Foundations of Classical Electrodynamics
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Lecture - 05
Spherical Coordinate System, Line, Surface and Volume Element

So, hello students to the foundation of classical electrodynamics course. So, we are still in module 1 studying the mathematical preliminaries. So, today we have lecture number 5 and we will be going to cover the spherical coordinate system that we started in the last class and then like to understand what is line surface and volume element in a given coordinate system. Today we have class number 5.

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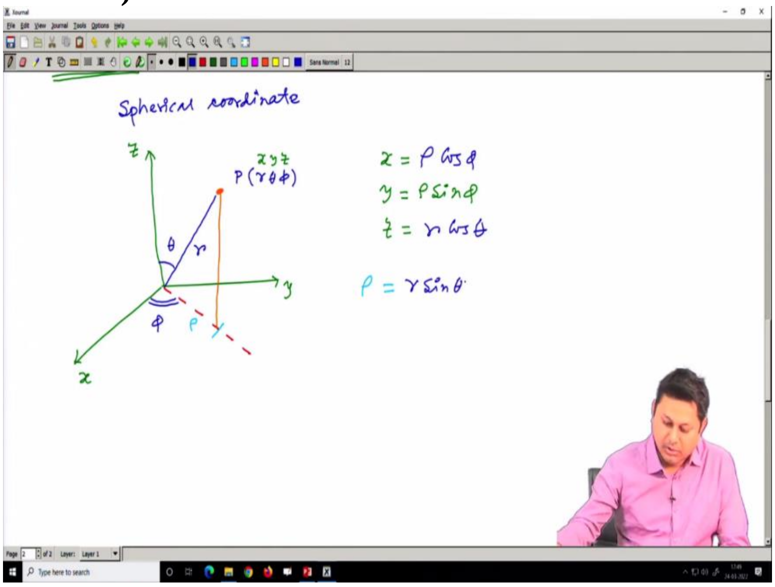

So, we already started the concept of spherical coordinates, so, let me do it once again. So, in spherical coordinate, we define a coordinate system here a point here r, this is the point. Its coordinate is say, r, θ and φ in spherical coordinates, but this point can also be written in terms of Cartesian coordinate and I should write it x, y, z. Now, what is θ, what is φ here if I join this.

So, this value is r, this angle is θ and this angle is φ. These are the 3 coordinates that basically define a particular point p on this system. Suppose, this is my x coordinate, y coordinate, z coordinate in Cartesian coordinate system, but I can also define a point in terms of r, θ, φ, which is the spherical coordinate system. So, if I now try to understand what is the relationship between x, y, z and p, r, θ, q, then I can write it here.

So, x is simply equal to ρ cosφ because this projection whatever the projection r is making let us so, from here to here this is ρ suppose, then we convert this ρ to r then we have y, which is simply ρ sinφ, and I should read it in different colour but anyway and z = r because r is already there so z components should be r cosθ. Now, what is ρ? ρ equal to because this is ρ, if this angle is θ, then I can simply write ρ is nothing but r sinθ.

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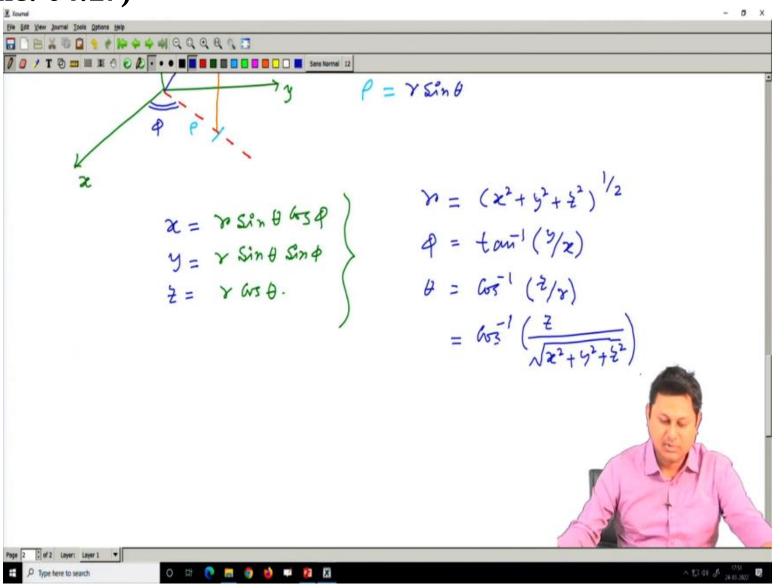

Now, x, y, z, I can write in terms of so, this is my x, y and z in term of r, θ, φ, it simply gives us r sinθ cosφ, r sinθ sinφ just replacing ρ and r cosθ. These are the relationship between x y z and r θ φ. We can also have the other way that means, let me write it what is r here, what is φ here and what is θ here in terms of x y z that we can also write. What is r? r is simply x^2 + y^2 + z^2 whole to the power $\frac{1}{2}$.

φ, you can see from here it is simply $\tan^{-1} \frac{x}{y}$ like we had in the cylindrical system. θ because this angle is θ here. So, θ is $\cos^{-1} \frac{z}{r}$, which is cos inverse and then we have z divided by the value of the r that we already get here, $\sqrt{x^2 + y^2 + z^2}$. So, this is the transformation from x, y, z to r, θ, φ in a spherical coordinate system.

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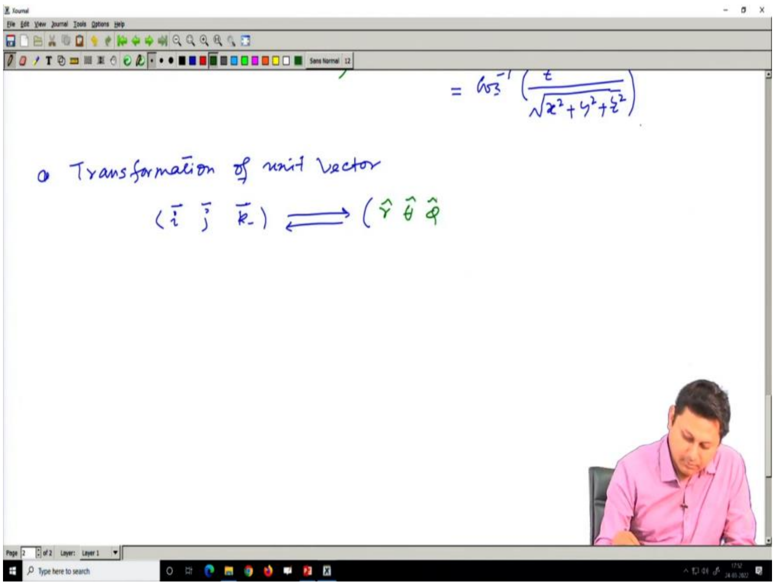

Now, let us do what we did last time for other coordinate system and that is transformation of unit vector. With that we do for say Cartesian coordinate we have i, j and k unit vectors and I want to understand that how these things are transformed in spherical coordinate system with the unit vector r θ φ. So, let us draw the picture that will be useful.

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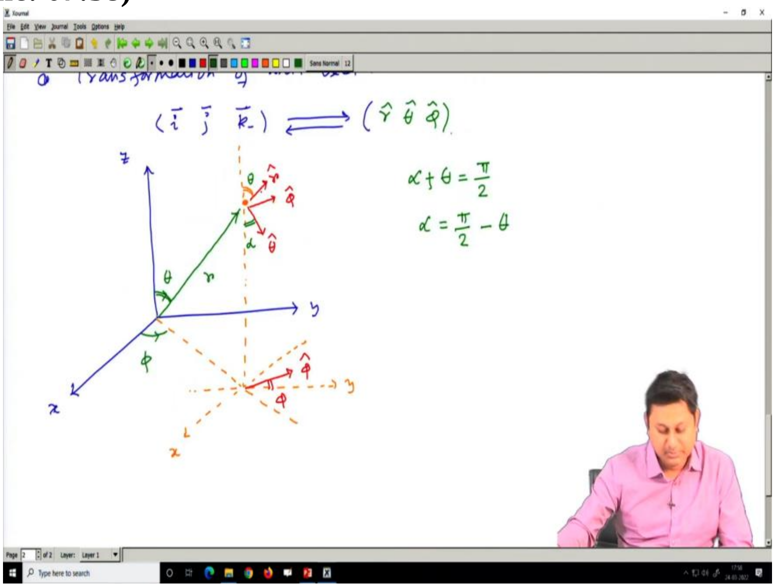

So, these are the coordinate system. This is my x, this is my say, y and this is the z and let us draw a line here and a point here and then this length from here to here is my r. This is my r, this angle here is θ and this angle is my φ. So, these are the 3 coordinates. So, this is r, this is θ, and this is φ. Now, in terms of unit vectors if I try to understand what should be the unit vector associated with these coordinates.

So, the unit vector r will be along this direction. So, this is r unit vector, which is making an angle here this angle should be θ by geometry. What is the unit vector of this θ that will be if

I draw a curve here tangential it should be downward. So, it is like this. Say this is my θ unit vector. So, I am having angle here. So, let us name this angle say this is α and what is the φ unit vector? φ unit vector should be again tangentially.

If I am having a surface here curved surface, tangentially it should be along this direction. So, that is my φ unit vector. Please note that r unit vector, θ unit vector, φ unit vector, they are forming orthogonal system here. So, these angles whatever the angle they are making here at this point are 90°. But we need to you know find out what is the component. If I now decompose this r unit vector, θ unit vector and φ unit vector along x y z.

Then I should have a relationship with i, j, k and r, θ, φ, that is my goal here. In order to do that, we can extend few things. So, for example, so, this is the line I am having so, parallel to x, I can have one line here and parallel to y, I should have a line here, so, that it will be easier for me to decompose everything. So, now, if I just you know put this θ here, because θ this φ unit vector.

If I just make a projection of this φ unit vector, it should stay like here, just I am dragging this φ vector here to here in this and this angle should be φ because whatever angle we have, because they are parallel these are the parallel. So, this should make an angle φ here as well. So, this is unit vector φ. It will be easier for me to you know decompose along the directions.

So, mind it, this is the negative this is the x direction, this is the y direction because parallelly I am just drawing the line and then I just replaced these things here it will be easier to decompose that is the reason. Now, we can have another relationship with the α because α is the angle that I need to resolve I need to use certain relationship because it is following certain relationship and that is $\alpha + \theta$ it has to be $\frac{\pi}{2}$ from the geometry it is clear.

So that α is $\frac{\pi}{2} - \theta$. You can see that r and θ unit vector they are making 90°. So, $\alpha + \theta$ has to be 90° because this is a straight line. So, with this picture now, we can decompose one by one. So, let us do that, let us do that.

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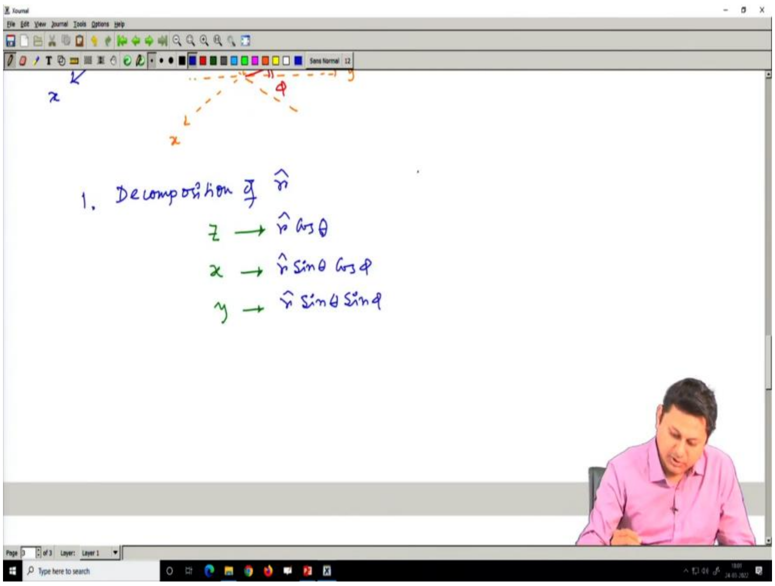

So, first thing, decomposition of r unit vector whatever the r unit vector we are having which is hanging in the space I used to decompose along x direction, y direction and z direction to find out its components in terms of i j k that is the goal. So, if I want to find out what is the z component of r then it is easy, because you can see that it is making an angle θ along this z direction.

So, it should be simply r unit vector cosθ of that. Then I want to know what is the x and y component. Now you can see that r is hanging here, but I can make a projection of r over this space over this plane xy plane and if I do, I simply have the projection simply have this value. Let me draw it in a different write in different colours. This is simply r sinθ is the projection of this r vector whatever the r vector we have on this xy plane.

And then I am now having r vector here this line in the xy plane now, if I want to find out what is the x and y components it will be easy. The x component will be simply cosφ and the y component should be sinφ sinθ. This is the projection and sinφ. This is the decomposition of r. Now, I want to do the same thing decomposition of I like to have the decomposition of r then let me do this here. So, to make some space wait I will make it here.

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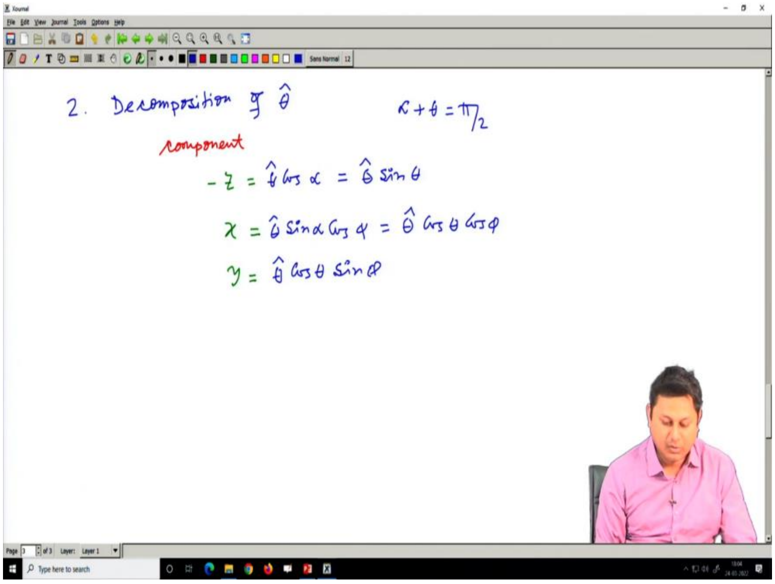

So, to know I θ. So, these are the components mind it, these are the components we are looking for z component, y component, x component and y components, so, component. So, which component I go here first say let us first check what is θ, θ is you can see that θ is hanging here with an angle α. So, the first thing that we can do, we can decompose this θ along the z direction, but you can see it is going the downward side.

So, the component that we are getting is the minus z component. So, minus z component along the opposite direction is simply θ unit vector and making an angle α. So, θ unit vector making an angle φ here. So, it is θ unit vector then cosα. So, I am having θ unit vector cosα. Now we have a relationship with α and φ that $\alpha + \alpha$ and θ that is $\alpha + \theta$, $\alpha + \theta$ was $\frac{\pi}{2}$.

So, we are going to exploit this expression and that leads to θ unit vector. So, sinθ that is my z component. What about the x and y component? So, the x component again I can have the projection over this xy plane and if I make a projection of this θ over xy plane, it should be θ sinα because cosα was the projection along z directions. On the xy plane, the projection will be simply θ unit vector sinα and then this is φ.

So, this is making an angle φ. So, it should be simply cosφ. So, eventually we will be going to have that θ unit vector cosθ because sinα = cosθ and cosφ. In a similar way, for y I simply have θ unit vector cosθ and sinφ. This is the component that we have θ unit vector component that I can distribute in x, y, z, because these are hanging this vector is hanging all these 3 vectors are hanging in the space.

What we are doing that we are decomposing its component along x, y, z, the Cartesian coordinate system with the geometry that we are having here. So, 2 decompositions are done the final one.

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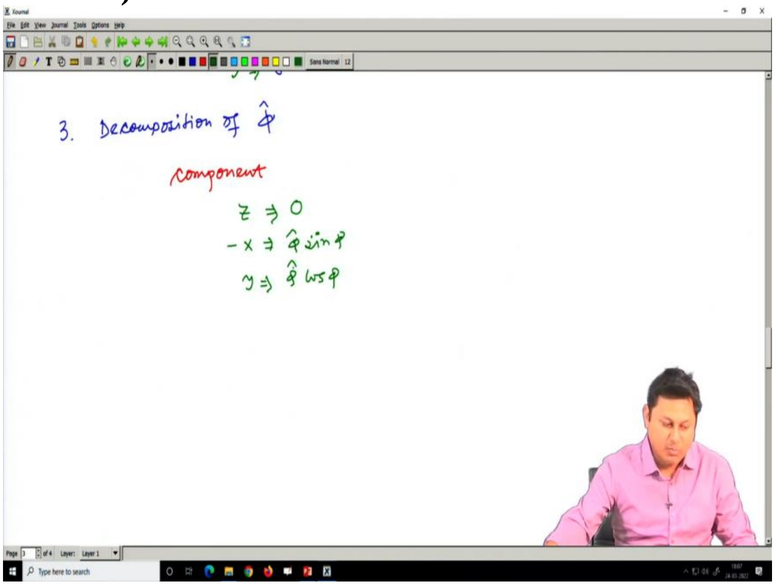

Decomposition of φ unit vector. So, I have the component first. So, let us start with z component and then what is z component here for φ unit vector you can see that the φ unit vector is line here, which is parallel to the in all cases it should be parallel to the x and y plane. So, that is why we should not have any kind of z component of this φ unit vector because φ is tangentially over this if we have a curve like this and it is tangentially around this direction.

So, we should not have any z components. So, z component of the φ unit vector is simply 0. Now, already we make a projection because it is parallel to xy plane, so, I am having a projection is lying in the xy plane rather now, this angle is φ now, these things are very simple. So, I just now divide this along this y direction and along this x direction and if I do you will simply get this.

So, minus x component we will first find because it is in opposite because you can see that it is in this direction. So, first you have this component here, which is the minus x direction and that component is simply φ unit vector and then the sinφ and y component is φ unit vector or I should write it as a this is not there is a component I am dividing cosφ that is all. So, should make it this sign.

So, that these are the way I do this is a component. So, 3 different components I divide. Now, after having the division of component now, we are in a position to write the entire thing in a matrix form that the way we did. So, this component z is nothing but the k unit vector, x is i unit vector and y is j unit vector. So, I can write any vector in this way, i, j, k, in terms of r, θ, and then I can make a transpose and get the other ones.

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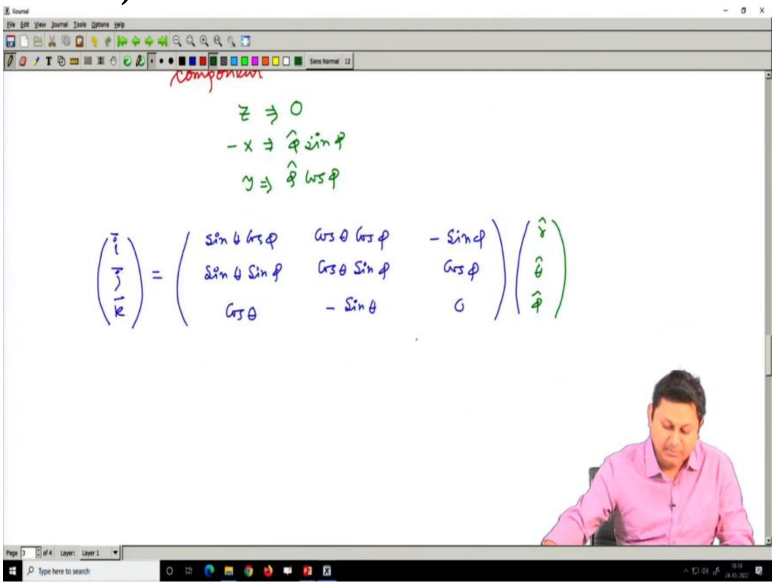

So, if I write now, finally my i j k these are the 3 vectors, which I decomposed and this decomposition whatever I decompose if I now write then all the x components what we get here, I need to gather that and then I will find. So, it should be simply sinθ cosφ, then cosθ cosφ because I am just putting the components, which the component I divide, I decomposed. x component sinθ cosφ.

Then what x component sine cosθ cosφ and then x component minus of sinα, minus of sinφ. So, here I should have minus of sinφ. What about the y component? Similarly, sinθ sinφ, then cosθ sinφ and then cosφ. These are the y component and z component all the 3 cases cosθ, then minus of sinθ and 0 final case is 0. This side I should write r unit vector, then θ unit vector and then φ unit vector.

In the books, you will only get this matrix form but how it came normally it is not given that is why I thought that I should do all these things explicitly. So, that you can have an idea that how from one coordinate system how the unit vectors are changing, transforming in another coordinate system. So, here you just check by yourself that how these components are decomposed and then from that I can prepare the matrix.

So, from i, j, k, I can from r, θ, φ, this is the relationship with i, j, k to r, θ, φ, but I can also do the vice versa because this is an orthogonal transformation.

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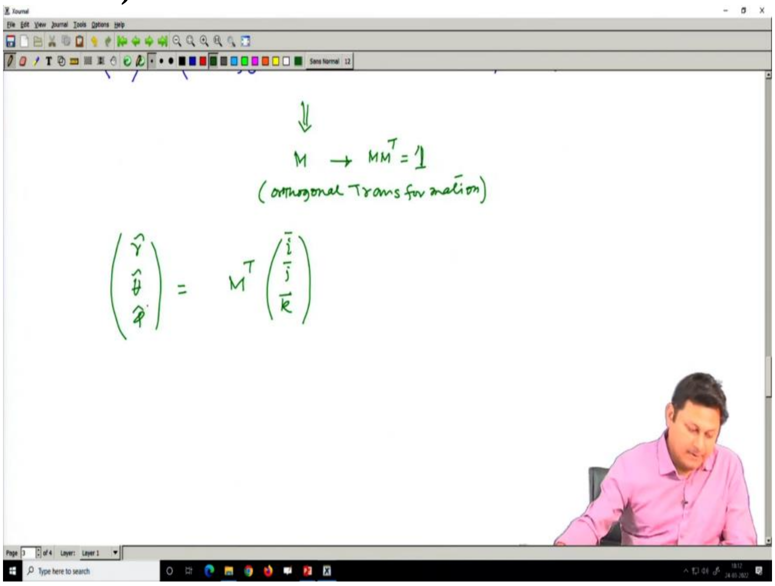

Last time, we mentioned that if we are having a matrix for orthogonal transformation, they are following certain rule, so, that if M is my matrix that is making this transformation, so, it is following these things M M^T is equal to unity. So, that means, this is because this is orthogonal transformation. So, I can, so you just make a transpose and you are going to get, so I am not doing that.

So, if you want to get that r, θ, φ, you will just need to put the M^T here and you will get i, j, k this side. This is how one can extract the other information from r, θ, φ unit vector how get i.

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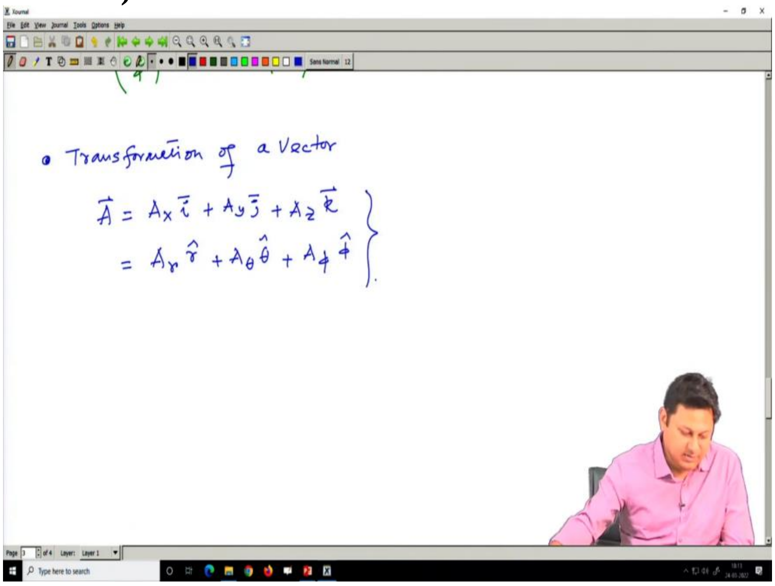

And in the similar way, let us quickly find out how the vector we are going to transfer. So, now transformation of a vector. So, A vector, a given vector can be written in terms of the Cartesian coordinate system like A_x i + A_y j + A_z k, but the same vector in polar coordinate spherical polar coordinate I can write it as $A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$. And this process is similar.

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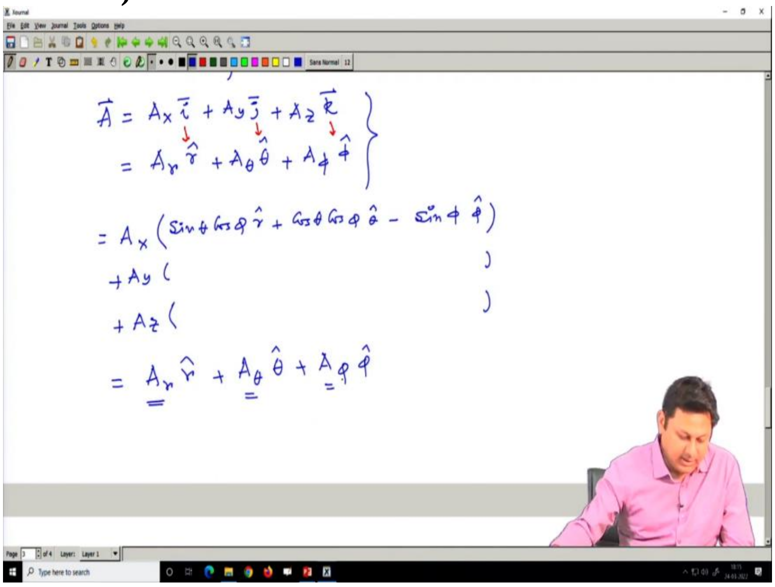

So, that is why I am not going to do the detail calculation. So, the next thing what you need to do, because now you know the relationship between A and r, j and how you know this how these things are transformed. So, that information I am going to use here and then let me just write so, these things is A_x in terms of in place of i, I just simply write the when i is decomposed in r, θ, φ. So, that matrix is there.

So, I can use this matrix and I am going to get this, $\sin\theta \cos\varphi \, \hat{r} + \cos\theta \cos\varphi \, \hat{\theta} - \sin\varphi \, \hat{\varphi}$ that is one term. Then you should do the next one and finally, we will have the last one. Now, this entire thing again you can decompose like A multiplied so, these things are equivalent to $A_r \hat{r}$. So, you can gather all the $\hat{r} + A_\theta \hat{\theta} + A\varphi \, \hat{\varphi}$.

You gather every all these things inside A_r, A_θ and A_φ and after doing that. You will simply find that I am just writing the final expression, which is same actually, after doing all this exercise.

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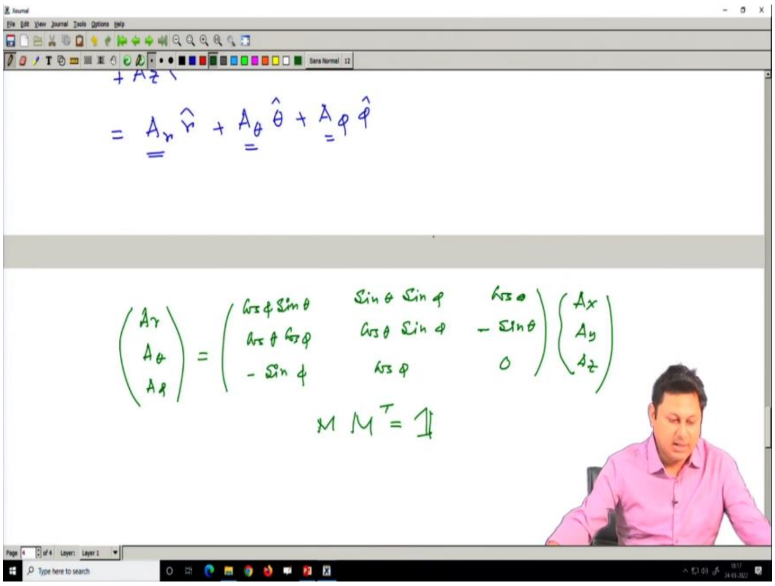

You can write simply the relation A_r A_θ A_φ is equal to $\cos\varphi$ $\sin\theta$, $\sin\theta$ $\sin\varphi$ and then $\cos\theta$. Here you have $\cos\theta$ $\cos\varphi$ and - $\sin\varphi$ and then $\cos\theta$ $\sin\varphi$ here - $\sin\theta$ and you have $\cos\varphi$ and 0. These matrix you will get the same matrix actually and this side you should write A_x A_y A_z. This is your M matrix. So, you can also have the MM^T will be 1, using that you can also have the other relationship, which is same.

I believe you can do that. Now, quickly I go to the next topic that I want to just start, line element, volume element and surface element.

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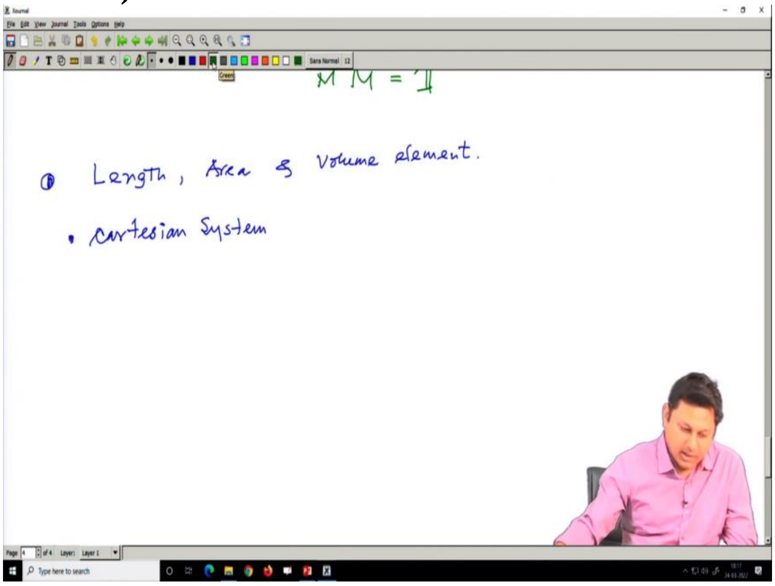

So, the next topic is how to calculate the line element, volume element. So, length, area and volume element in different coordinate system. So, let us start with Cartesian coordinate system first. In Cartesian system, these things are very simple very straightforward.

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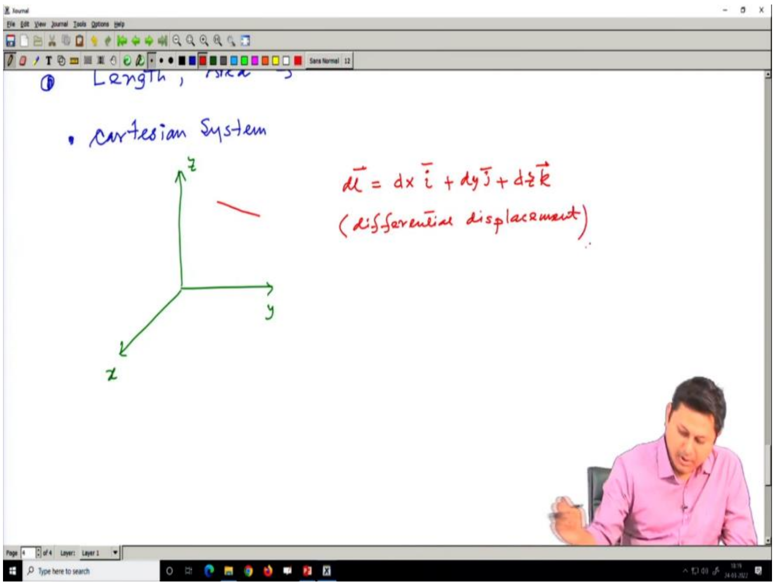

So, I am having a Cartesian coordinate system x, y and z, and any length here given, suppose this is a length element and this length in terms of vector simply given as ∂x i + ∂y j + ∂z k. This is the vector we are having. This is simply the differential, this is called a differential displacement. What is the surface element then.

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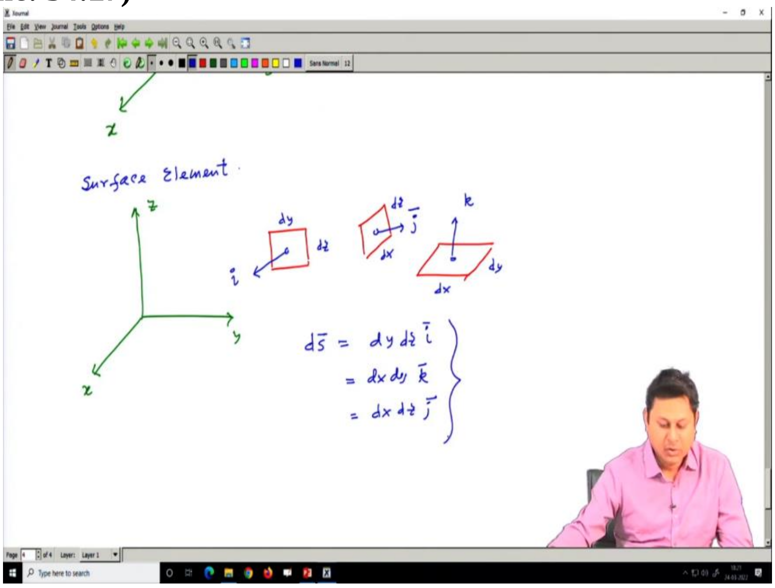

Length element we understand, surface element. This is again the coordinate system x, y, z and you can have surface like this. Let me draw the surface, one surface you can have this, if I consider the box, another surface you can have this and another surface you can have this. This surface, which is where this is my i direction, this surface this is my j direction and this surface this is my k direction.

So, this length is along this dy and along this dz, this is along this. This is dx and this along this, this is dz. For this we have dx here and dy here. These are the vector quantities mind it.

So, my surface element ds is like dy dz I, it can be dx dy k or dx dz j, these are the differential surface.

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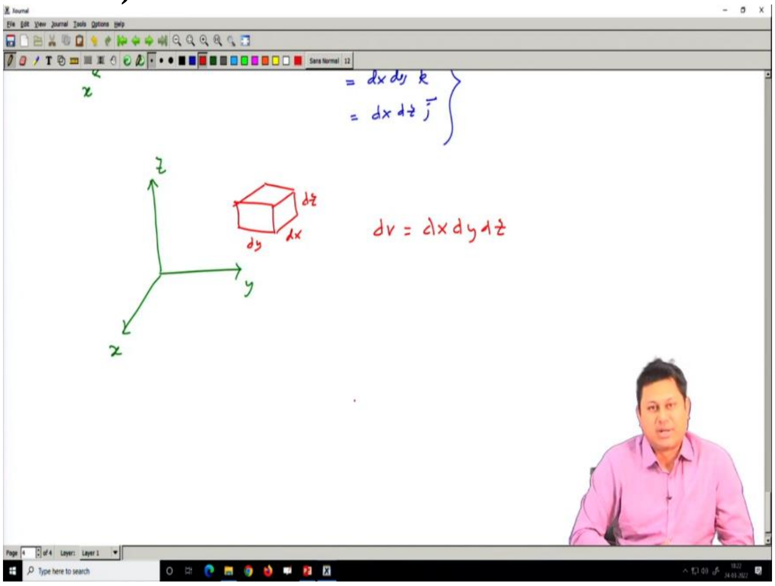

What about the volume element? Again, volume element is very, very simple. This will be volume element is a scalar quantity, so, you should not bother much about the direction, where the surface the direction is important. So, if I have a box here simply like this a volume element I can define. So, this along this direction this is dx, along this direction this is dy, along this direction this is dz.

So, my volume element simply dv = dx dy dz as simple as that. So, this is called a differential volume element. It looks quite simple and very straight forward for you know Cartesian coordinate system to find the line elements, surface element and volume element. But in the next clas,s we find that it is not that straightforward when we are going to other coordinate system like spherical polar coordinate system or cylindrical coordinate system, it is not straightforward.

So, in the next class we will discuss how to define these volume elements, surface element or line element in those coordinate system. With that note, I would like to conclude here thank you for your attention.