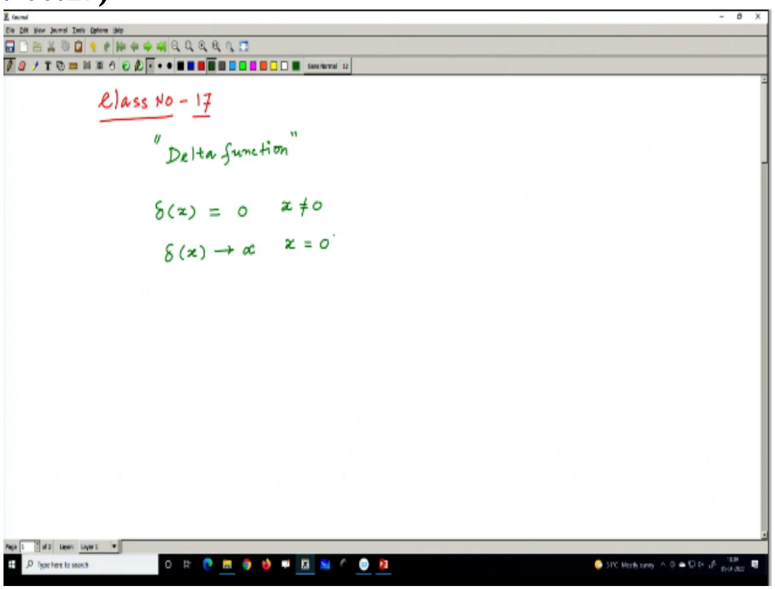

Today, we have class number 17 and we discussed delta function last day. So that will going to continue. So, delta function by definition as it says that δ(x) is 0, when x is not equal to 0 and δ(x) become infinity when x = 0.

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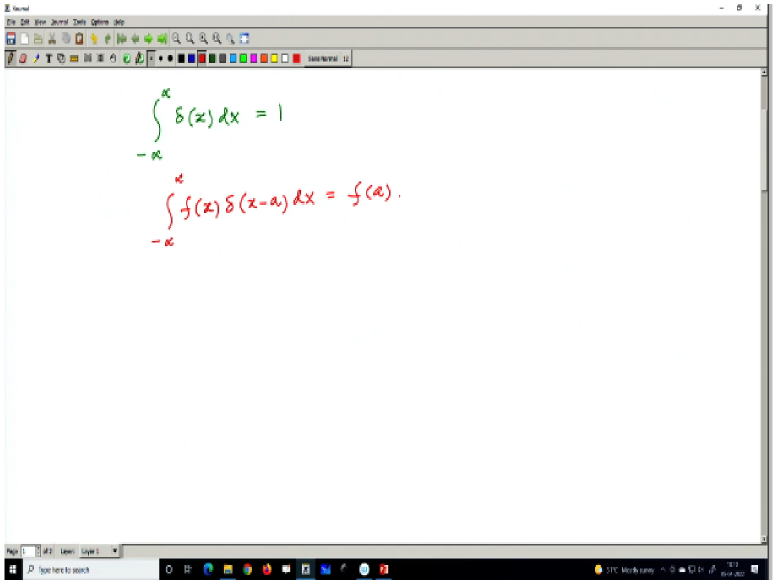

And if I integrate the entire x axis in 1 dimension, this gives us 1, finite value, better to write 1 in a normal way because this looks quite matrix kind of thing. So, simply 1. So, these are the 3 basic properties not properties this is by definition this is the delta function and also we find that in general a function if I want to find out this integral it gives us this.

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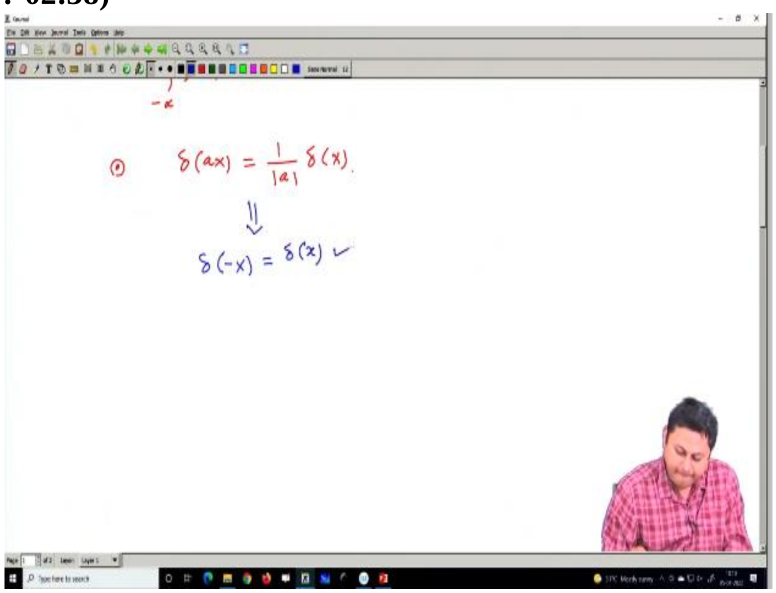

Last day also we try to understand few properties of the delta function one very important property was $\delta(ax) = \frac{1}{|a|}\delta(x)$ we prove it actually. The consequence of this expression tells us $\delta(-x) = \delta(x)$, that is. So, today we will be going to extend our discussion on delta function is a very important function in electrodynamics and also in other branches of physics. So, I thought that I should take some elaborate I should give some elaborate discussion on delta function.

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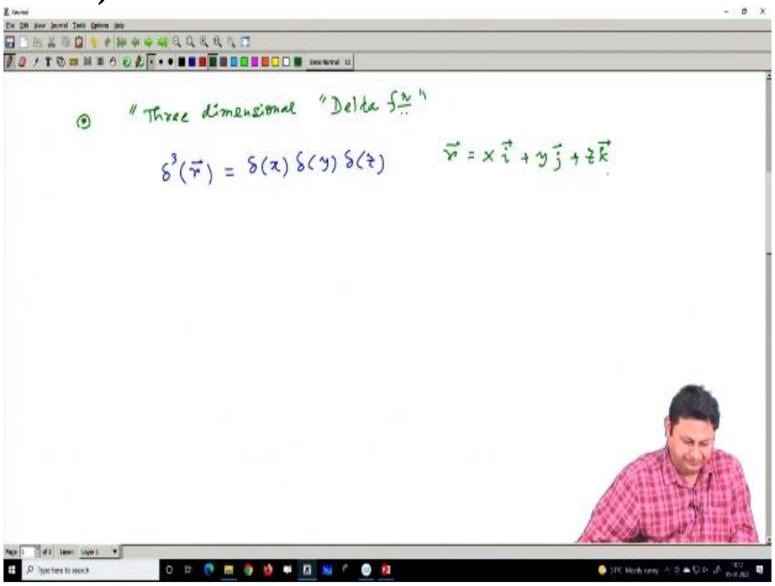

So, now, today we will be going to discuss 3 dimensional delta function. So far we are dealing with 1 dimensional delta function but 3 dimensional delta function we will do today in detail. So, normally we represent this 3 dimensional delta function as delta 3 in the argument we write vector r and that is simply in Cartesian coordinate system the multiplication of 3 delta function with this where r is the simple position vector.

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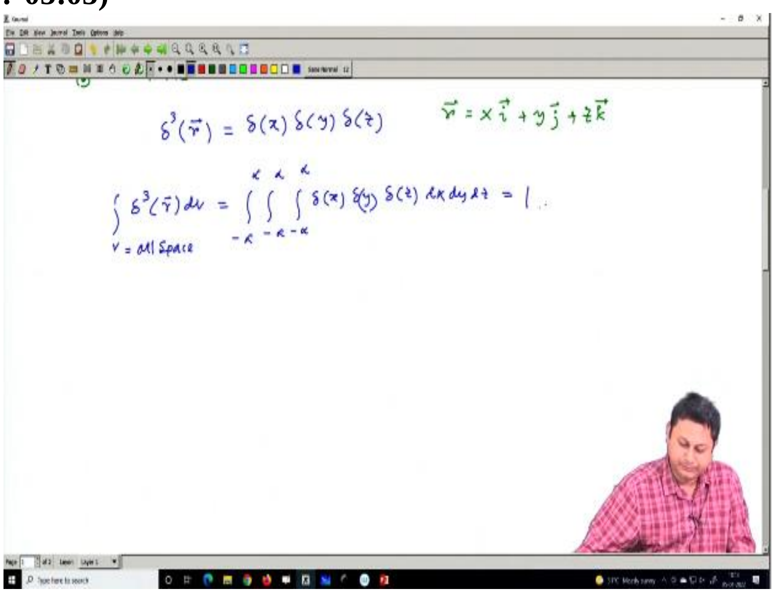

Now, if I want to integrate this delta function $\delta^3(\vec{r})$ vector dv over interspace v. So, this is all space. Then this is simply minus infinity to infinity then we have minus infinity to infinity and then minus infinity to infinity δ(x) δ(y) δ(z) and the volume element dx dy and dz that is equal to 1 that is the property.

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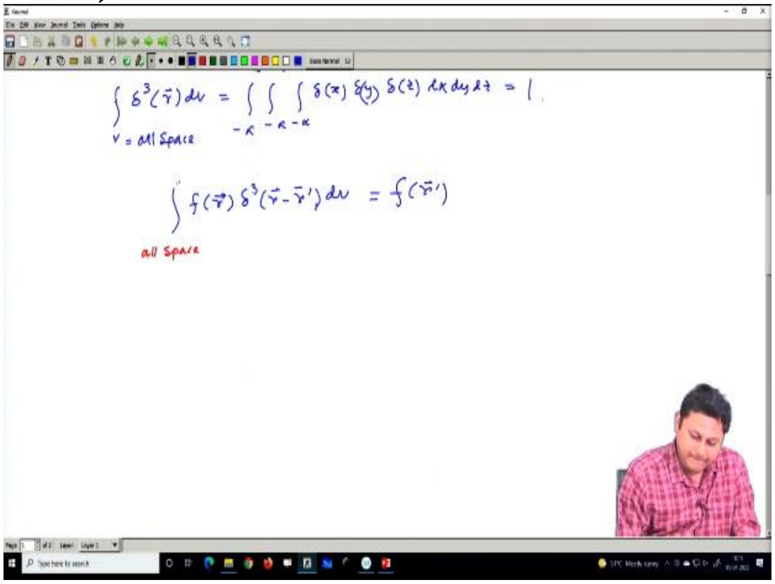

Now, if I integrate this over a function in all space that will do in 1 dimension for all space and then integrate this function. We will simply have function evaluated at r' point. So, this is the 3 dimensional way to represent delta function in Cartesian coordinate system however, we can generalize these in other coordinate system but the thing is again we need to deal with the scale factor that we learn in curvilinear coordinate system.

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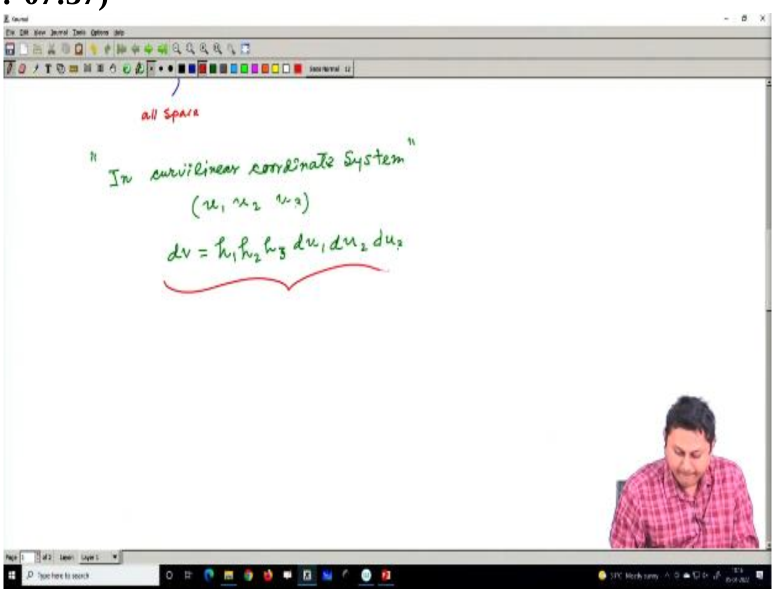

So, in curvilinear coordinate system what happened that we have the variable u_1 u_2 u_3 and the volume element there, if I define the volume element this volume element if you remember this is h_1, h_2, h_3 and du_1, du_2 and du_3, this is the way we define the volume element. This is the volume element in curvilinear coordinate system. If I now define the delta function here.

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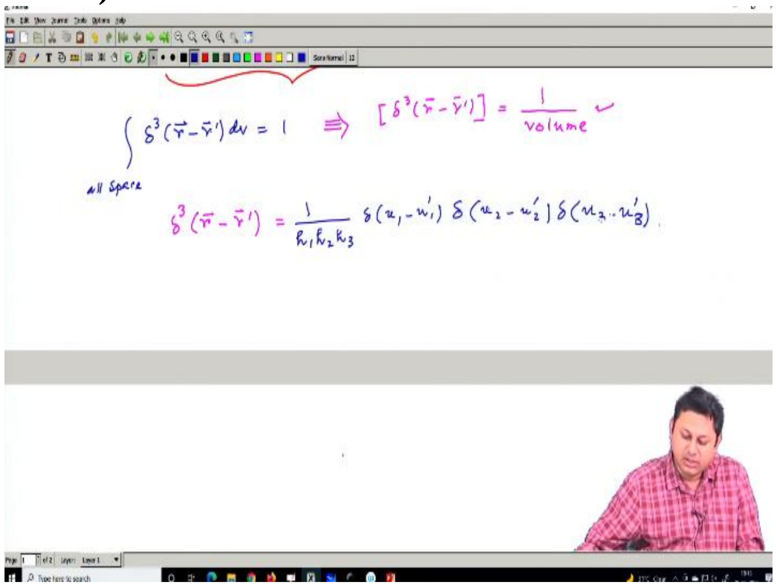

So, in general the delta function in 3D If I integrate it over all space and it is like $\delta^3(\vec{r} - \vec{r}') \, dv = 1$, which I mentioned in the last class that from where we can find out the unit of the this delta function if somebody want to ask what should be the unit. So, from this expression you can tell that $\delta^3(\vec{r} - \vec{r}')$, the unit is 1 by volume here and in general, when we define the delta function in this curvilinear coordinate system.

So, $\delta^3(\vec{r} - \vec{r}')$, we should have here the right-hand side this is the general form $\frac{1}{h_1 h_2 h_3}$ and then $\delta(u_1 - u_1')$ $\delta(u_2 - u_2')$ and $\delta(u_3 - u_3')$. So, this is the general form of the delta function in any coordinate system and curvilinear coordinate system I just mentioned.

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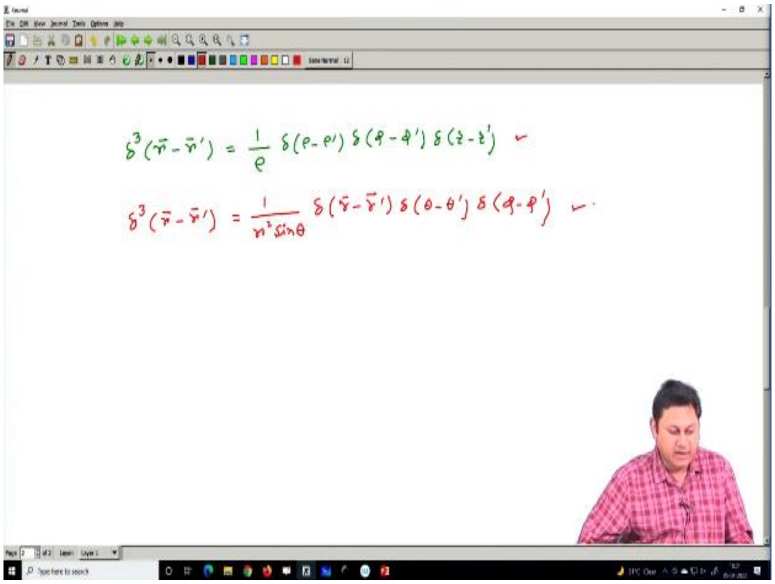

So, then if I now represent this cylindrical coordinate system then the delta function should be in cylindrical coordinate system if I represent the delta function, sometimes the function is given in the cylindrical coordinate system with the delta function and in that case you need to use this. So, it suggests that it should be $\frac{1}{\rho}$, because if you calculate the scaling factor h_1 h_2 h_3 is 3 multiplication gives you $\frac{1}{\rho}$ we discussed this several times in earlier class.

So, it should be $\delta(\rho - \rho')$ $\delta(\varphi - \varphi')$ and $\delta(z - z')$ in the similar way in cylindrical coordinate system if this is the case, then in spherical polar coordinate system I can define this delta function having this dimensional form and there I should have $\frac{1}{r^2 \sin\theta}$ because the scaling factor you need to calculate if you go back here the scaling factor is here $\frac{1}{h_1 h_2 h_3}$.

So, h_1 h_2 h_3 if you multiply in case of spherical coordinate then it should be $r^2 \sin\theta$. Then δ we should have $(r - r')$ $\delta(\theta - \theta')$ and $\delta(\varphi - \varphi')$. So, this is the way we can define the delta function in cylindrical coordinate system and in spherical coordinate system. After that now we try to understand the delta function, but with another identity. We will try to derive, but before that let us quickly do few problems regarding delta function as an exercise.

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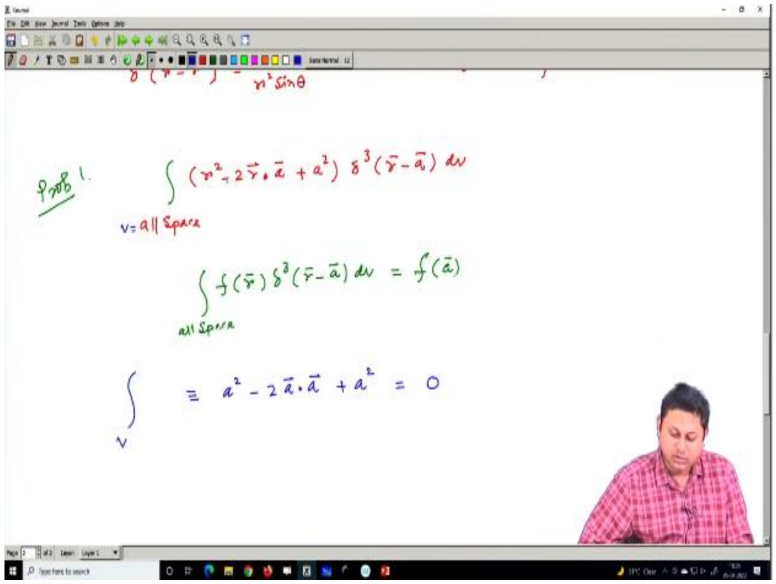

So, we will have problem 1 and this problem 1 is saying that find out the value for all space the function is defined in this way this is $r^2 - 2\ \vec{r} \bullet \vec{a}$ and $+\ a^2$ and then you have $\delta^3(\vec{r} - \vec{a})$ dv, this is. So, we know that the delta function is simply if this is the case it is given as a function of r. So, when we have the integration like function as a vector r and then $\delta^3(\vec{r} - \vec{a})$ dv, we simply put the if this is for all space then we simply have this.

So, we are going to use that so, whatever the integration we have here that overall space so, v overall space. So, that thing should be equivalent to simply by replacing a in place of r. So, we have r^2 here. So, it should be simply a^2 and then we have minus of $2\ \vec{a} \bullet \vec{a}$ because there it was $\vec{r}$ • I just replace r to a and then plus a^2. So, this quantity is simply 0. So, this integral should be 0 when we try to calculate for all space.

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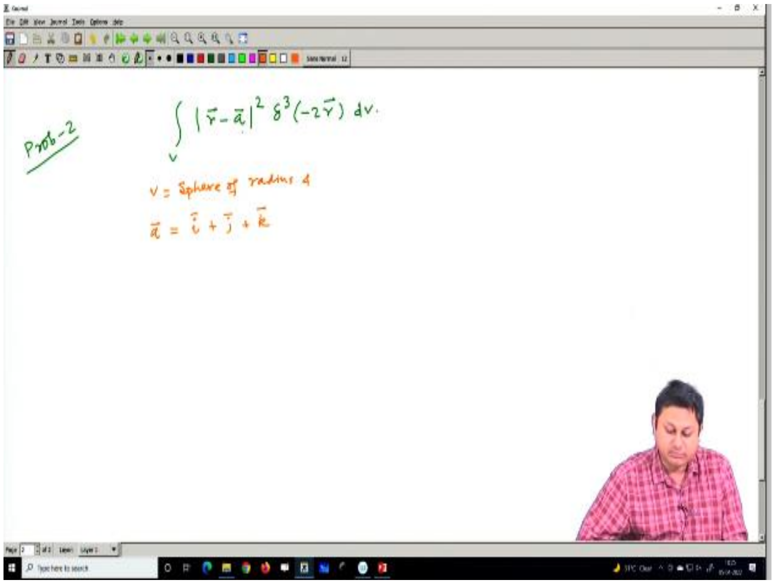

In problem 2, we have a similar kind of problem. Here we have integration v and then the function is $|\vec{r} - \vec{a}|^2$ and I want to evaluate delta and in the argument it is (-2$\vec{r}$) dv. Now, this v is defined here this v the space is defined here and it is saying that this is a sphere of radius 4. So, in the space we are integrating these things of a sphere of radius 4 and now a vector is also defined a vector is also given that is important and this a vector is i + j + k.

So, now, first we need to check that this a vector where the delta I mean this what is the magnitude of this a vector, first let us find.

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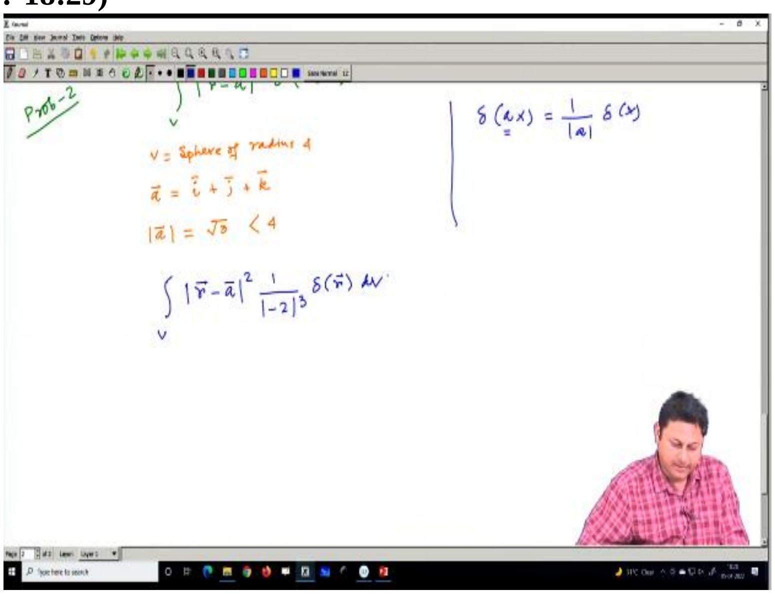

So, the magnitude of a vector is $\sqrt{3}$. This quantity if you calculate it is less than 4. So, that means, if a is that means we have whatever the integral we want to calculate inside that the a sits. So, then the integration if I calculate this integration, we can have this volume, which is the sphere of radius you know 4 and this quantity $(\vec{r} - \vec{a})^2$. Now, here inside the delta we are having -2r, so that is something so we already had the vector.

And some identity regarding delta and that is $\delta(ax) = \frac{1}{|a|}\delta(x)$. If I want to get rid of this constant and then that is the relation. Now it is δ^3 here sitting. So, that means here if we have δ^3 then this quantity when we have δ^3 here, so, this quantity should be multiplied by I mean like it should be a cube here. So, I should have $\frac{1}{|2|^3}$, because it is 3 dimension δ^3 and then I have $\delta(\vec{r})$ and dv.

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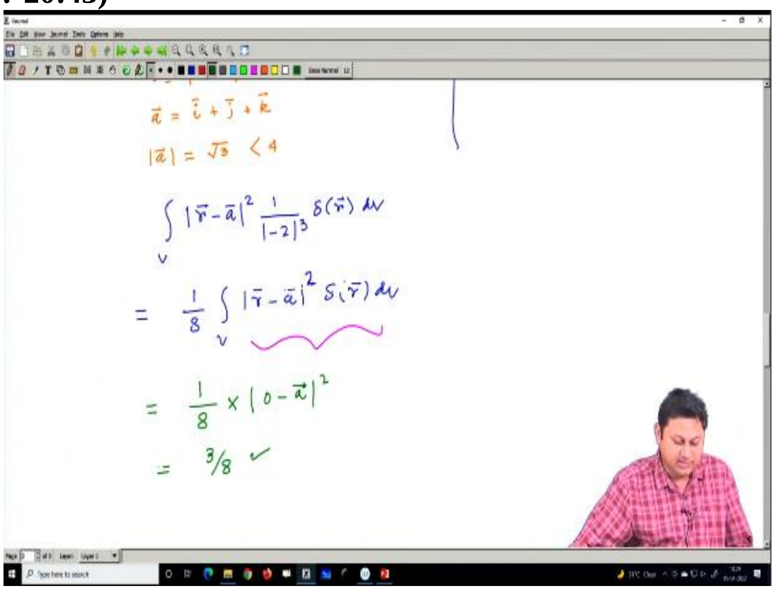

So, now, like is simple. So, now, I can take this $\frac{1}{8}$ outside and rest of the integration is straight forward because integration $|\vec{r} - \vec{a}|^2$ and $\delta(\vec{r})$ dv. So, that is the standard integration we are having a delta function with a function. So, I just replace here r = 0 and then whatever we get that is the result. So, if I put r = 0, we have simply $\frac{1}{8}$ multiplied by if r = 0, if I put then it should be $|0 - \vec{a}|^2$.

So, $|\vec{a}|$ we already calculated and that is $\sqrt{3}$. So, this quantity is simply gives me $\frac{3}{8}$ that is the value of this integration. So, this is a typical kind of problem that you can use, if I put a simple function in 1 dimension and then put a delta function then you just simply replace this argument whatever the argument here is 0 and then we can find out the valves, but there is some problem where instead of having a simple x here you can have a function. So, if that is the case then how to deal with this issue?

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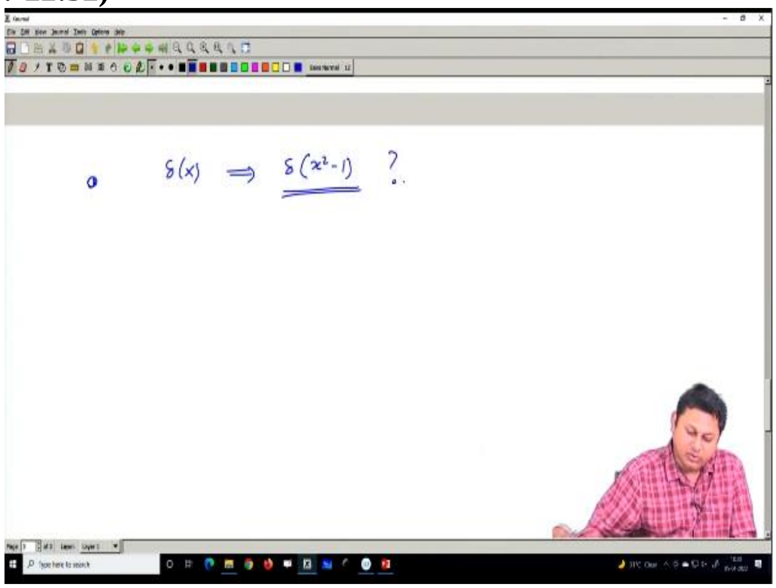

So, that I will now be going to do. So, another very important identity regarding the delta function is this, which is useful. When we have the instead of having delta for example, instead of having delta x you have you are having some kind of functional form like x^2 - 1 in the argument. So, then how to deal with these kinds of delta function that is in question. So, in order to handle that, so, let us understand this identity.

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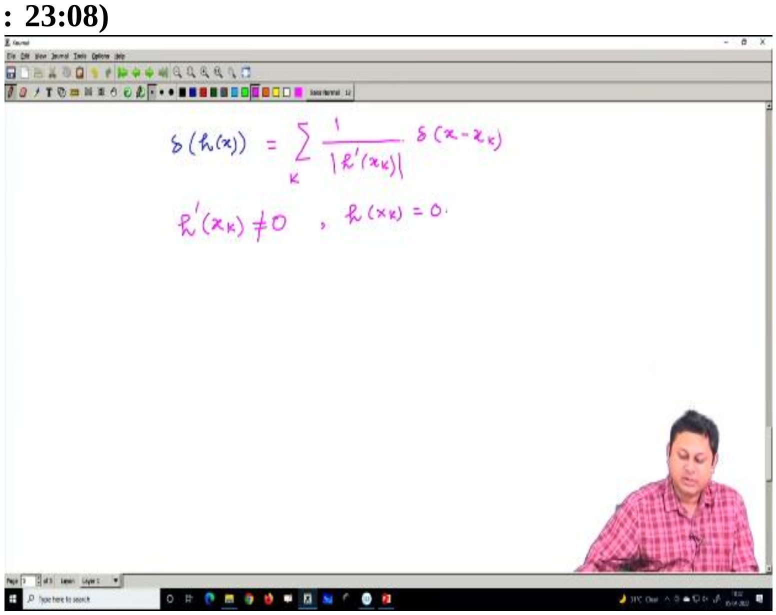

So, the identity is saying that if we have delta say function of h, which is a function of x. Then what happened. So, we are having delta function and inside the argument we have another function inside the function we have another function. So, you can understand this in this way, I mean I just simply try to derive this. So, let me first write down the result here, what should be the result?

The result should be $\sum_k \frac{1}{|h'|}$, h' is a derivative of this function and I put x_k where x_k is the roots of this equation, whatever the equation we have, if we find the root where this function vanishes that value and then $\delta(x - x_k)$. Under the condition that when you put the root of the derivative of this function, this thing should be not equal to 0. This thing should be otherwise this and since it is a root x_k is the root. So, we have $h(x_k) = 0$, because this is a root.

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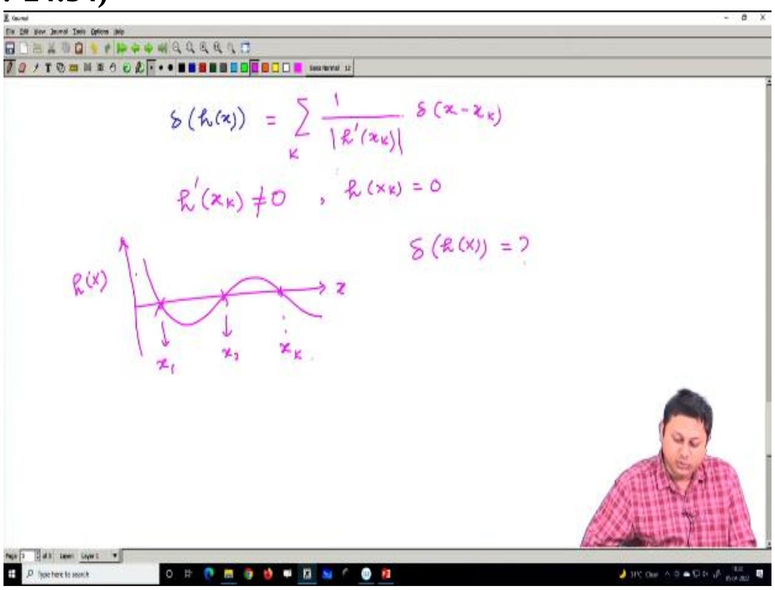

So, we can have a function if I considered this function h_k. So, we can have a polynomial kind of function having multiple 0, so, these points, so, I am plotting x here and in this side I am plotting h, which is a function of x and try to find out this quantity delta in the argument we have the function h this one and we want to figure out what is this the result already is written. So, the point is these are the roots say x_1, x_2 and so on.

So, this is my k^{th} root suppose, so, these are the roots, where the function vanishes if you know that, then you just put the value you just make a derivative of this given function, put the value of these roots there and make a summation and that should be your result.

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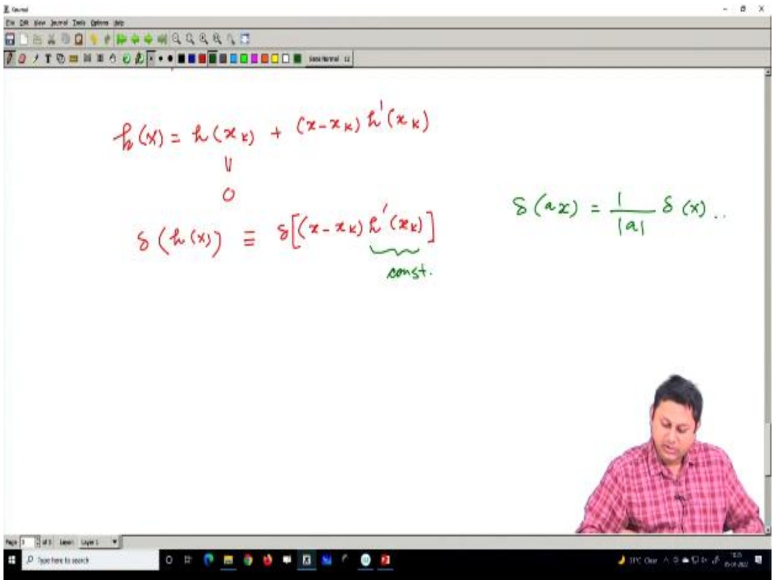

So, you can quickly prove it by just expanding this h. For example, if I expand this h, h(x) if I expand this, so, it should be Taylor series expansion around this root point. So, it should be the x_k and then we have x - x_k then the derivative of this quantity at x_k point and so on. If I take only up to 2 and now, you look at this is a root. So, this quantity should be 0 because that point I already mentioned here this 1 that x_k is the root and I am expanding my Taylor series around this root point this points and then this value should vanish.

If this point vanishes, then we can have delta and then h(x) is equivalent to delta of x - x_k multiplied by h say prime and then x_k this one. Now, if you look carefully this is a constant term because I evaluate h' at x_k point. So, this is a constant. So, I know that when we have delta function and we take a constant and x it should be $\frac{1}{|a|}\delta(x)$ that we proved last day.

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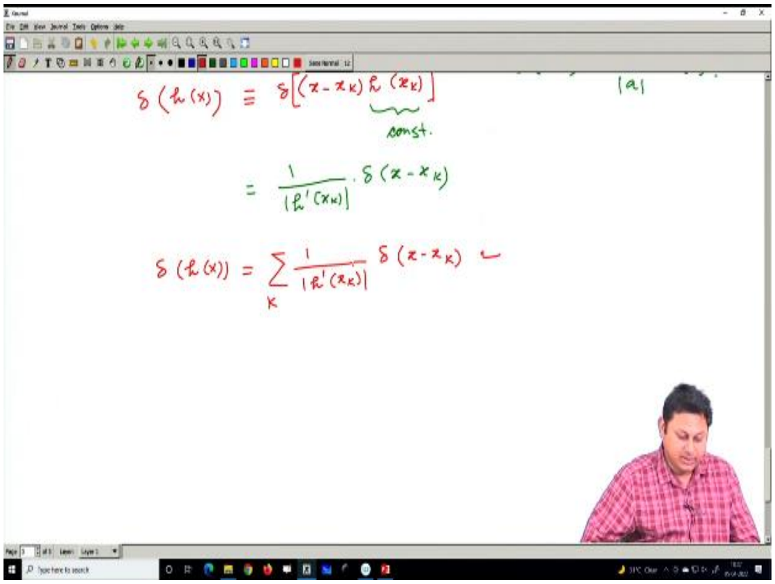

So, we are going to use this to find out and in that case, it should be simply $\frac{1}{|h'(x_k)|}$ and then δ(x - x_k). But, this is done for only 1 root for all other roots, what we get is the summation of the entire thing, because here we have 0 where delta function is meaningful here we have 0 the argument is 0, when x h function is 0 that means, the argument of the delta function is 0 and that is the meaningful point for delta functions.

So, these are the points where we have the meaningful results. So, in order to get the entire picture, I should have a summation and then it becomes simply δ(h(x)) is simply summation for all the roots, how many roots are there? I need to calculate and for each case I am going to calculate this 1 under the condition that this is not equal to 0. This quantity what I am writing right now is should not be 0, if it is 0.

Then we should have other technique for simplicity let us considered this is not equal to 0 and we are getting this. So, this is a very important identity when we have a delta function and now, the next thing is to apply this stuff to find out some problems. So, what kind of problems one can so, let us do one problem.

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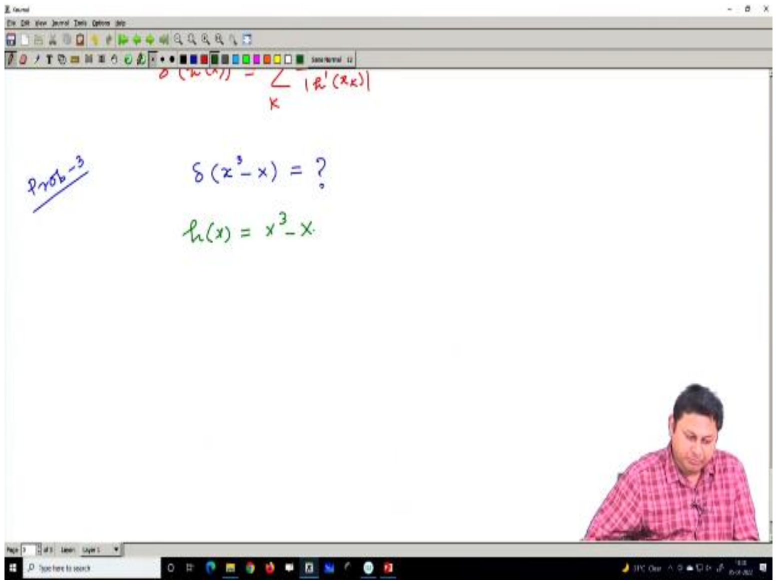

So, this is problem 3, in problem 3, maybe one can ask what should be the value of this quantity? The first problem is what is this quantity, I mean how I can write this delta function, in terms of I will get rid of this argument and the result is one can derive using whatever the expression I just wrote here, this is h(x) here is so, let me write it here. So, my h(x) here, this function x^3 - x.

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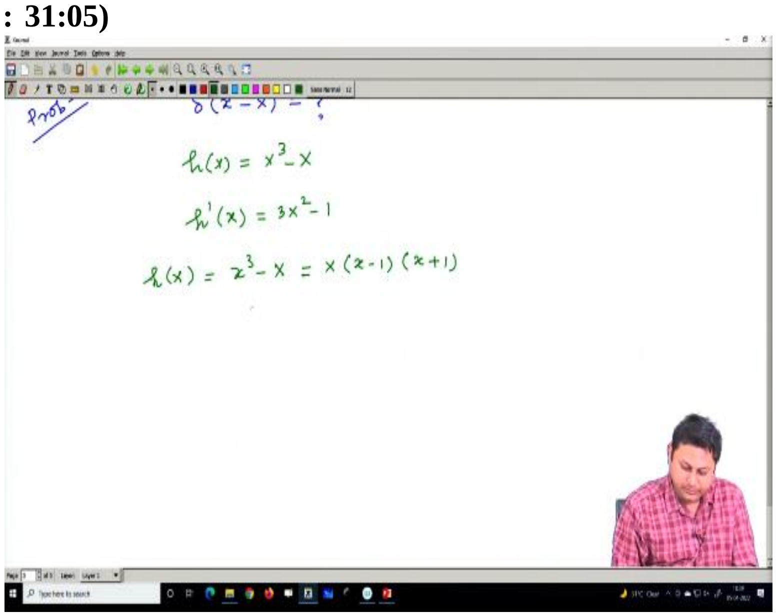

So, I need to find out what is my h' because every time I need to calculate h' = $3x^2$ - 1 this is my h' fine. Now, let us find out what are the roots, so, sorry. So, h(x) given as x^3- x, which should have 3 root and I can write it as x multiplied by (x – 1) and (x + 1).

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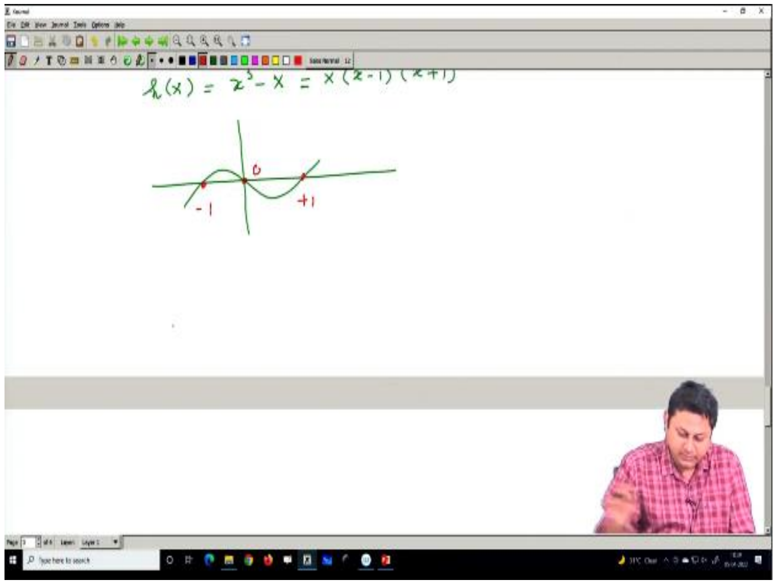

So, the roots are say at -1, 0 and +1. So, we have a root here at -1, have root at +1 and this is 0, these are the 3 roots. So, once we know the roots then we are done.

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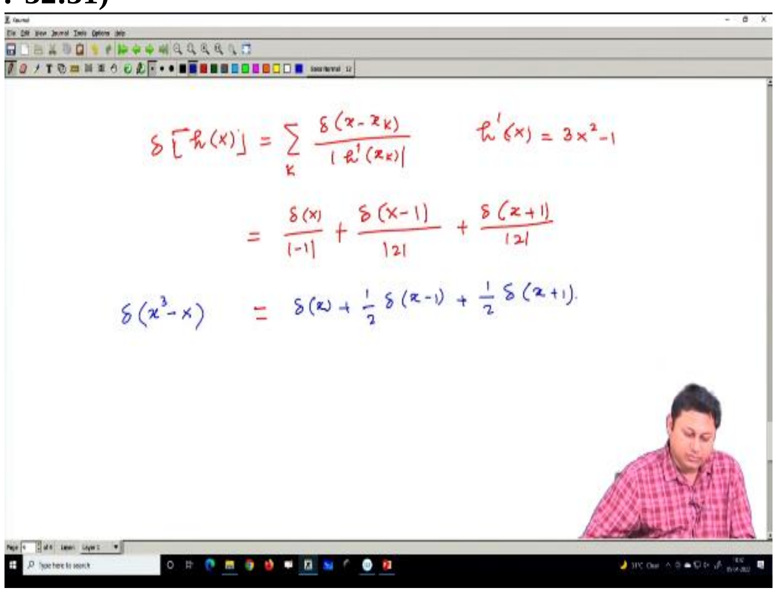

So, according to our expression we have δ [h(x)] is $\sum_k \frac{\delta(x - x_k)}{|h'(x_k)|}$. So, that quantity I know so, for each cases, so, the first case when $x_k = 0$. So, I have $\delta(x)$ here and the denominator if I could, so, what is my h' I should write here $h'(x)$ is $3x^2 - 1$, that we already figured out, $3x^2 - 1$. Now, I put $x = 0$, if I put $x = 0$, here in this equation.

I should have $|-1|$ plus what is the next value for next root $\delta(x-1)$ next root is same 1, so, it should be x - 1 divided by I will put 1 here then we have $|2|$ plus delta next root is -1. So, I should have x + 1 and again I should have here |2|, when I put this value 1 then I should have -1 then I should have this. So, eventually we have this is equivalent to.

So, delta so, finally what we get? Finally, we get that $\delta(x^3 - x)$, which is a function itself is equivalent to if I split this into 3 deltas. So, 1 delta function plus half delta function at minus 1 + half delta function that is plus 1. So, now, since you know this, so, I am giving you a homework that is the last part of this today's work.

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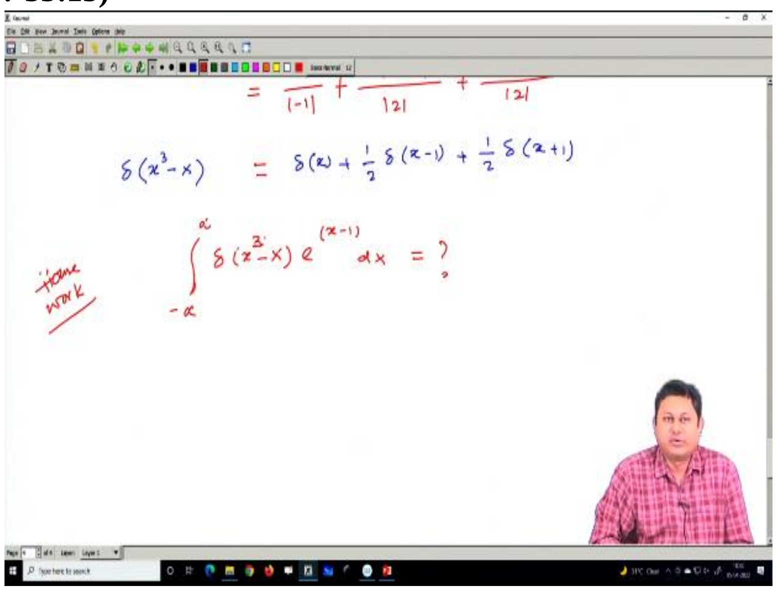

So, this is homework and please evaluate this integral $\int_{-\infty}^{\infty} \delta(x^3 - x)e^{(x-1)}$ dx. Please evaluate this integral integration when we have a delta function having a function in the argument. So, with this note I am not going to have much time. So, with this note I am going to conclude here. So, thank you for your attention and see you in the next class.