Foundation of Classical Electrodynamics
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Lecture - 45
Electrostatic Boundary Value Problem

Hello students to the foundation of classical electrodynamics course. So, under module 2 today, we have lecture 45 and today we will go to learn the electrostatic boundary value problem mainly today we will go to solve the Laplace equation in different condition in different dimension and try to understand that how the potential can be calculated with this given boundary conditions.

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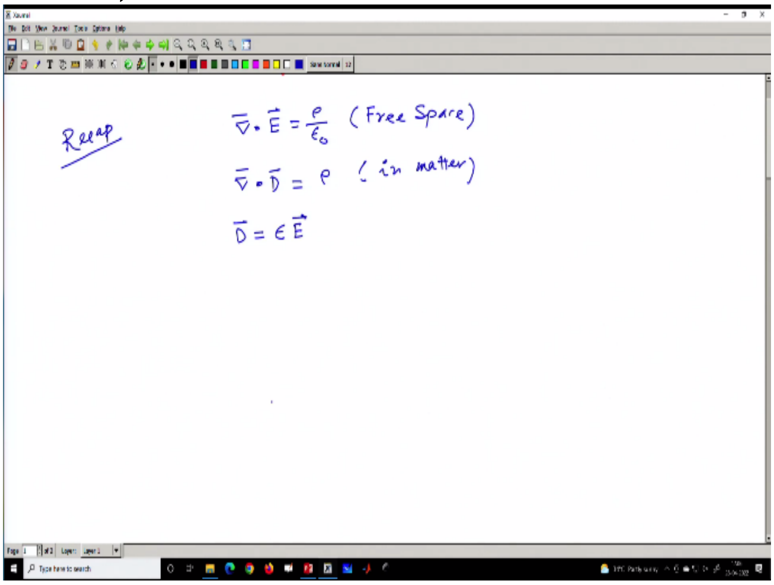

So, we have class number today 45 and our today's topic is electrostatic boundary value problem or in short boundary value problem. So, in boundary value problem before going to do the problem directly let us recap and try to understand how we calculated the portions and Laplace equation. So, first let us start with this equation this is $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and that is for free space we have this expression when electric field is in matters and $\vec{D}$ is $\epsilon\vec{E}$.

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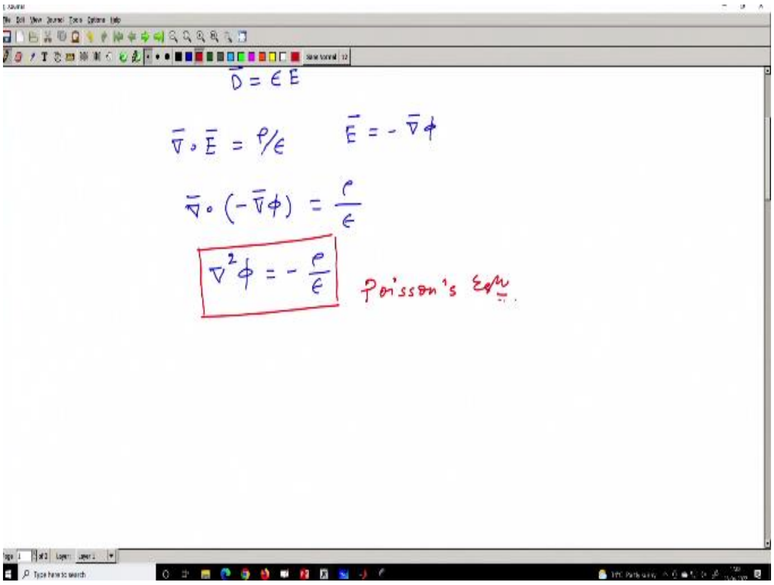

But if I only concentrate on the $\vec{E}$ part, so $\vec{\nabla} \cdot \vec{E}$ is $\frac{\rho}{\epsilon}$ and $\vec{E}$ is $-\vec{\nabla}\phi$. So I can write it as $\vec{\nabla} \cdot (\vec{\nabla}\phi)$ that is $\frac{\rho}{\epsilon}$ and this quantity is simply Laplacian or ∇^2. So, $\nabla^2\phi = -\frac{\rho}{\epsilon}$ this equation is our Poisson's equation.

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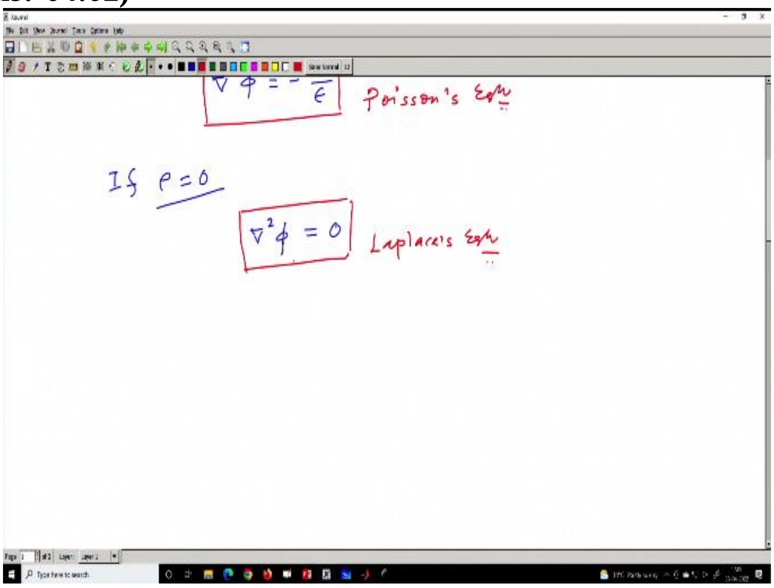

Now, if $\rho = 0$ then we have right-hand side 0 and we have it term without any source in the right-hand side and that is called the Laplacian equation so, these 2 equations we already derived earlier so, this is not new. So, now, what we do that to try to you know solve this Laplace equation because this is nothing but a second order differential equation. So, I can solve this for a given system where two boundary conditions are there so, that we will go to do. So, before that let us write down the form of the Laplace equation let us remind the form of the Laplace equation in 3 coordinate systems.

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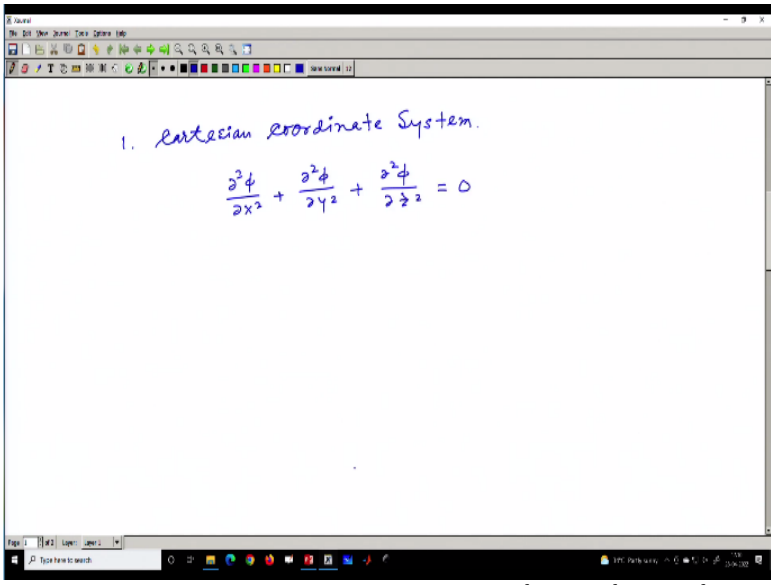

So, 1 in Cartesian coordinate system in 3D it is simply $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$ where ϕ is the potential, this is in Cartesian coordinate system.

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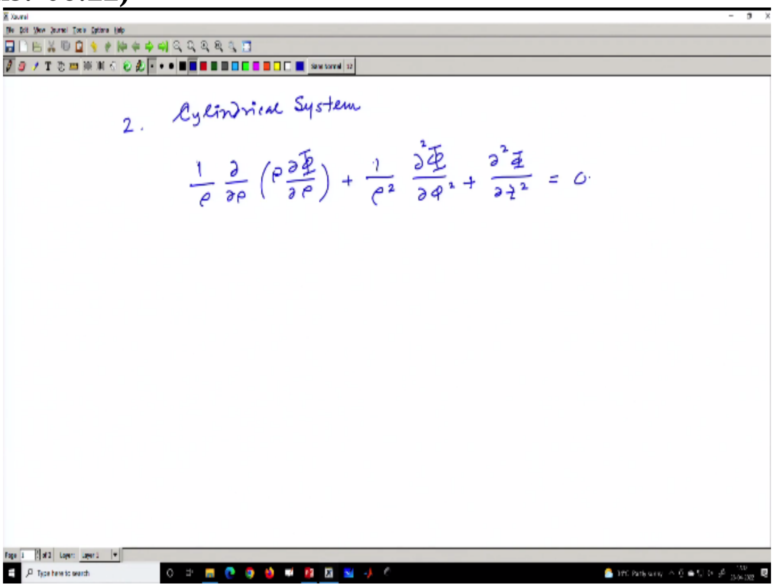

In cylindrical system, it is $\frac{1}{\rho}$ and then $\frac{\partial}{\partial \rho}$ and then ρ $\frac{\partial \phi}{\partial \rho}$ we did it long ago now, we will be going to use it this φ now, I have it so, this ϕ let us put it is a ϕ and I should have $\frac{\partial^2\phi}{\partial z^2}$ so, I should have a second order derivative here $z^2 = 0$.

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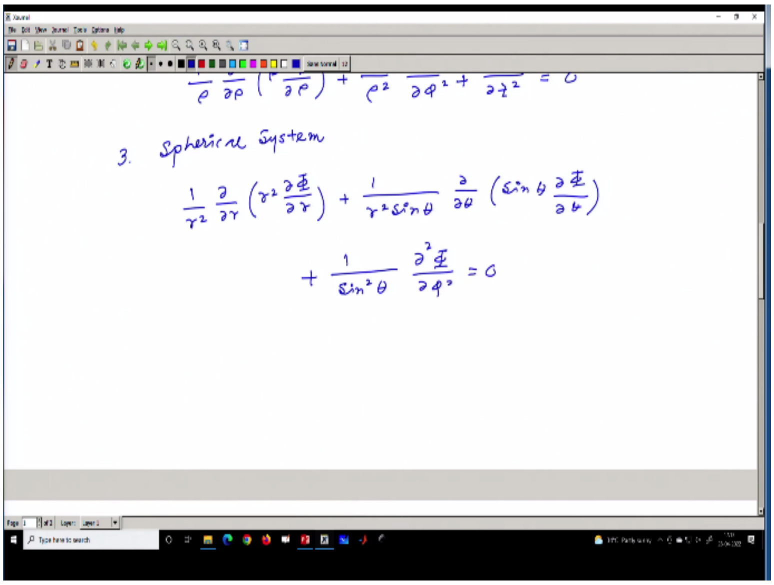

And in you know spherical coordinate system this is $\frac{1}{r^2}\frac{\partial}{\partial r}$ then $r^2\frac{\partial \phi}{\partial r}$ + $\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}$ and then sin θ $\frac{\partial \phi}{\partial \theta}$ and then + $\frac{1}{sin^2\theta}$ I think $\frac{\partial^2 \phi}{\partial \varphi^2}$ = 0. So, I think these are the three coordinates these are the expressions. Now, let us start a problem then only we can understand that how we can exploit this and how can how we can solve this?

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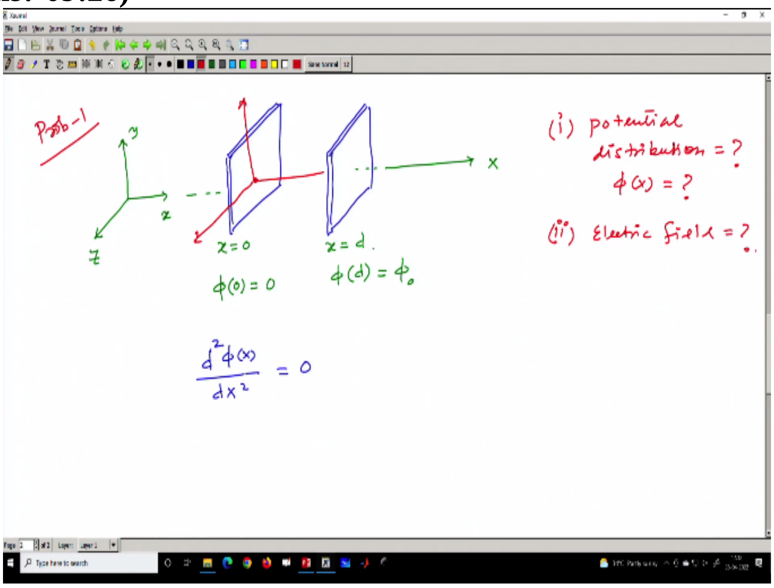

So, let us do problem 1 say so in problem 1 it is saying that the statement of the problem is like that. So, consider a parallel plate capacitor filled with air. So, let us consider a parallel plate capacitor where, you know this is one plate this is another plate so, these 2 plates are parallel and suppose we have this is x direction and it is placed over this axis defined by say x, y and z so, 2 parallel capacitor is there and now this is filled with air.

So, inside there is air and bounded by the conducting plates at x = 0 and x = d. So, this point is at x = 0 and this point is at x = d. So, this is just in the left hand side I am showing that coordinate system, but the origin is sitting here. So, this is my origin so, this, this, this is my actual origin x = 0, point 1 parallel at 1 capacitor is placed and another is at a distance x = d now, given that the potential has the values 0 and ϕ_0 on the left and right boundaries.

So, here the value of the potential is also given and simply the value of the potential ϕ at x = 0 is 0 and potential here is ϕ at (x = d) is ϕ_0 this is given and the question is the left-hand and right-hand boundary solve the Laplace equation to obtain the potential distribution in the electric field between the plates how the potential is distributed here because here we have potential 0 here we have potential ϕ. So, how the potential are going to be distributed in between this space that we need to figure out.

So, basically we need to solve the 1 dimensional Laplace equation in Cartesian coordinate system and that is $d^2\phi$, which is a function of x because it is changing over this and dx^2 equal to 0. So, if I solve this boundary this differential equation and then we can find out how the potential is changing not only that the second part is so, let me write down the problem one by one so, first we need to find out 1 potential distribution this is my first problem is what that is I want to know what is $\phi(x)$.

And second the electric field between the plates these 2 I want to find. So, if I solve this equation then I can find out the explicit form of ϕ and then from that I can calculate the electric field so, that should be the recipe.

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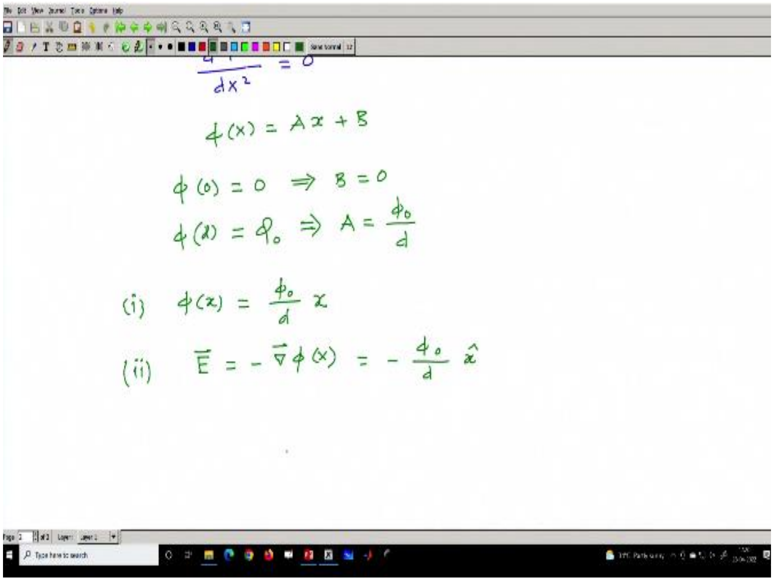

So, if this is 0 then we can readily write that $\phi(x) = Ax + B$ where A and B are constants. Now, let us find out the I mean just put the boundary condition and boundary condition is saying that ϕ_0 is 0, which gives us readily that the B has to be 0 and ϕ d because this distance is d, ϕ d = ϕ_0, which simply gives that the value of A = $\frac{\phi_0}{d}$ very simple. So, the overall my $\phi(x)$ is simply $\frac{\phi_0}{d}$ multiplied by x. So, it is linearly changing and this should be the form. So, the answer of the part 1 is done.

What about part 2? About the electric field $\vec{E}$ we know that it is simply minus of this quantity $\vec{\nabla}\phi$ so, we have $-\frac{\phi_0}{d}\,\hat{x}$. So, that is the solution very simple and very straightforward problem. So, this is just one more problem we do by just solving this Laplace equation under certain condition, now go to problem 2.

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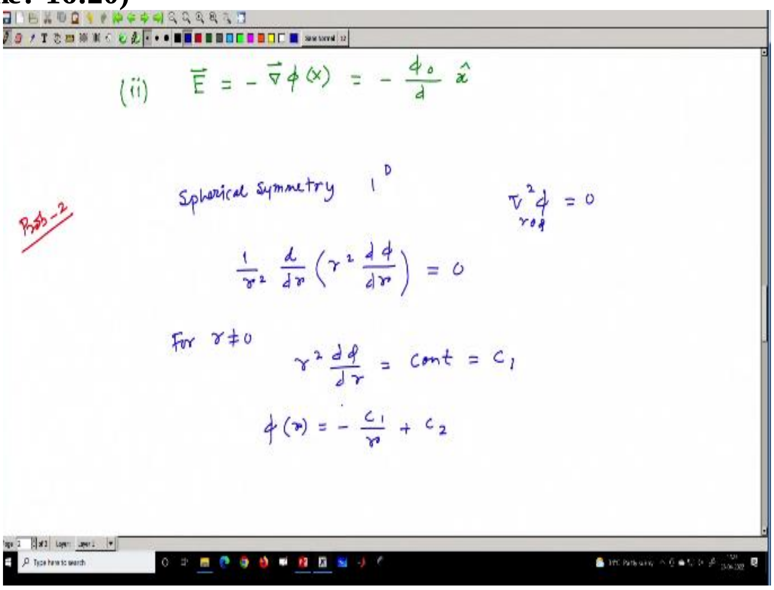

The problem 2 is saying that before going to problem 2, so, maybe we can you know for spherical symmetry, so, let us make a note here in 1 D so, we have the Laplace equation of the form $\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr}) = 0$ this is the form of this Laplace equation in 1 D when the operator is given in r θ φ in 1D θ and φ we can because there is a symmetry so, ϕ should not depend on θ and φ so, we should have only it depends on r then I can write this is my equation.

So, the general solution I can write for r not equal to 0, because if r is 0 then so for r not equal to 0 the general solution so, this becomes $r^2\frac{d\phi}{dr}$ to be some constant say this constant in C_1 so, I can simply find that ϕ as a function of r is you know $-\frac{C_1}{r}$ plus another constant C_2 so, that should be the general you know general solution you just put $\frac{d\phi}{dr}$ is $\frac{C_1}{r^2}$ and then integrate it and you will be going to get simply this. After having the information after just deriving the general solution we are going to use this for this given problem.

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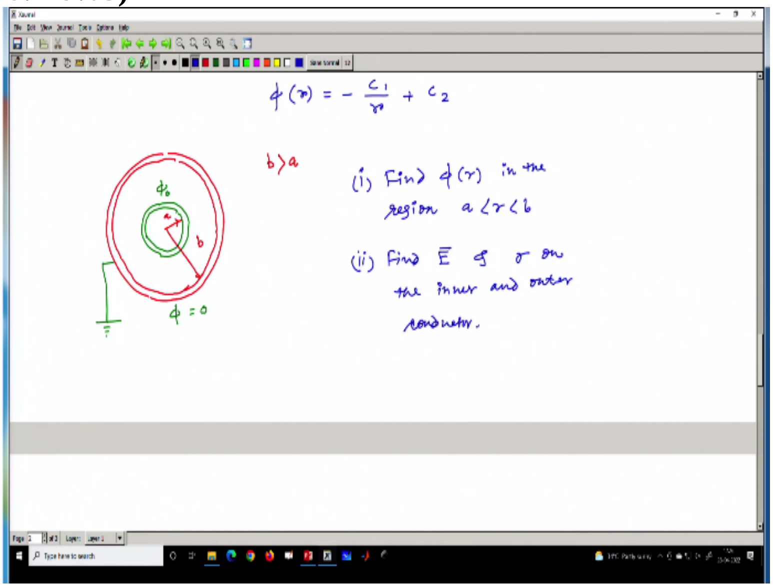

Now, the problem 2, suppose a concentric spherical conductor of the radius a and b are there. So, I have a spherical conductor so, this is the spherical shell suppose it has and 2 concentric spherical conductors are there, so, this is one and another conductor is inside that, where the radius is a and b? So, from here to here it is a and from here to here it is b so, b obviously greater than a, what else the inner one is kept at a potential ϕ_0. So, this the potential here is ϕ_0 and the outer one is grounded.

So, obviously, so if it is grounded so, the potential of this outer sphere has to be 0 find the potential in the region. So, the problem is this is the statement and the problem is find the potential as a function of r in the region a less than r less than b so, in this region we need to find out the potential and also find E and σ on the inner and outer conductor.

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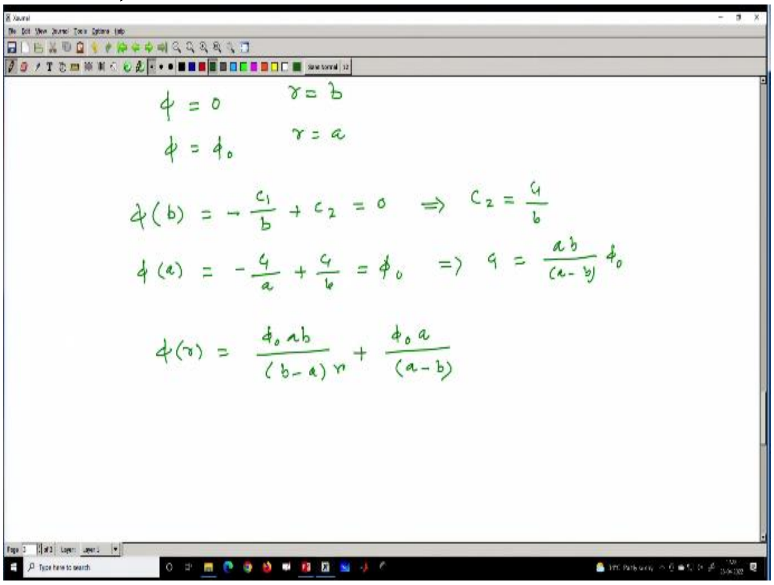

So, let us quickly write the boundary condition, which is given so, ϕ is 0 when r = b, ϕ is ϕ_0 when r = a. So, from that we can because the general solution is here I have already derived it so, I can simply write ϕ(b0 = $-\frac{C_1}{b}$ + C_2, which is equal to 0 that simply gives us C_2 = $\frac{C_1}{b}$, $\phi(a)$ = $-\frac{C_1}{a}$ + C_2 I replace because C_2 I know this is $+\frac{C_1}{b}$, which is ϕ_0 and from simply from here we can find that $C_1 = \phi_0 \frac{ab}{a-b}$ from here and then we have ϕ_0. So, C_1 C_2 I calculate.

So, $\phi(r)$ is simply ϕ_0 ab divided by because this is a negative sign so, I should write b - a and plus because this is divided by r also and then this is $\phi_0 \frac{a}{a-b}$ just simply put the value of C_1, C_2 to the general solution that we figured out this is my general solution. So, here I just put the value of C_1, C_2 and get the result that so that is why I calculated this earlier once you know the value of $\phi(r)$, we can simplify this further and I mean just take (b – a) common or (a – b) common and then do the rest.

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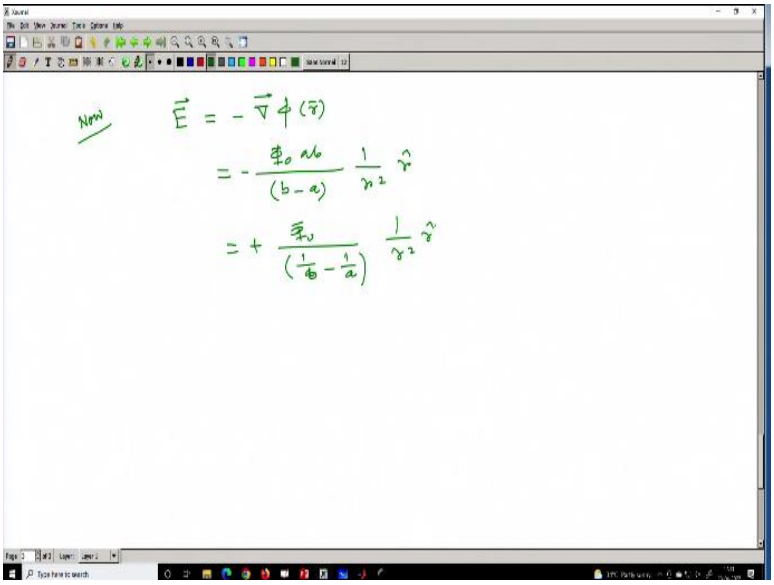

But anyway, so now if I want to find out the electric field $\vec{E}$ because that is another part of the question. $\vec{E}$ is $-\vec{\nabla}\phi$, which is a function of $\vec{r}$ and which just we calculate. So, that thing is seems to be $\phi_0 \frac{ab}{b-a}$ whatever is there and then I have a negative sign $\frac{1}{r^2}$ $\hat{r}$ because we are making simply the derivative with respect to partial derivative with respect to r that is all. So, this should be my E we can simplify it.

So, we can write simply $\frac{\phi_0}{\frac{1}{b}-\frac{1}{a}}$ and already 1 negative sign b minus I make these b - ϕ so, minus and minus so, that should be this minus is there I need to put this minus sign and it should be a minus so, then if I write this then it should be simply plus anyway $\frac{a}{b}$ I am just dividing this with $\frac{a}{b}$ that is all when you divide $\frac{a}{b}$ the b b will cancel out we have $\frac{1}{a}$ - $\frac{1}{b}$ but make this negative sign I make $\frac{1}{b}$ - $\frac{1}{a}$ this negative sign need it seems to me and then I have then $\frac{1}{r^2}$ and this.

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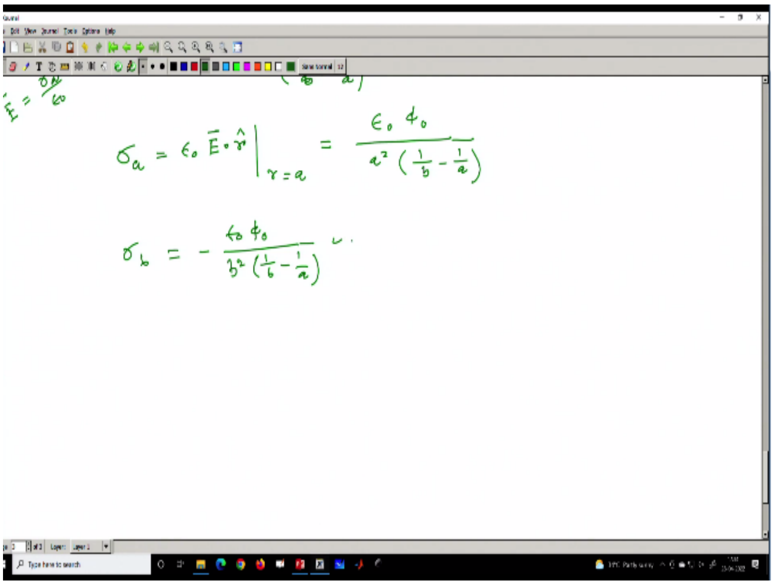

Now, the question is the charge density. The charge density of the inner conductor σ_a that we know because $\vec{E}$ is $\frac{\sigma_a}{\epsilon_0}$ along $\hat{n}$ if I make both the side if I make a dot productive $\hat{n}$ then σ_a comes up to be ϵ_0 then $\vec{E}$ dot here it is simply $\hat{r}$ because in the direction and $\hat{r}$ the same direction and that will be evaluated over the surface this quantity and that simply turns out to be ϵ_0 and then $\vec{E} \bullet \hat{r}$ means this quantity at a.

So, simply $\frac{\Phi_0}{a^2}$ into 1 by whatever I get $-\frac{1}{a}$ it seems to be something like this with I should not put any kind of vector direction because it is a surface charge I already make $\vec{E}$ dot product with $\hat{r}$. Similarly this is for outer surface, for inner surface here in the question is mentioned that inner and outer σ on the inner and outer conductor. So, for inner surface of the outer conductor σ_b should be similarly if you calculate we should have I think a negative sign.

Because r, now will go to defer it will go to -r and you will get a result like it is evaluate at b^2 $(\frac{1}{b} - \frac{1}{a})$. So, this is the way you calculate please check it and do some other problems. Now, we do 1 dimensional problem for you know this Cartesian coordinate system then we did for spherical coordinate system.

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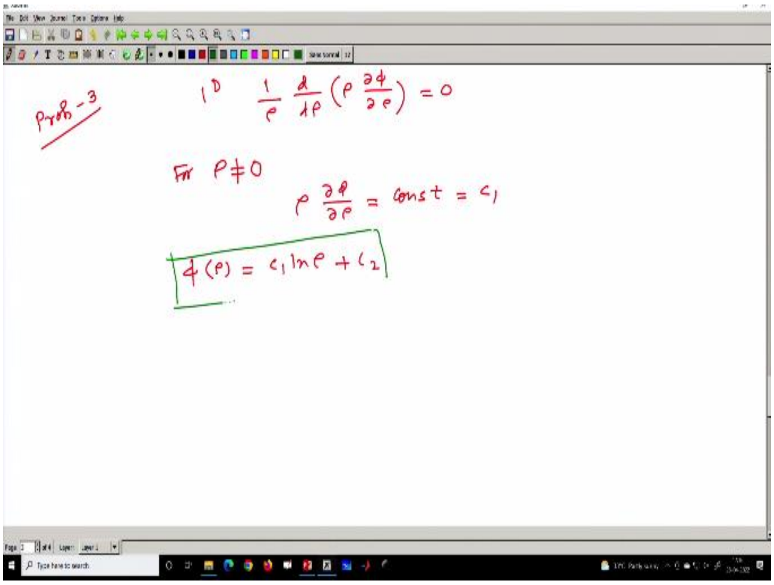

And now, we will continue with this 1 dimensional problem for a cylindrical coordinate system problem 3. So, for cylindrical coordinate system, let us first calculate for 1D cylindrical coordinate system the Laplace equation comes up to be $\frac{d}{d\rho}$ and then ρ $\frac{\partial\phi}{\partial\rho}$ = 0 that is the Laplace equation in 1 dimensional form. Now, the general solution for ρ not equal to 0 the general solution is simply by making $\frac{\partial\rho}{\partial\phi}$ this is constant like before and say this constant is c_1.

If it is c_1 then simply we can write my potential ϕ, which is a function of ρ should be c_1 ln ρ + c_2 very simple if I put this ρ here and then integrate it should be simply ln ρ + c_2 where c_1, c_2 are constant and evaluated from the boundary condition. So, the general solution here for 1 dimension I already figured out, I do not need to do it once again so, this is the general solution for cylindrical. Now, I need the boundary condition for a given problem and then I just resolve the value of c_1 and c_2. So, the problem now it is given like this.

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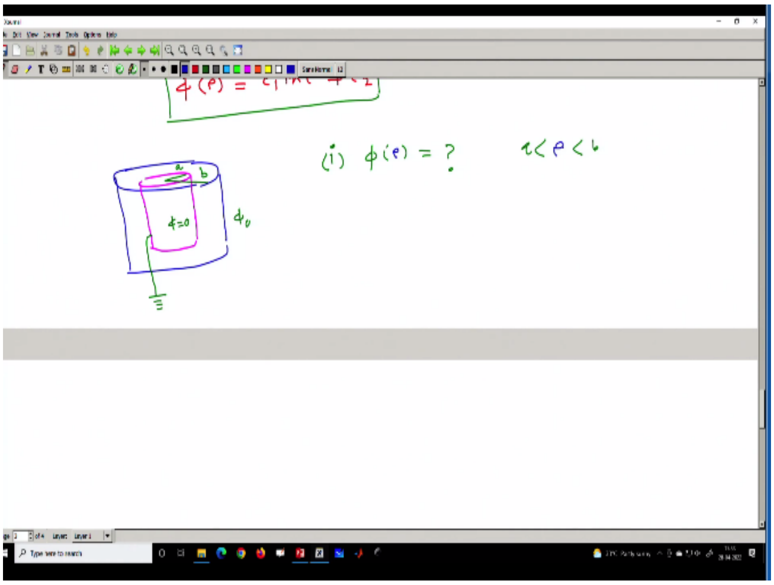

So, I have a 2 concentric cylinder like this one is this another concentric now, the inner radius is a and the outer radius is b. Now, 2 coaxial cylinders are there of radius a and b potential of the inner one now, this is grounded it is saying that the inner one is grounded. So, obviously, the potential here $\phi = 0$ for inner and the outer is ϕ_0 this is ϕ_0. So, what should be the potential in the region? So, the question is simple like before calculate $\phi(r)$ in the region when r is less than b greater than a again a very straightforward problem.

Because the general solution is already known. So, I just here I should not write r rather I write ρ so, this is ρ so let me quickly do that it will not be going to take much time.

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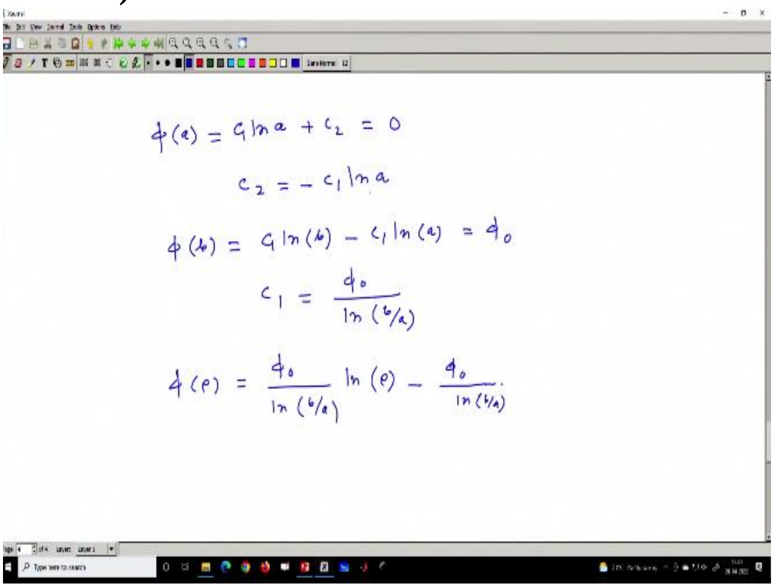

So, $\phi(a)$ this is simply $c_1 \ln a + c_2$ and according to the problem this is 0. So, we simply have $c_2 = -c_1 \ln a$, $\phi(b)$ I should have c_1 here and then ln b and then I write c_2 in terms of c_1, which I already evaluated, it is a c_1 ln a and that value is simply ϕ_0 according to the problem. So, from here I simply write c_1 is $\frac{\phi_0}{\ln\frac{b}{a}}$. So, this is ln $\frac{b}{a}$. So, what is my $\phi(r)$ or $\phi(\rho)$?

$\phi(\rho) = \phi_0$ general solution, I am just writing just putting the value of c_1 ϕ_0 ln $\frac{b}{a}$ and then ln ρ - c_2 I calculate this is c_1 the c_1 is ϕ_0. So, I again I write ϕ_0 ln $\frac{b}{a}$ and then ln a. So, this eventually gives me a solution like ϕ_0 ln $\frac{b}{a}$ and then ϕ_0 then ln ρ – ln a so this is ln $\frac{\rho}{a}$. So, that is the solution in the region when ρ is in between a and b. So, here ρ is in between b and a so, we have done all the problems related to you know the first related to the 1 dimension.

So, now, in the next to we will go for a 2 dimensional problem. So, you will see that there is a significant amount of change will happen when I go from 1 dimension to 2 dimension, because the differential equation will be now 2 dimensional and boundary condition become more instead of having two boundary conditions we should have 4 boundary conditions and we need to deal with that to find the final answer. So, it will be a lengthy problem. So, let us do that. So, this problem I think it is given in the Griffiths book, but I am redoing it here.

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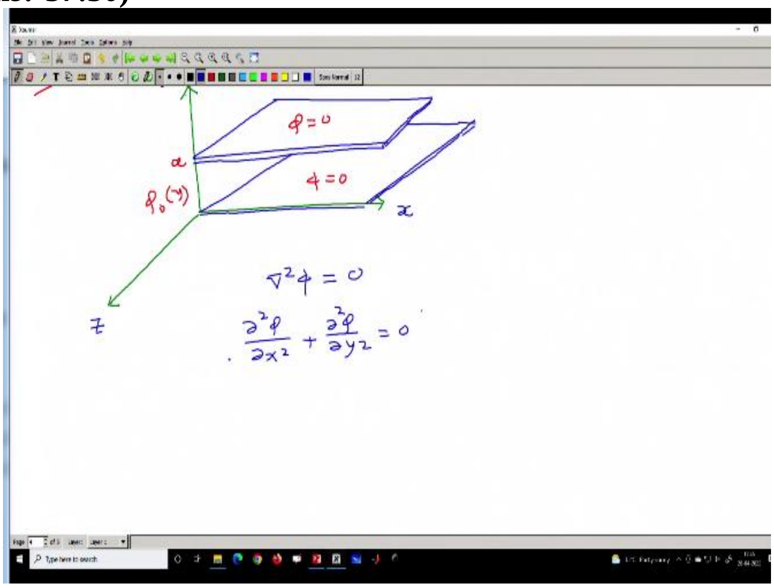

So, say problem 4, in problem 4 we will do the problem in 2 dimensional problem it will be 2D problem with Cartesian coordinate 2D Cartesian coordinate. So, the system as defined is this is the coordinate and we have 2 plate like this is one plate and another plate is this. So, let me put this as z axis, x axis and y axis and the plate is like this so this is 2 plate parallelly placed. Now the potential here it is now boundary condition are given.

So, potentially here for this plate is 0 potential here for this plate is 0 and then this is lengthy from here to here is *a* and this is y axis. So, ϕ_0 the potential is ϕ_0 the function of y the statement of the problem is 2 infinite grounded metal plates both are grounded. So, that is why the potential of the 2 plate is 0 lies parallel to xz plane and lifting at x = 0. So, this is at x = 0 if closed off with an infinite strip insulated from the plate, so, here we have an insulation here.

From the 2 plates maintained at the specific potential ϕ_0. So, this is maintaining a potential ϕ_0 into find the potential inside the slot so, what should be the potential inside this slot? So that we need to figure so, this is a 2 dimensional problem as I mentioned. So, let us now construct this problem. So, I need to find out the potential in between and the lefting at x = 0 as it is mentioned here in this region.

So, infinite strip insulated from the 2 plates, a 2 plate is insulated by strip maintaining as a specific potential $\phi_0(y)$. So, this potential is maintained at ϕ_0 and depends on the value of the y. So, this is the condition that is given. So, I have the Laplace equation we know because there is no source I can use this and this equation in 2D simply I can write at this, because this is z independence, there is nothing at the plate is extended infinitely along z. So, the z coordinate should not be here.

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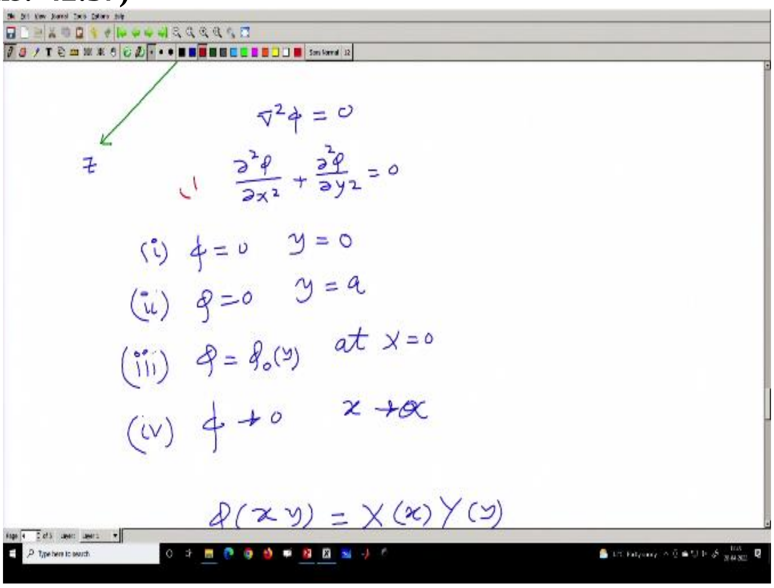

Now the boundary condition one by one if I write the first boundary condition is ϕ = 0 at y = 0, second is ϕ = 0 at y = *a* and then third condition that ϕ is maintaining a potential like this at x = 0 that is this region here and finally, I have ϕ tends to 0 as x tends to infinity. So, if I go for a very large distance is expected that the potential going to die out. So, that is the trivial boundary condition that we need to use here to solve this problem.

Now, we are going to use the separation of variable because now ϕ is a function of x and y. So, x and y, ϕ I can write as X and Y this is the way we use the standard separation of variable and I can put this in this equation 1 let us write this as equation 1.

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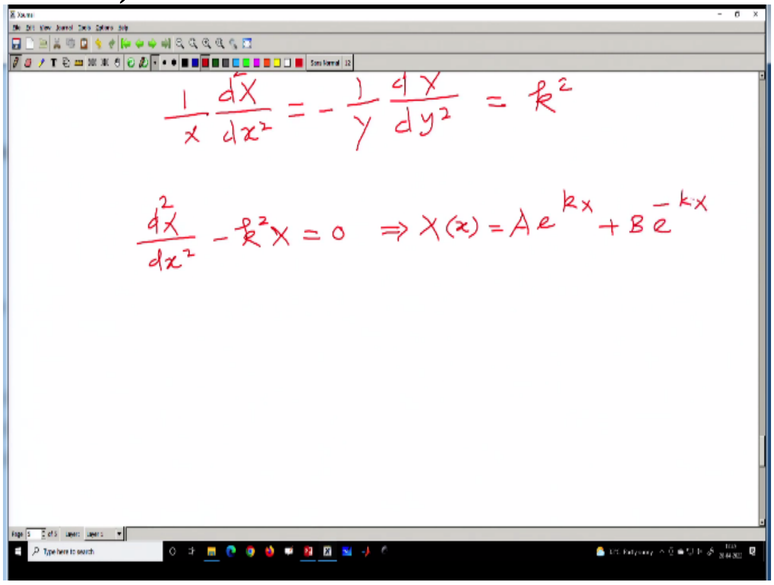

So, if I put this in equation 1, then what I get is something like this $\frac{1}{X}\frac{d^2X}{dx^2}$ that is equal to $-\frac{1}{Y}\frac{d^2Y}{dy^2}$. So, now we can see that left-hand side is it complete function of X, right-hand side is complete function of Y. So, they have to be equal to some constant and let us write this constant k^2 some constant this is a standard way. Now, from that I can have an equation for X like $\frac{d^2X}{dx^2}$ – k^2 X = 0 that gives me a straightforward solution that X(x) is A e^{kx} + B e^{-kx} we know that when we have plus k^2, so, very standard solutions are this.

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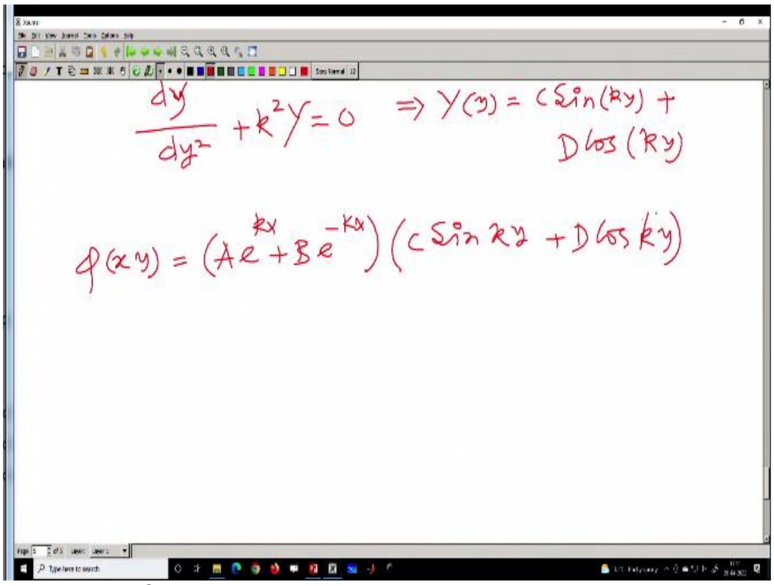

On the other hand, I have $\frac{d^2Y}{dy^2}$ + k² Y = 0 that leads to a sinusoidal solution because if this is plus we know that so, I can write is solution like C sin ky + D cos ky so, I have the general solution in my hand. So, ϕ, which is a function of x and y now can be written like a combination of these 2 solution multiplication of these 2 solutions e^{kx} + B e^{-kx} bracket close and C sin ky + D cos ky.

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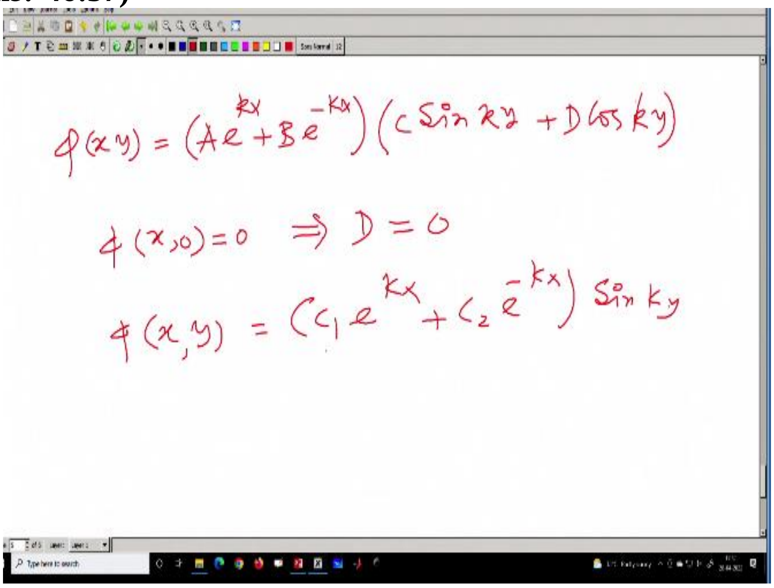

So, now, I am going to use the boundary condition and the boundary condition is saying that ϕ at (x, 0) = 0 that means ϕ at (y = 0), 0. So, that means, if I put y = 0 here, so, this term is not there, and this term we will have only D and that is 0 and this is true for all x values. So, that basically gives me simply D = 0. So, now, I can simplify because my D = 0. So, the equation like C_1 e^{kx} because C is here so, I can multiply C A and C B and write another constant C_1 and C_2 e^{-kx} multiplied by sin ky.

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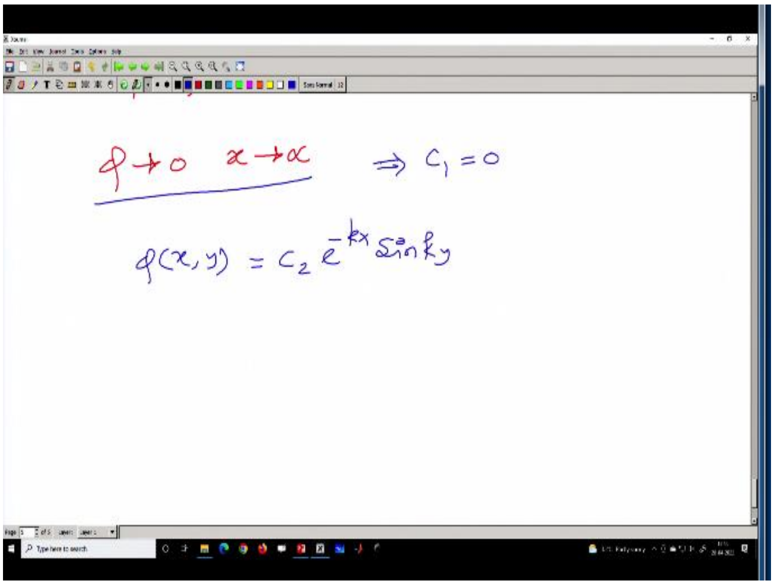

So, now, I am going to use the other solutions and other boundary condition is saying that ϕ tends to 0 when x tends to infinity. So, if I put that then obviously, at x tends to infinity you can see that the term related to C_1 is growing exponentially. So, that is not desired. So, that means again C_1 has to be 0. So, from here I can see that $C_1 = 0$ otherwise this will blow up at x tends to infinity. So, my solution is even become simpler. Now, I have C_2 and then e^{-kx} with a sinusoidal term ky.

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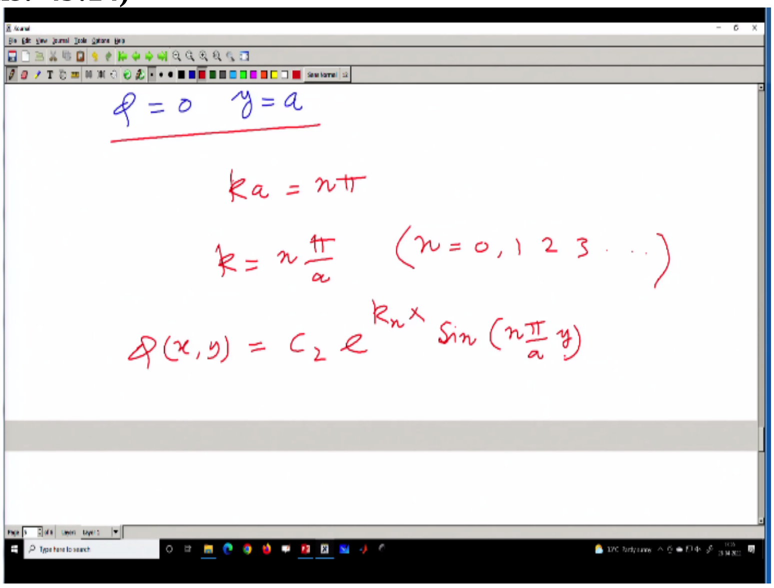

Now another boundary condition that ϕ = 0 when y = a, this is the boundary condition let me highlight with red colour. So, if that is a boundary condition, so, I need to put this boundary condition here and we will see that ka has to be equal to nπ because all the value is 0, C_2 is not equal to 0. Because, if C_2 is 0 the entire solution then we will collapse. So, that means k into a when you put y = a then that is 0. So, that means ka has to be nπ.

So k now having a value like $\frac{n\pi}{a}$ where the value of the n is say it can be 0, 1, 2, 3 etc. So, my solution is finally, if something like C_2 e^k is a function k is changing with so, I should write k_n here because at different n we can have different values, so, k x and then sin $\frac{n\pi}{a}$ that is k and y that is the form of the solution. So, now, the general solution so for different n I can have different solutions. So, we know that when for different n we have different solution than the general solution is a superposition of all the solutions.

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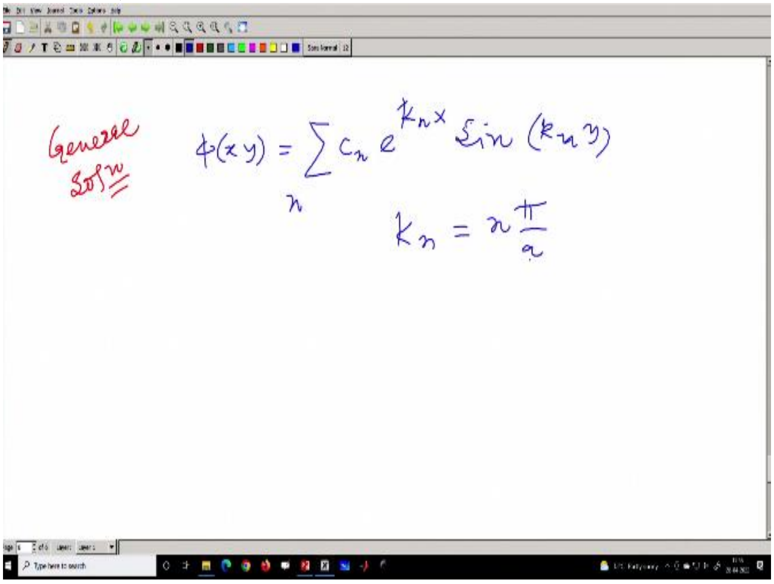

So, the general solution has to be the superposition of all the solutions and eventually I should I have ϕ as a function of x y is equal to all the combination of the solution with different n so it should be sum over n C_n because for different n value we should have different constant I just put it as C_n $e^{k_n x}$ and then sin (k_ny) where k_n is simply $\frac{n\pi}{a}$ this is the way we can have.

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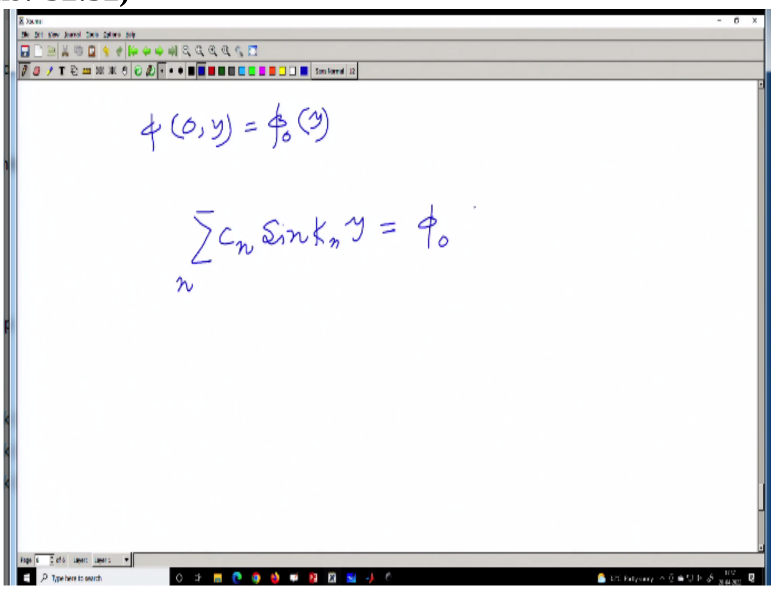

Now, the boundary condition 3 is saying that $\phi_0(y)$ that is at x = 0 the entire the potential is something like this. So, if I now put here at x = 0 then I can have an equation like C_n when n sin (k_n y) is simply $\phi_0(y)$. Now, the challenge is how to calculate C_n from this equation?

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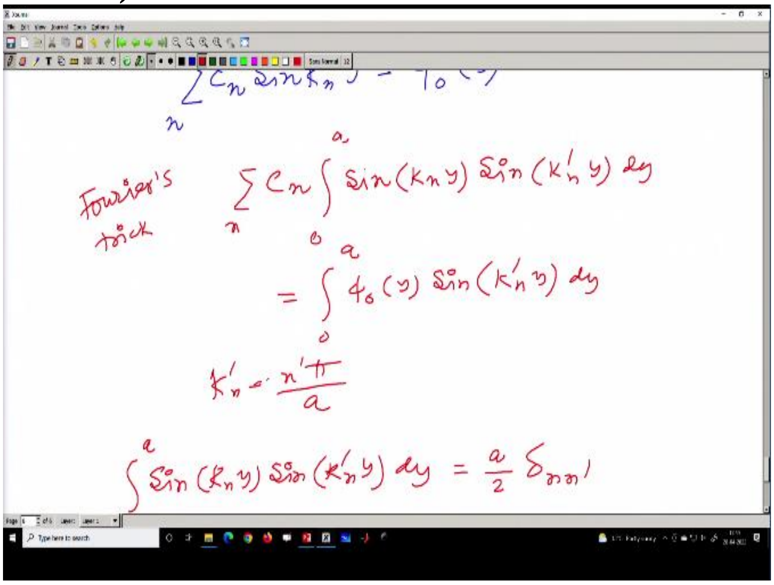

So, there is a very elegant way and which is called the Fourier trick or Fourier transform, Fourier trick where we find this C_n by multiplying you know the by using this the complete set this is called the complete set. So, what we do is this C_n and we integrate 0 to *a* and whatever this function here we are having we write sin (k_n y) and then multiply with sin (k'_n y) integrate over dy and do the same thing in the right-hand side.

Integrate 0 to *a* and whatever the function we already have multiplied with this function sin (k'_n y) and dy were k'_n is for another $\frac{n\pi}{a}$. So, this is n I should write it correctly. So, now we know from these complete set functions 0 to *a*, this quantity sin (k_n y) sin (k'_n y) dy is simply $\frac{a}{2}\,\delta_{nn'}$, this deals with the delta function. So, if that is the case, so, when we know sum over this all n only the meaningful thing is when you know these n and n' are same.

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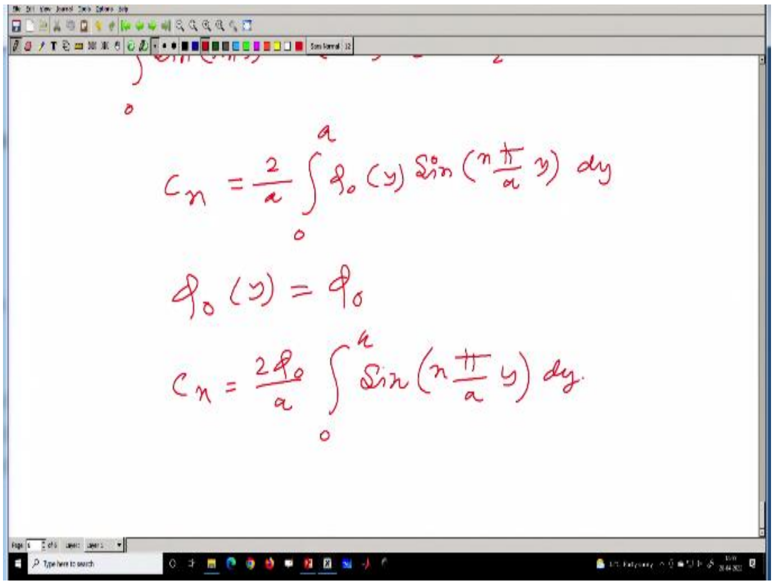

So, that simply makes C_n to be because when n' = n then only we have the value so, I can extract the C out of that and simply I have ϕ_0, 0 to a integrate and y and then sin $(\frac{n\pi}{a}$ y) as usual dy. Now, if $\phi_0(y)$ is a constant say ϕ_0 I have my C_n to be $\frac{2\phi_0}{a}\int_0^a \sin\left(\frac{n\pi}{a}y\right)$ dy.

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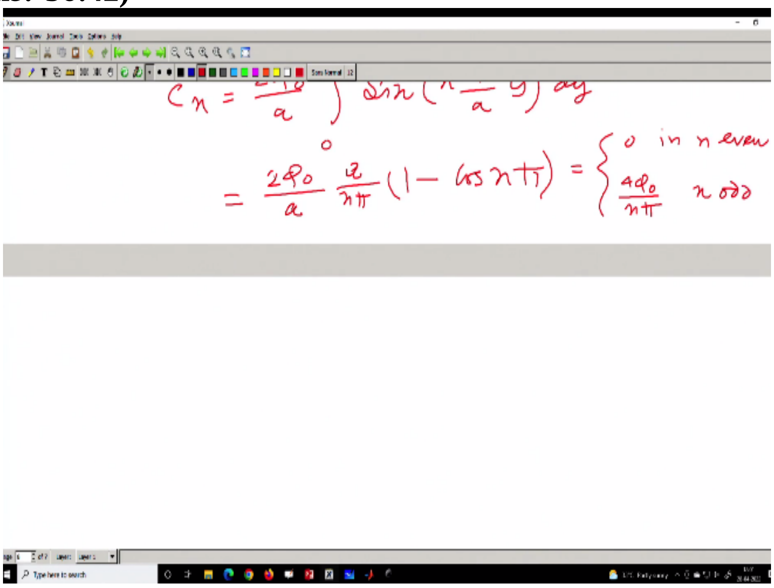

So, now this integration we can further evaluate and if you evaluate it simply comes out to be $\frac{2\phi_0}{a}$ and $\frac{a}{n\pi}$ if I do this integration and simply you have (1 - cos $n\pi$) and that thing is equal to 0 when n is even, and $\frac{4\phi_0}{n\pi}$ when n is odd, please check it.

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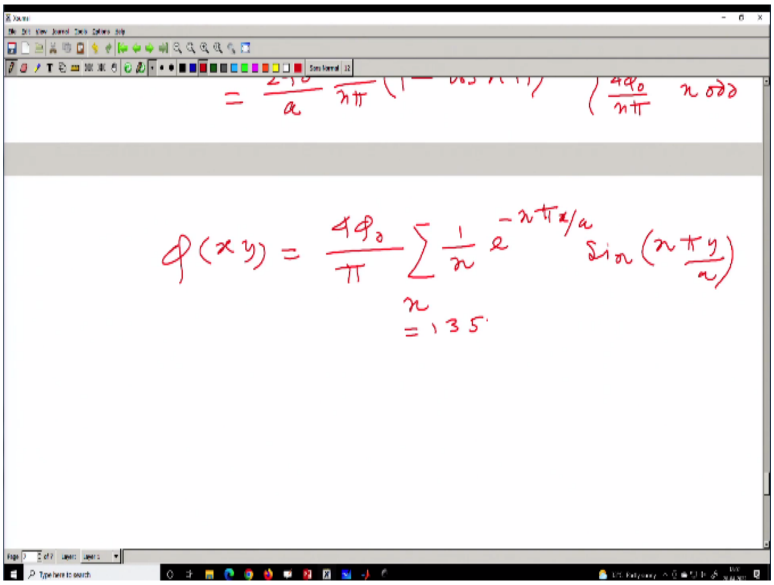

So, finally, what is my solution? My solution is after doing all these calculation lengthy calculation we finally get a solution like this. The point is for 1 dimension the problem was very straightforward, but as soon as you move to 2 dimension it becomes very you know, lengthy I should not say complicated I should simply say it is lengthy and you need to use few tricks to find out the constants that is thing it is a very important problem I believe, you can understand once you do that by your hand so that should be the value 135.

Because other case it is 0, the odd value is meaningful. So that should be the form of the solution. I already spend 1 hour to share for this class because this is a very lengthy problems that are associated with that. But the point is you need to you know, practice by yourself at least do 1 or 2 problem with this boundary value. In the next class again, we will continue with this boundary value problem with more examples.

And then maybe you will be in a position to solve the problem by your own that is the main goal you know in this course that you can do the problem by yourself. I am doing almost all the problem in live mode so that you can have the idea of how to solve this. With this note, thank you very much for your attention and see you in the next class.