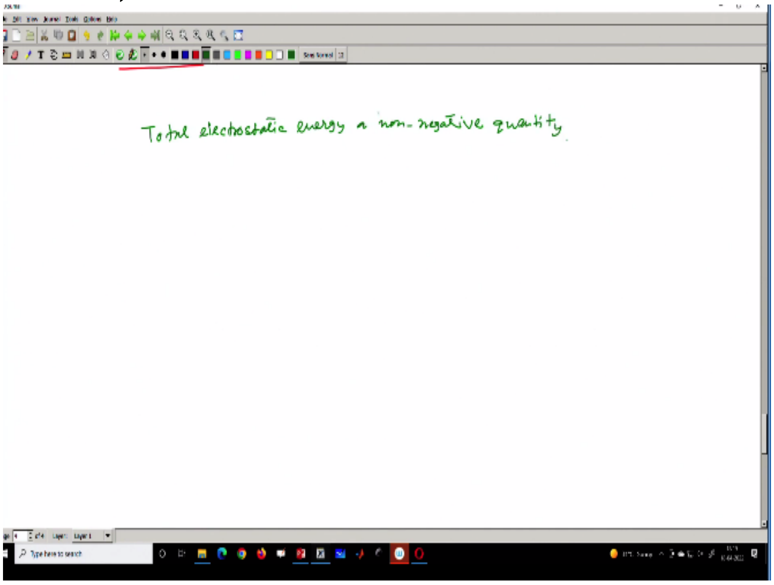
Foundation of Classical Electrodynamics
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology – Kharagpur

Lecture - 31
Electrostatic Energy Calculation

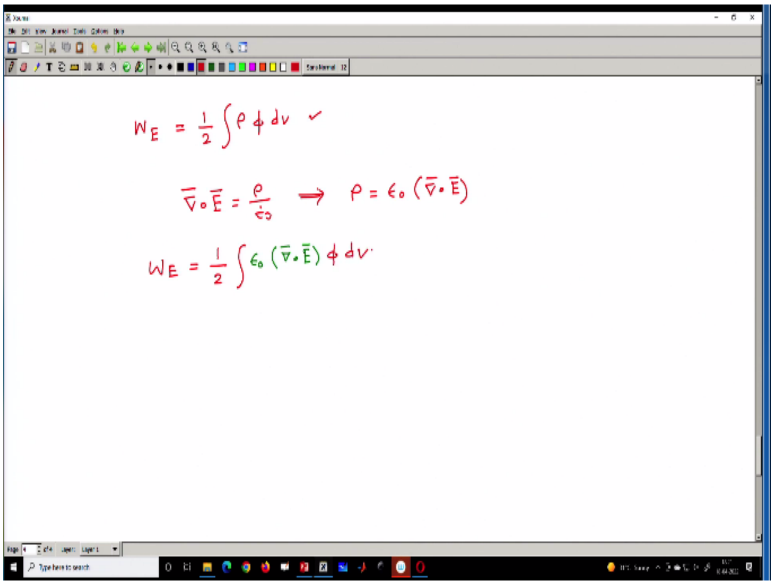
Hello students to the foundation of classical electrodynamics course. So, under module 2 today, we have lecture number 31. And in last couple of classes we discuss about the electrostatic energy and Green's reciprocity theorem etc. So, today we will be going to continue with these concepts of electrostatic energy and try to calculate few problems where the charge distribution is given to you.

(Refer Slide Time: 00:48)



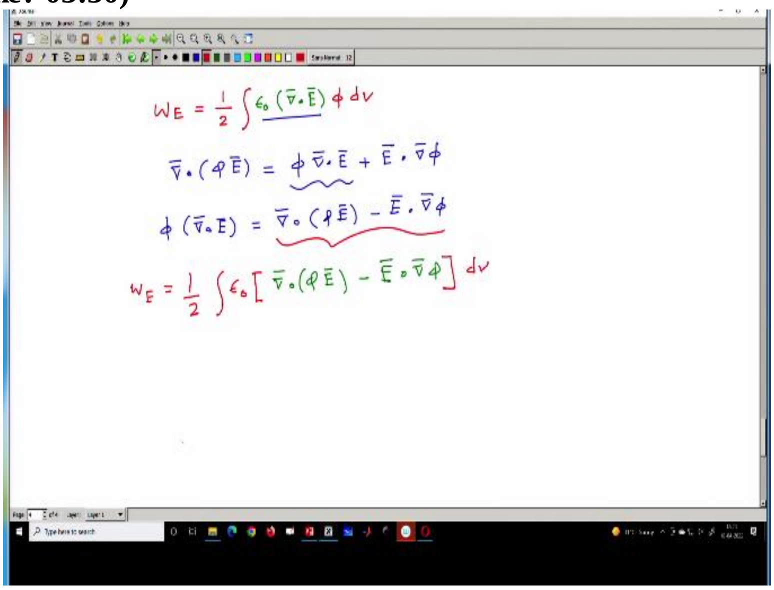
So, today we have class number 31 so before going to the calculation, so let us understand one thing that the total electrostatic energy is a non-negative quantity you need to show that actually that this is non-negative and there are ways to show it.

(Refer Slide Time: 02:01)



So, as per our definition this is $\frac{1}{2}$ ρ ϕ dv. Now we will try to show I mean another way these things, so, the $\vec{\nabla} \bullet \vec{E}$ is $\frac{\rho}{\epsilon_0}$ that we know and that leads to the value of the ρ in terms of $\vec{E}$ and that is ρ = ϵ_0 multiplied by $\vec{\nabla} \bullet \vec{E}$ that thing I should plug in this equation. So, then we get $\frac{1}{2}$ then integration ρ in place of ρ I just write it ϵ_0 then the $\vec{\nabla} \bullet \vec{E}$ that part and then rest part is as usual ϕ dv.

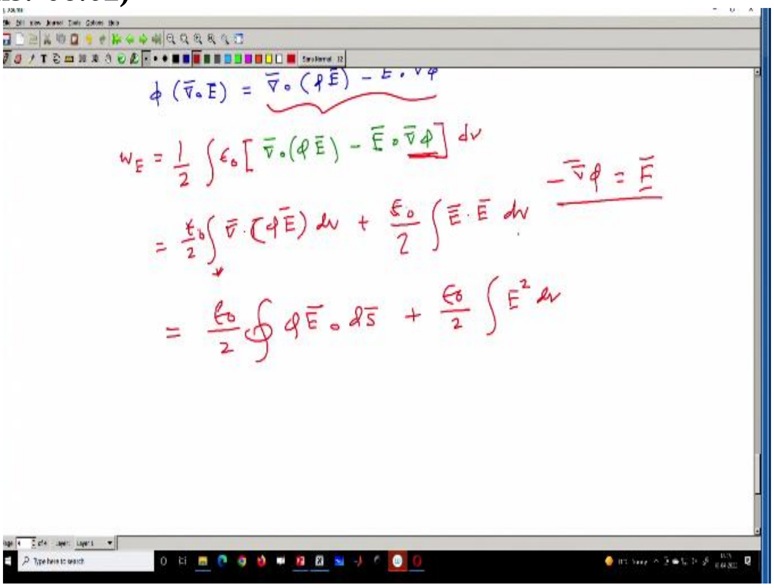
(Refer Slide Time: 03:30)



Now, I am going to use a vector identity. So, the vector identity is the divergent of a scalar field multiplied by the vector field, this is a very important and that we know that this is the ϕ the scalar field multiplied by the divergence of the vector field and then plus the vector field dot the gradient of the scalar field that we know. So, I can from here I can see this term here gradient of vector field multiplied by so, this quantity is sitting here so, I replace this with other 2 so that means ϕ multiplied by $\vec{E}$ is simply minus that thing I replace.

Then I will simply have it is half integration then I can have this enter quantity. So, ϵ_0 then in the bracket I should have this quantity whatever is written here. So, let me do it in different colours. So, it should be this, bracket close and then dv this is the term.

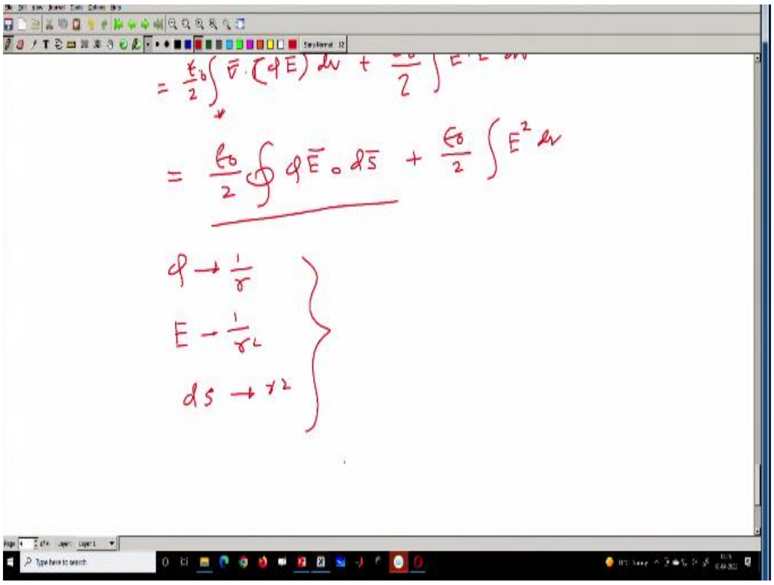
(Refer Slide Time: 06:02)



And so, I can divide this into 2 parts so, half integration of say ϵ_0 I can take it outside and then the other term is this $\vec{\nabla} \bullet (\phi\vec{E})$ dv and then minus of $\frac{\epsilon_0}{2}$ then integration of this term here I am having this quantity, which is you know $\vec{\nabla}\phi$. So, the $\vec{\nabla}\phi$ is simply the minus of the electric field $-\vec{\nabla}\phi$ this quantity so, that if I put them I should have a plus sign and simply $\vec{E} \bullet \vec{E}$ here with dv.

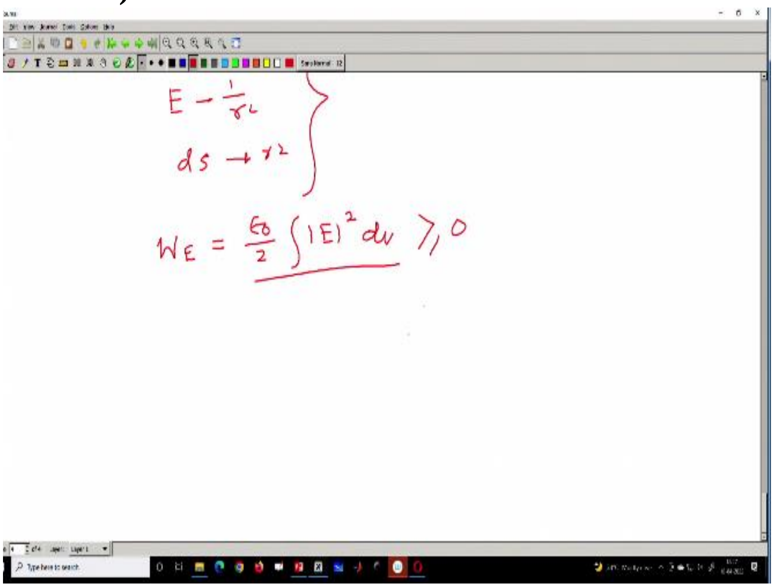
Now, this is a if I make a this is over volume integral so, I can make it a closed surface integral according to the divergence rule. So, which is $\phi\vec{E}$ over the surface $d\vec{s}$ on the other hand the right-hand side other term is $\frac{\epsilon_0}{2} \int E^2$ dv because it is $\vec{E} \bullet \vec{E}$ now, if you look carefully this term.

(Refer Slide Time: 08:21)



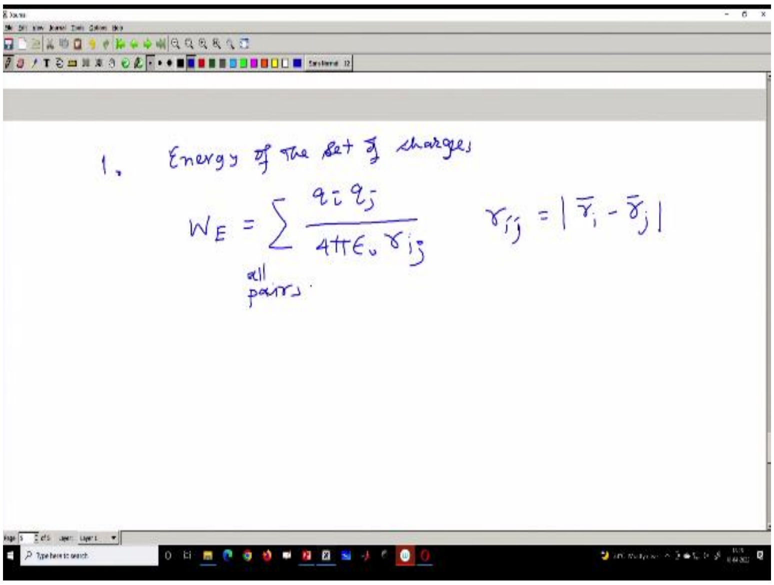
Then for a point charge the ϕ varies as $\frac{1}{r}$, E varies as $\frac{1}{r^2}$ and ds is the surface element for polar coordinate that should varies as r^2. So, that means, if I take an infinitely large or very large surface then the contribution of these things will be 0 the contribution of this part will be 0.

(Refer Slide Time: 08:50)



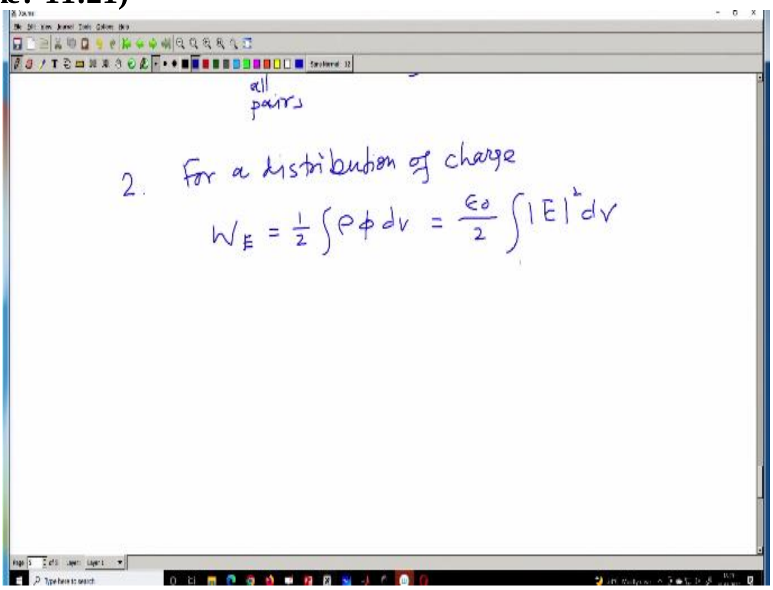
So, that means, the W_E is simply equal to $\frac{\epsilon_0}{2}$ $\int |E|^2$ or E^2 and then dv and that quantity is obviously greater than equal to 0. So that basically proves that the electrostatic energy is a non-negative quantity. So, this is one way to show that and also this expression that we derived from here that value is $\frac{\epsilon_0}{2}$ E^2 it should be useful and we will be going to use that maybe in future calculations.

(Refer Slide Time: 09:47)



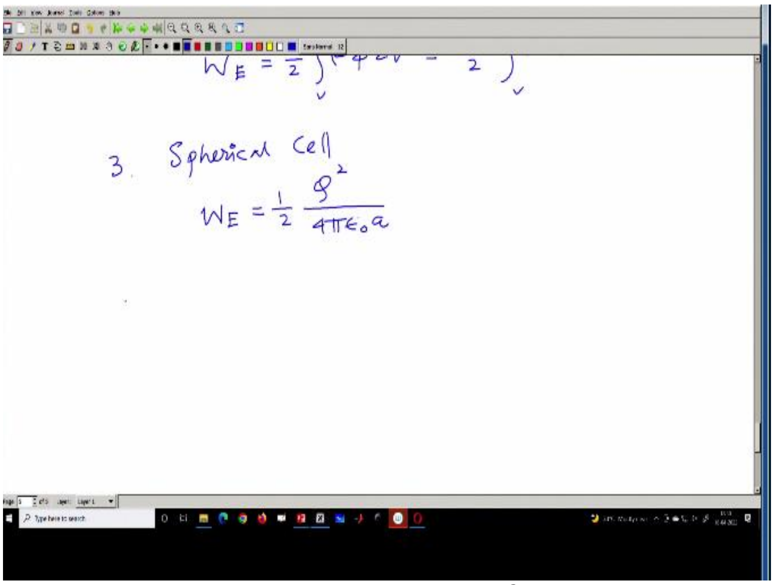
After that for electrostatic energy for different system if I whatever the information we have so far if I write it down. So, energy of the set of field set of discrete charges set of charges is $W_E = \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$, r_{ij} is nothing but the distance between these 2. So, many places it is written in this way. So, that is why I am writing r_{ij} is $|\vec{r}_i - \vec{r}_j|$ and this is for all pairs. So, that we already discussed.

(Refer Slide Time: 11:21)



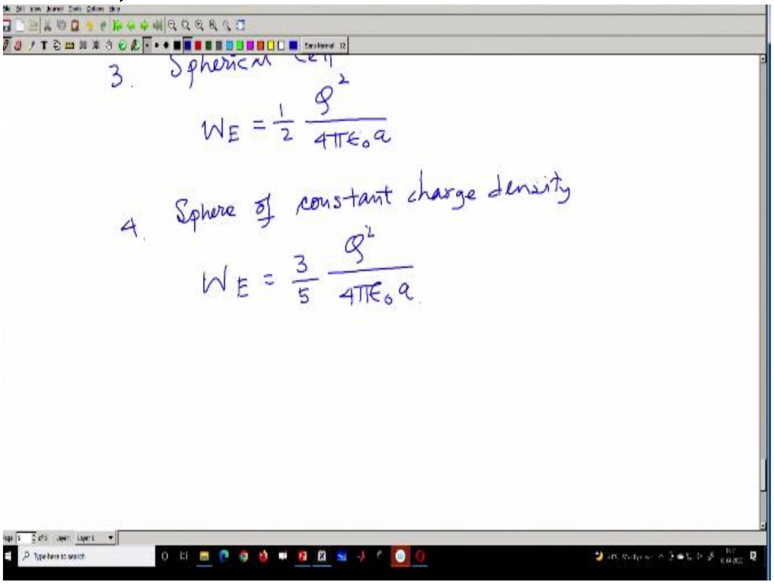
So, 2 now, today we are going to make a problem with that, we will show 1 problem where we are going to use this expression. So, for a distribution of charge what we have is this quantity this is $\frac{1}{2}\int \rho\ \phi$ dv and that quantity we just before calculate this is $\frac{\epsilon_0}{2}\int |E|^2$ dv or E^2 dv or E dot dv over the volume integral that is the value.

(Refer Slide Time: 12:21)



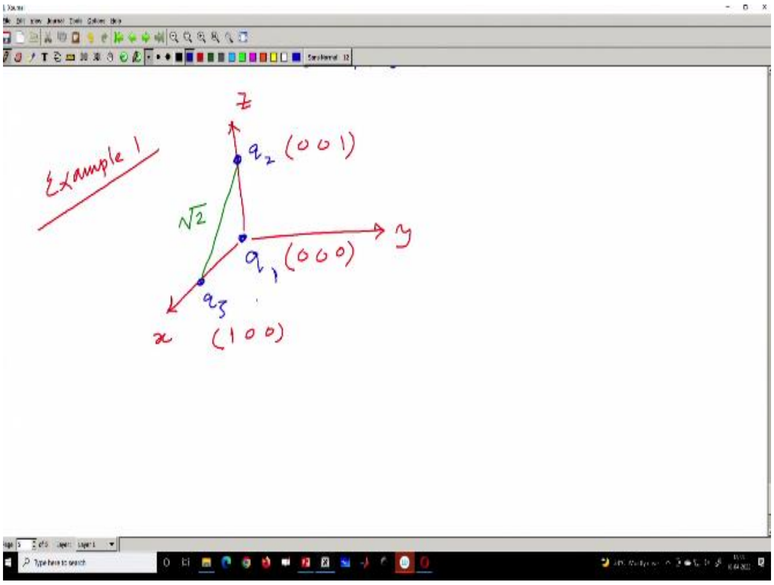
Third is for spherical shell will show that the W_E is $\frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 a}$ where a is the radius of the shell.

(Refer Slide Time: 12:59)



And finally, the sphere of constant charge density that value is $\frac{3}{5}$ and then $\frac{Q^2}{4\pi\epsilon_0 a}$ that is the values I just write it here. So, now one by one we are going to see the first thing is the example 1.

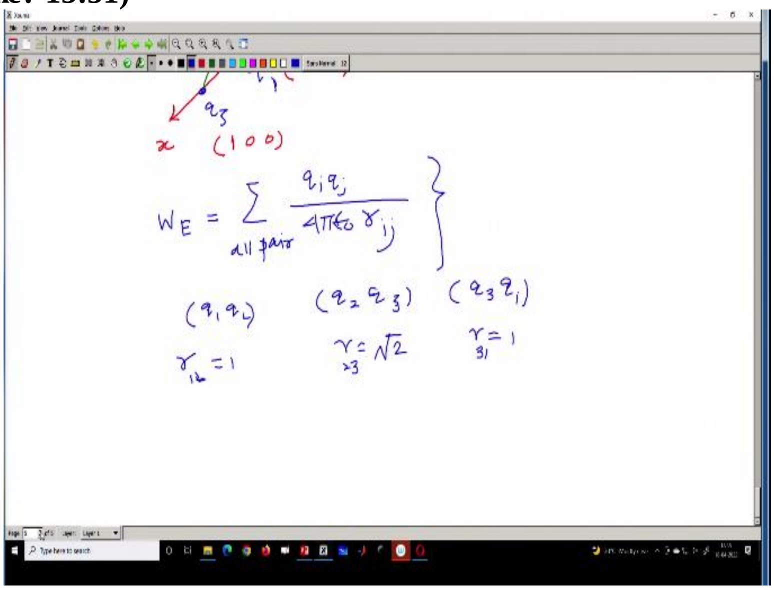
(Refer Slide Time: 14:06)



So, now first I will show example 1 and that is the point charge thus the point to find out so, this is the distribution of the charge that is given. So, one charge is sitting here say this is q_2 another charges sitting here say q_3 another charges sitting in the origin say q_1 so, this distance so the location of these points are given so, this is q_1. So, the location is write here, the coordinate, this is 0 0 0 the coordinate of this point is say if this is my x axis, this is my y axis and this is my z axis.

Then it is 0 0 1 and the coordinate of this point q_3 is over x axis so, this is 1 0 0. So, the distance between these 2 points I can calculate easily, which we may be requiring and that is root over of 2. Now, the question is if I am having these 3 charge distribution q_1, q_2 and q_3 then what should be the energy.

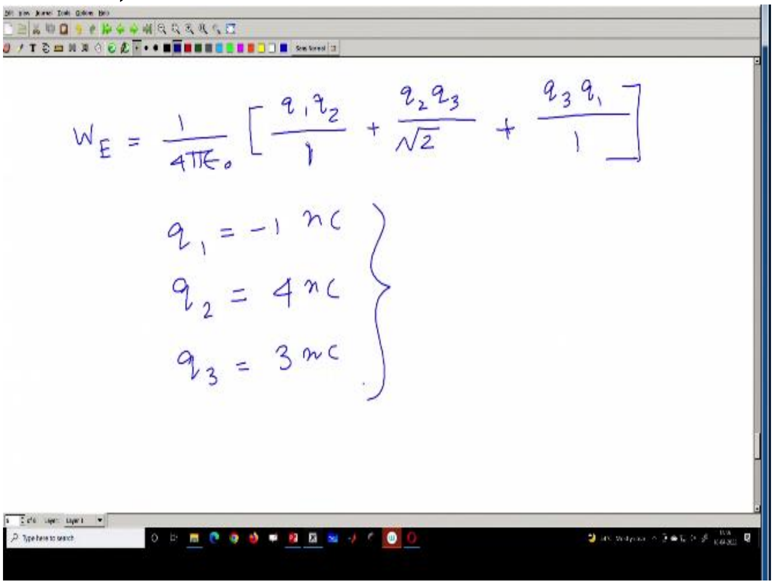
(Refer Slide Time: 15:51)



Now, this I know, so, this is the formula for discrete charge, all pair and then $\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$ this is the formula we know. So, we are going to use that here. If we use those, I should have 3 pairs one pair is q_1, q_2 another pair is q_2, q_3 and another pair is q_3, q_1. So, the distance between these 2 pairs q_1 and q_2, I can also find out q_1 and q_2 the distance is 1, for q_1, q_3, the distance is again 1 and for q_3, q_2 the distance is root over of 2, this is root over of 2 and the distance between these 3 q_3.

So, I should write this is 1 2, pen is not working properly this is 1, 2, this is 2, 3 and this is 3, 1 for 3, 1 this is 1.

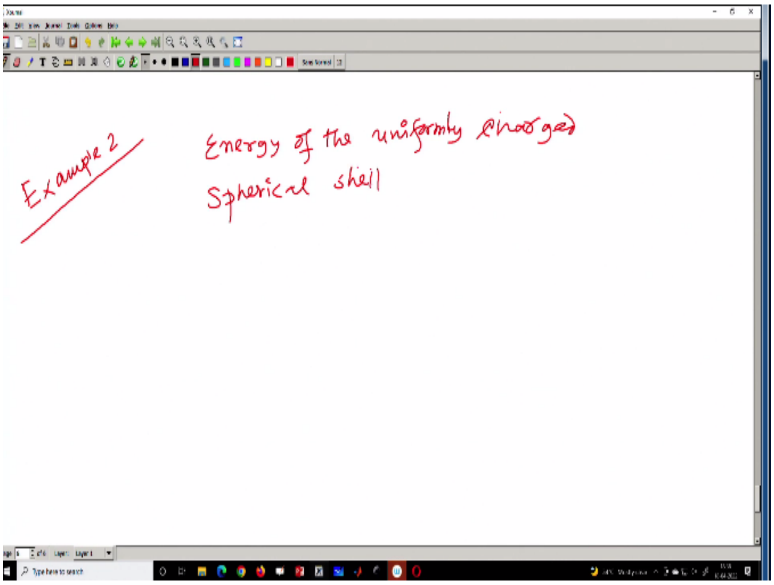
(Refer Slide Time: 17:55)



Now, I will just simply use the formula and if I do then I simply get $\frac{1}{4\pi\epsilon_0}$ this is my W_E, $\frac{1}{4\pi\epsilon_0}$ then $\frac{q_1 q_2}{1} + \frac{q_2 q_3}{\sqrt{2}}$. So, this is 1 and plus $\frac{q_3 q_1}{1}$ so, that should be my answer. Now, in the problem the value of q_1, q_2, q_3 can also be given and I am just putting the value here suppose q_1 is given as minus of 1 nano coulomb, q_2 is given I am just putting some arbitrary values q_2 is say 4 nano coulomb and q_3 is say 3 nano coulomb.

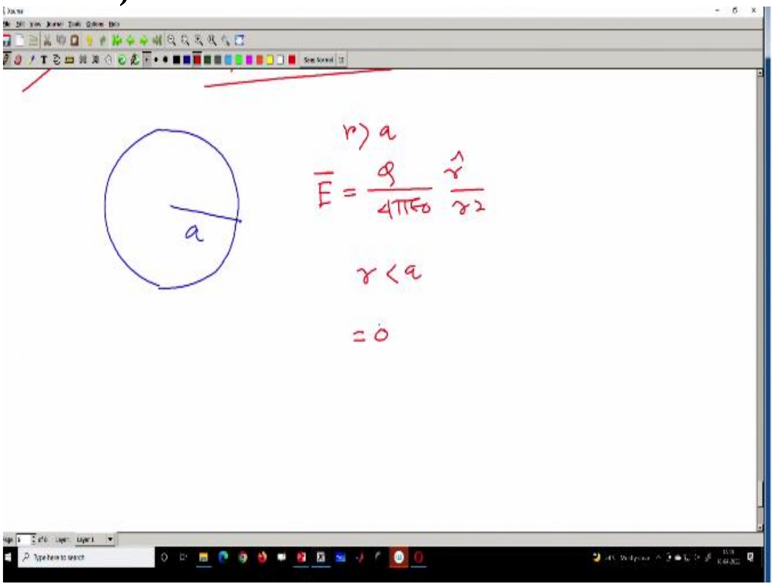
So, these are the value of the point charge and if this value is given and this is the distribution that is given. So, you can calculate what should be the total energy of the system just put the value of q_1, q_2, q_3 here in this equation and you will get the result.

(Refer Slide Time: 19:28)



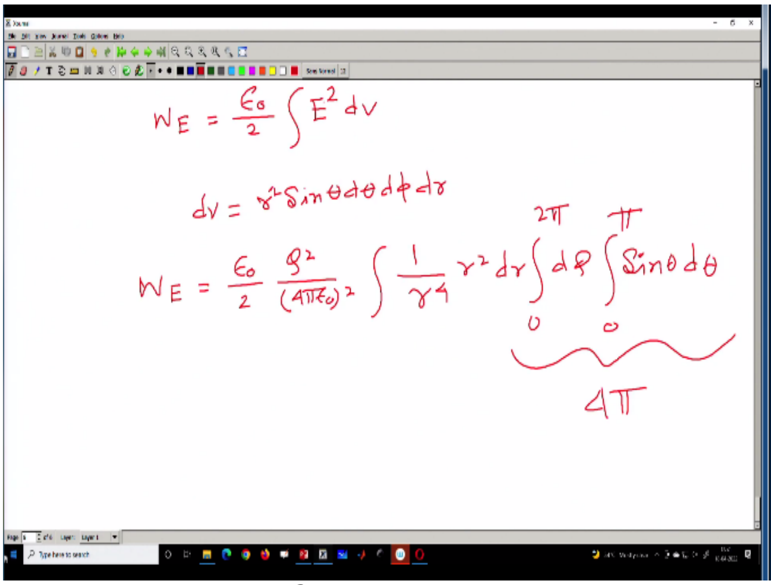
Now, let us go to another problem example 2 this example is try to find out the uniformly so energy of the uniformly charged spherical shell is a very standard problem and most of the textbook it is given, for example, in the Griffiths, you will find this problem.

(Refer Slide Time: 20:31)



I am just trying to do this. So, this is the spherical shell with radius a, we already had this expression, just before I wrote this that the value should be $\frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 a}$ here, I have already mentioned this. So, now I am going to show it meticulously how to calculate? So, the electric field we know for this shell at the point r greater than a is simply using the Gauss's law, it should be $\frac{Q}{4\pi\epsilon_0}$ (because Q is the total charge) and $\frac{\hat{r}}{r^2}$ that should be the value. And for r less than a this value is 0. So, there will be no electric field inside the shell, which is charged.

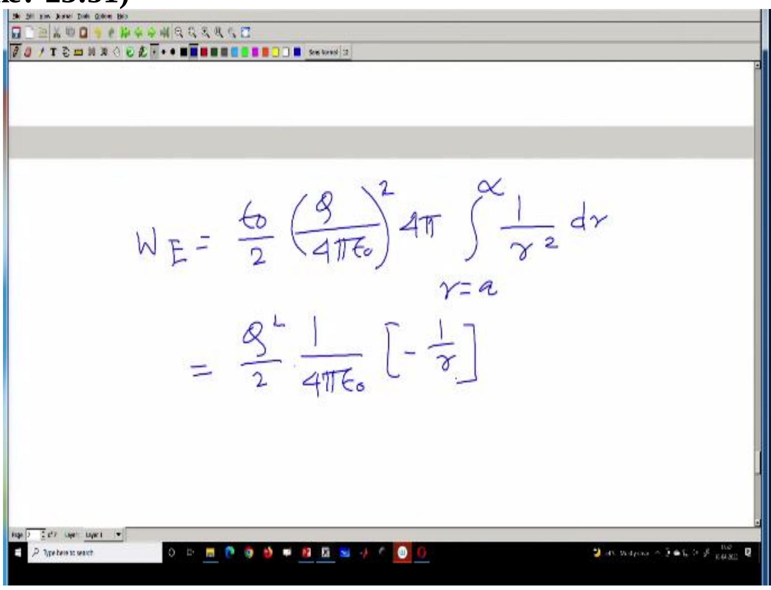
(Refer Slide Time: 21:37)



So, the energy I know the expression E^2 dv so, I just need to calculate this. So, dv here is a spherical shell. So, dv should be the volume element due to the coordinate in the spherical polar coordinates, it is $r^2 \sin\theta\, d\theta\, d\varphi\, dr$ this is the volume element we known. Now, the E value is known is over here. So, that means the energy should be, so whatever the electrostatic energy we have from the infinity to $2a$.

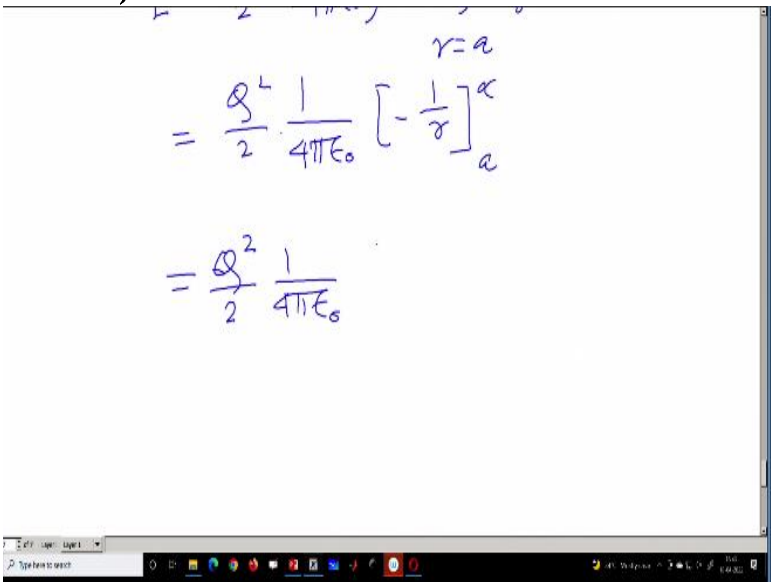
So that means here and then so, I have E^2, so I have $\frac{Q^2}{(4\pi\epsilon_0)^2}$. And then I need to have this integration, the integration inside I should have $\frac{1}{r^4}$ and then r^2 dr and then I have 0 to 2π, $d\varphi$ and 0 to π $\sin\theta\, d\theta$. So, we know that this contribution always comes up to be 4π. So, that thing I am going to use here, I am going to put here.

(Refer Slide Time: 23:51)



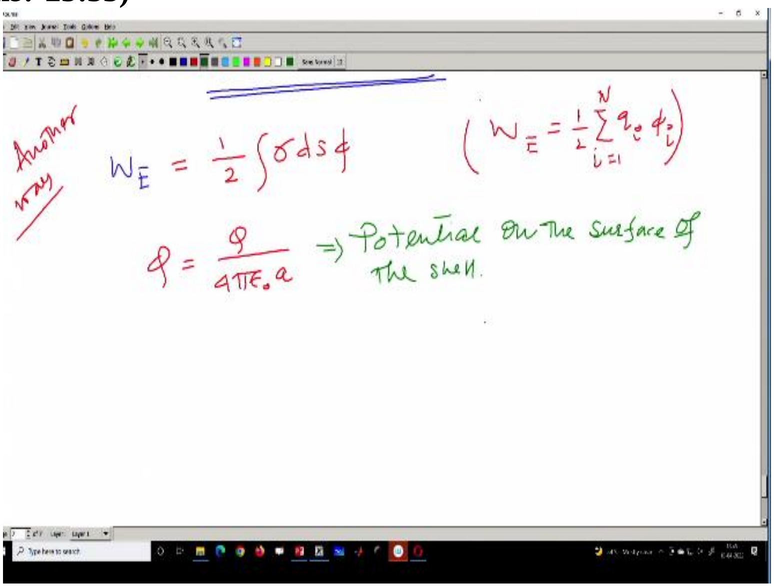
So, this quantity simply becomes $\frac{\epsilon_0}{2}$, then whatever we had $\frac{Q}{4\pi\epsilon_0}$ and the square of that quantity, which we had already here and then this 4π will come out integration and we have the integration r from a to ∞ $\frac{1}{r^2}$ and dr. So, this simply gives us Q^2, 4π one 4π will cancel out, one ϵ will cancel out and that should be $\frac{Q^2}{2}$ then $\frac{1}{4\pi\epsilon_0}$ and then this quantity is $-\frac{1}{r}$ then a to ∞.

(Refer Slide Time: 25:05)



So, now if I do the integration it should be simply $\frac{Q^2}{2}$ and then $\frac{1}{4\pi\epsilon_0}$ and then this quantity is $\frac{1}{a}$. So, that is the expression precisely that is the expression we are looking for.

(Refer Slide Time: 25:33)

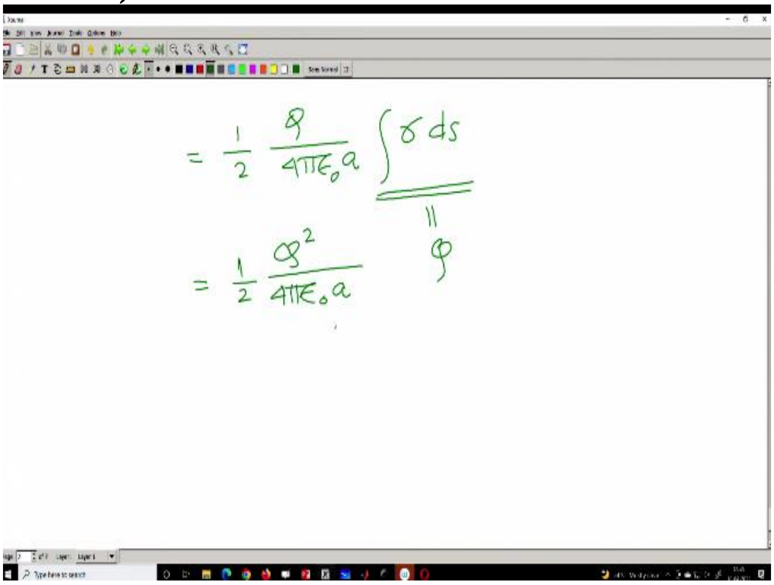


So, this is another way you can also do and that is this is another way, this quantity we know that for surface charge distribution, I can write it as σ and ds this is a charge and then ϕ potential because W_E for discrete charge I just use that concept that it is half and then this is sum over q_i

ϕ_i integration over i 1 to N this is for discrete charge and now, if we have a continuous charge distribution, so, this q_i charge can be represented as the surface charged density σ over ds.

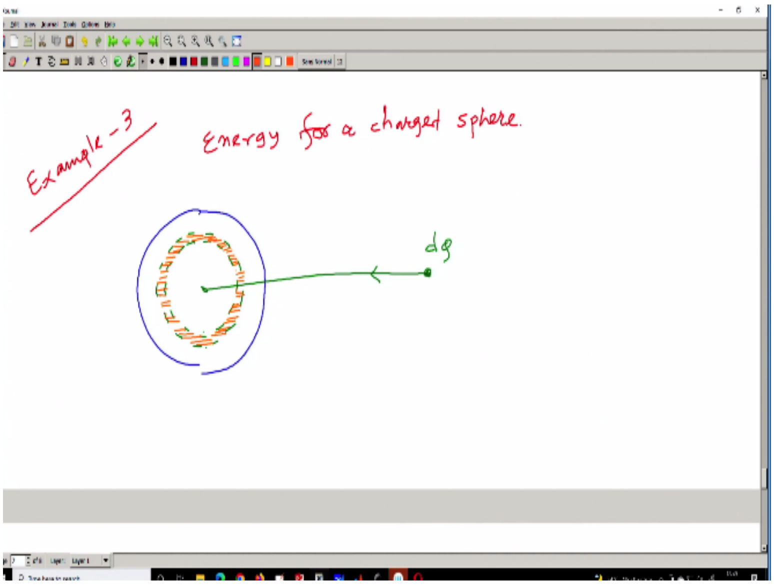
Now, this ϕ you know ϕ is the potential and that value is $\frac{Q}{4\pi\epsilon_0 a}$, which is the potential on the surface of the shell this quantity is nothing but the potential on the surface of the shell so, the potential is known, so, I just simply use that because this does not depend on the integration.

(Refer Slide Time: 27:28)



So, simply I have $\frac{1}{2}$ and this quantity is $\frac{Q}{4\pi\epsilon_0 a}$ and then I have integrations σ ds. Now, this σ ds is nothing but the total charge, this is the total charge. So, I can have $\frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 a}$, which is the same result that we had in the previous calculation the same thing in a different way $\frac{Q^2}{4\pi\epsilon_0 a}$ and then multiplied by $\frac{1}{2}$.

(Refer Slide Time: 28:33)

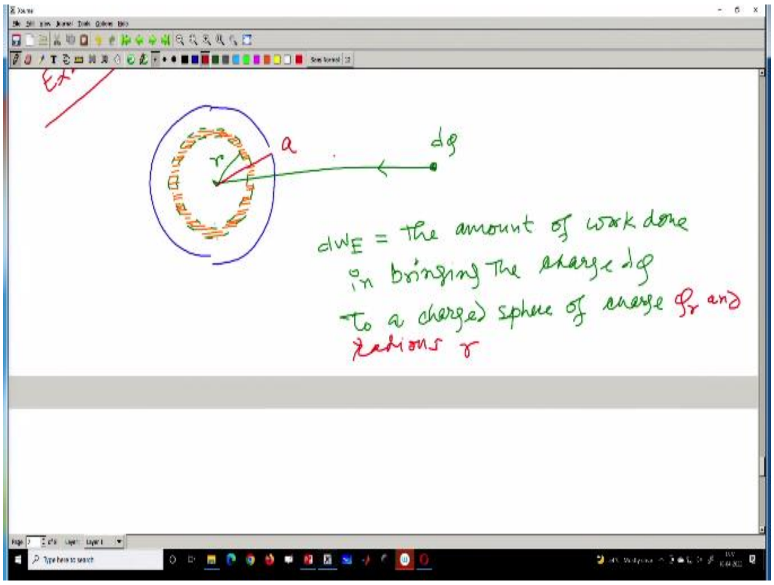


Now, finally, I will like to do the problem this is for spherical shell and now, I will have for this is example say example 3. So, now we calculate the electrostatic energy for a charged sphere so, the problem previously was shell. So, that means, I simply have a very you know tiny shell here like this but now it is sphere so, it is simply a sphere. So, the energy of the sphere if you calculate, so, it is simply the work done in gathering the charges together from the infinity, so I have a charge here.

So, I will gradually bring this charge to form this sphere and in order to do that, I need to do some work and that work should be equivalent to the electrostatic energy. So, for example, if bring you the first charge then there is nothing here, but when you put the second piece of charge, then it will experience some kind of field due to the charge that is already there and so on this process will go on and you need to do some work to bring more and more charges to form this sphere.

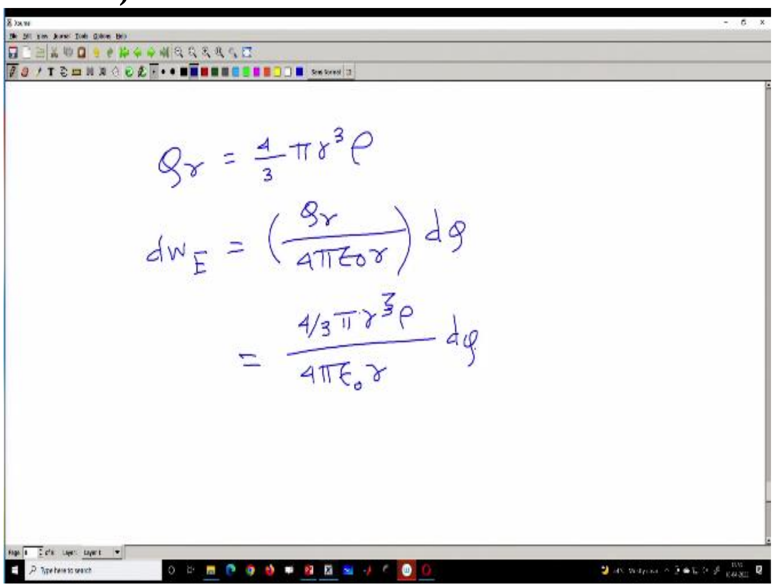
So, exploiting that idea we will calculate so that means, this is the amount of charge dQ I bring from infinity and then gradually form this sphere. So, let us make a cut here and make a shell here so, that is the region I want to fill with the charges dQ so this is the region of my interest. So, I bring the charge here to gradually build this surface sphere, but let us start with this so, this is the region first we are going to build. So, how I write mathematically?

(Refer Slide Time: 32:10)



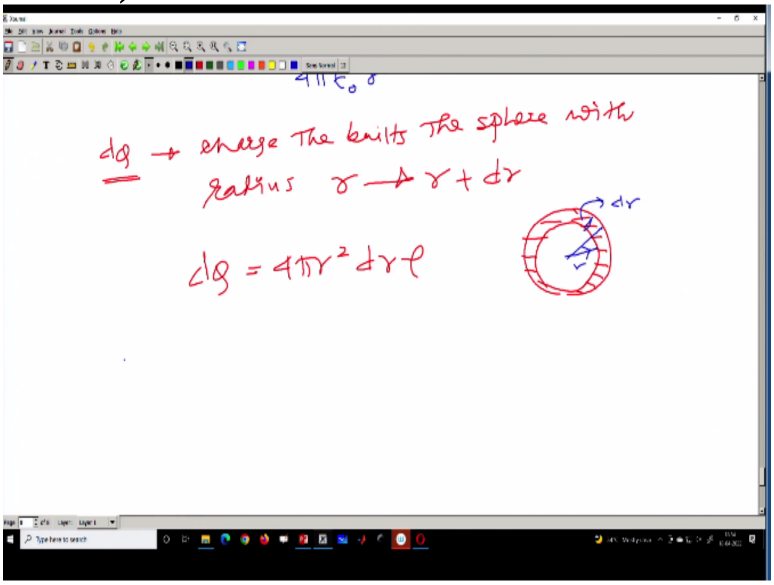
So, mathematically is dW_E, which is the amount of energy that so, the amount of work done this is basically, the amount of work done in bringing the charge dQ to a charged sphere of charge say. So, this is now r and already this amount of charge is there. And this is the radius *a* and already this charge is there. So, this charge Q_r, which is already formed is sphere of charge of radius r of charge Q_r and radius r so, this radius is gradually increasing when I bring the charge from infinity so, this is *a*. So, first we calculate what is the amount of the charge you already have? So, this is I am saying Q_r.

(Refer Slide Time: 34:21)



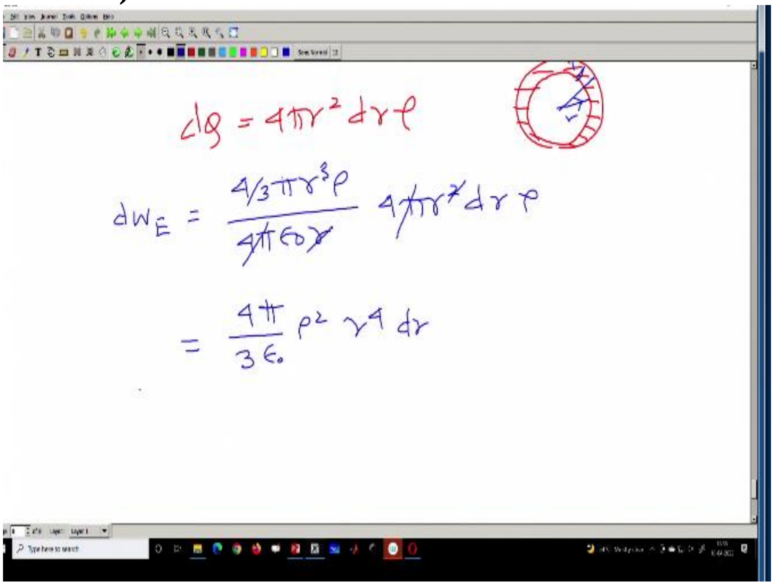
So, what is the charge already it has? So, Q_r is simply $\frac{4}{3}$ πr^3 and since it is a uniform charge distribution. So, ρ is the charge density. So, the amount of work that we do is simply the potential that it is generating that is $\frac{Q_r}{4\pi\epsilon_0 r}$ and the amount of charge I bring dQ, Q_r is already there. So, I have $\frac{4/_3\pi r^3\rho}{4\pi\,\epsilon_0 r}$ dQ. So, then dQ is the charge that build up this sphere with radius r_2.

(Refer Slide Time: 35:31)



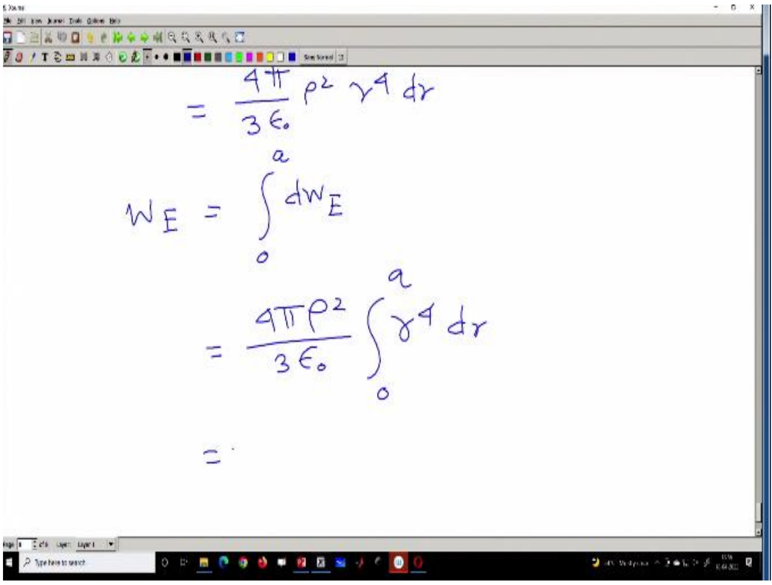
So, dQ is the charge so, dQ is the amount of charge the charge that builts the sphere with radius r to r + dr so, just to increase the amount of charge that increase the radius r to dr. So, what is dQ then? dQ in terms of ρ you can calculate this is simply $4\pi r^2$ and then dr and ρ that is the region. The amount of charge in this region having radius r to dr so, this is dr and this is r. So, this from here to here it is dr so, now, I have dQ.

(Refer Slide Time: 36:55)



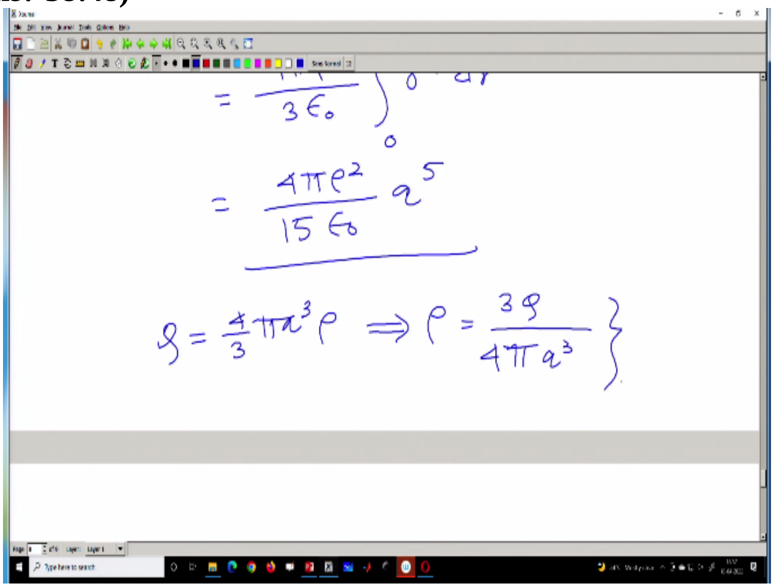
So, dW_E is simply $\frac{4}{3}$ whatever we had already $\frac{4/_3 \pi r^3 \rho}{4\pi \epsilon_0 r}$ and in place of dQ I simply write $4\pi r^2$ dr and then ρ. So, it seems that 4π 4π will cancel out and few more term will cancel out. So, one r and one r^2 will cancel out, ϵ_0 remain same. So, eventually I have 4π here and then divided by 3 ϵ_0 and then ρ^2 should be here and then I have r^4 dr that should be mine.

(Refer Slide Time: 38:06)



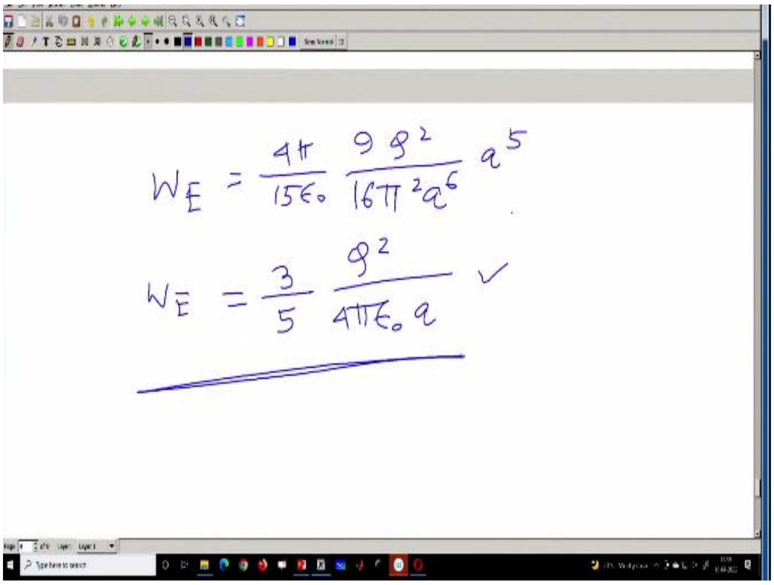
So, now, I need to create the entire sphere so, that means my W_E that is the total energy and that should be integration of the total thing with the limit 0 to a because I am forming the sphere of this radius from 0 to a radius. So, then I just simply integrate it so it is $\frac{4\pi\rho^2}{3\epsilon_0}$ this is a constant and then I should have r^4 0 to a dr and that quantity is $\frac{r^5}{5}$.

(Refer Slide Time: 38:48)



So, simply we have $\frac{4\pi\rho^2}{15\epsilon_0}$ and then a^5 that value. Now, I can return back because the total charge Q is there. So, I need to just change this ρ in terms of Q. So, Q is the total charge and that is $\frac{4}{3}$ πa^3 ρ. So, from here I can write ρ is simply equals to $\frac{3Q}{4\pi a^3}$ that is the value of the ρ in terms of Q and now I put this ρ here in this equation.

(Refer Slide Time: 39:50)



And if I do, then W_E will be simply $\frac{4\pi}{15\epsilon_0}$ and we have ρ^2. So that means the square of this quantity, so that becomes $\frac{9Q^2}{16\pi^2 a^6}$ and already a^5 is sitting here. So, few terms will cancel out, for example, 4 and this here and then 3, and so I should have $\frac{3}{5}$ here and then Q^2 should be here, divided by one 4 will cancel out another. So, I should have 4π here also. So, I missed that so that π will also cancel out.

So, simply I have 4 $\pi\epsilon_0$ and then a so that should be the value of the potential energy of a solid sphere having the charge density ρ and the total charge Q in terms of charge density, we already calculated, but in terms of total charge, it should be this quantity. So that again, we already wrote it earlier, that this should be in a listed way that what would be the energy, I think here we wrote it and then gradually, so this is the value I put earlier. So that again, we meticulously calculate and check.

Today, I do not have much time to discuss more. So, I like to conclude my class here. So, in the next class, maybe we will start a new topic like the multiple expansions on all these things, when we have a charge distribution, then what should be the potential at long distance that we discuss, and that potential can be divided into few terms, which we call the multiples, so that we discuss and do few problems also. So, thank you very much for your attention and see you in the next class.