Foundations of Classical Electrodynamics
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Lecture-55
Magnetic Vector Potential (Cont.)

Hello students to the foundation of classical electrodynamics course. So, we are in module 3 now and module 3 under lecture 55. We will be going to discuss more about the magnetic vector potential. So, the discussion will be continued today.

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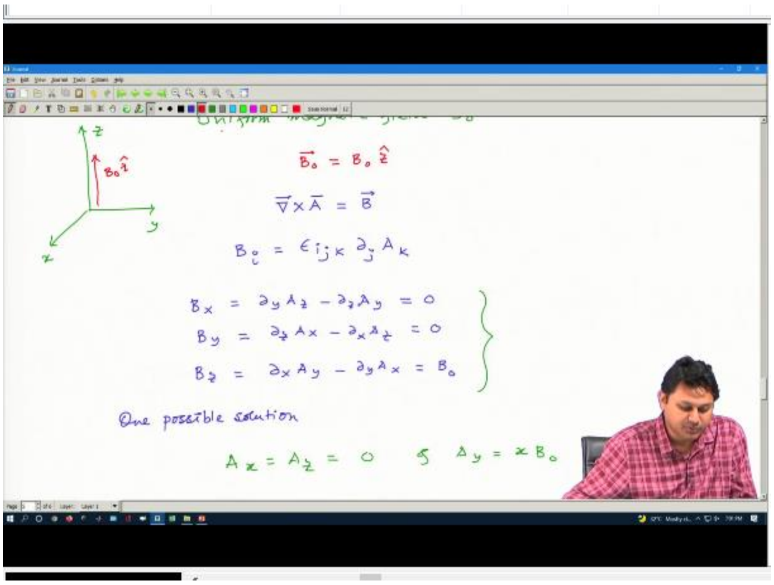

So, we have class number 55. So, our discussion on magnetic vector potential will continue. So, in the last class we calculated the magnetic vector potential for few cases. So, today we will be going to do another thing and that is try to find out the vector potential for a constant magnetic field. So, a magnetic field that is given is constant and corresponding to that magnetic field if I want to find out the vector potential, so, what should be the form of that that we consider.

So, let us consider a uniform magnetic field. So, uniform magnetic field is there $\vec{B}_0$ and we considered that this $\vec{B}$ is along the z direction. So, if this is my coordinate system x, y and z. So, $\vec{B}$ is along this direction it is uniform. So, we have $B_0\ \hat{z}$. So, that means, $\vec{B}_0$ vector is magnitude B_0 with $\hat{z}$ in this direction. Now I need to find out the $\vec{A}$. So, let us exploit this expression that $\vec{\nabla} \times \vec{A}$ is $\vec{B}$.

And so that means the i^{th} component of the $\vec{B}$ is simply $\epsilon_{ijk}\ \partial_j\ A_k$. So, if I want to find out the x component B_x, it should be $\partial_y\ A_z - \partial_z\ A_y$ and that is 0. If I calculate the y component of the $\vec{B}$ that is $\partial_z\ A_x - \partial_x\ A_z$ that component is also 0 and the B_z component is $\partial_x\ A_y - \partial_y\ A_x$ and that component is non zero and these values B_0. So, I have three equations in my hand, three partial differential equation in terms of A.

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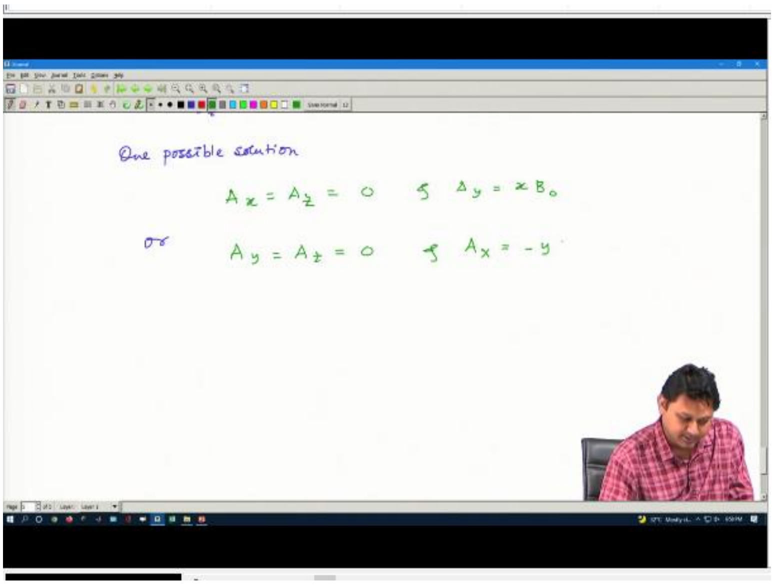

So one possible solution is this, let us consider A_x to be 0 with A_z to be 0. So, A_x A_z 0 and A_y = x B_0. If I put it you can see that it will go to satisfy because A_z is 0, A_x is 0. So, then A_y is x B_0. So, the first equation I am going to satisfy, second equation are going to satisfy and third equation is also going to satisfy because A_y is x B_0. So, when you make a derivative here, so you will get B_0 here and rest of the term will be 0.

So, all the three equations we are going to satisfy. Similarly or I can also construct the solution in this way, say A_y and A_z is 0 and A_x is -y B_0. You can check it that this solution can also satisfy these three equations what is given here. Now, if two solutions are there, the linear combination of these two solutions is also a possible solution and if you do that, then I can have another solution.

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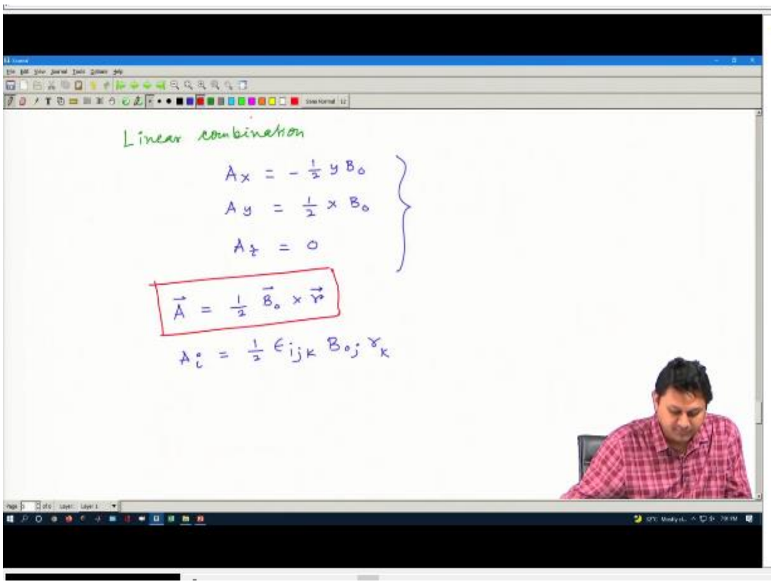

So, the linear combination of the solution if I make then I can have say a solution like $A_x = -\frac{1}{2}$ y B_0 and $A_y = \frac{1}{2}$ x B_0 and $A_z = 0$, the linear combination of these two equations leads to. A now in this case A can simply be written as these components are there. So, it can simply written as $\frac{1}{2} \vec{B}_0 \times \vec{r}$, this is the way you can write and now if I make a curl at both the site then you will see that it is satisfying, it is giving you the constant value B_0 as a magnetic field.

So, A_i is simply $\frac{1}{2}$ ϵ_{ijk} B_{0j} and r_k. So, this is my expression for constant magnetic field. I can always write A in this way. So, this is one of the solutions for constant magnetic field that I can use. If I now check carefully so B is along z direction according to this figure. So, let me draw this figure once again.

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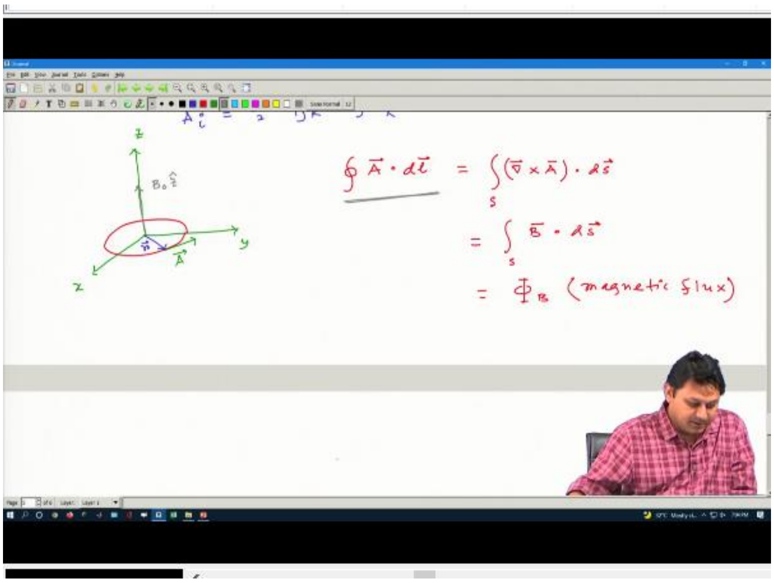

So, I have a coordinate system like this, this is x, this is y and this is z. And if I make a circle here and my $\vec{B}$ is along this direction which is $B_0\ \hat{z}$, $\vec{r}$ is this one this is my $\vec{r}$. Now, what is the direction of $\vec{A}$ then? This is $\vec{B}_0 \times \vec{r}$. So, it has to be in a tangential direction like this. So, this is the tangential direction. So, this should be the direction of $\vec{A}$. Every time $\vec{r}$ changes, so, this direction will be going to change.

So, if I integrate over this circle, as a close line integral, if I calculate this quantity $\vec{A} \cdot d\vec{l}$ and what I get? This quantity is simply the surface $\int(\vec{\nabla} \times \vec{A}) \cdot d\vec{s}$ and $\vec{\nabla} \times \vec{A}$ is nothing but $\vec{B}$. So, surface integral $\vec{B} \cdot d\vec{s}$ and $\vec{B} \cdot d\vec{s}$ is nothing but the magnetic flux. This is simply the magnetic flux. So, $\vec{B}$ is along the z direction and we can have the magnetic flux here.

And this magnetic flux can be calculated by this expression that if $\vec{A}$ is known, then from that exploiting that $\vec{A}$ one can calculate the magnetic flux. So, further we want to calculate another case say problem 3 or case 3.

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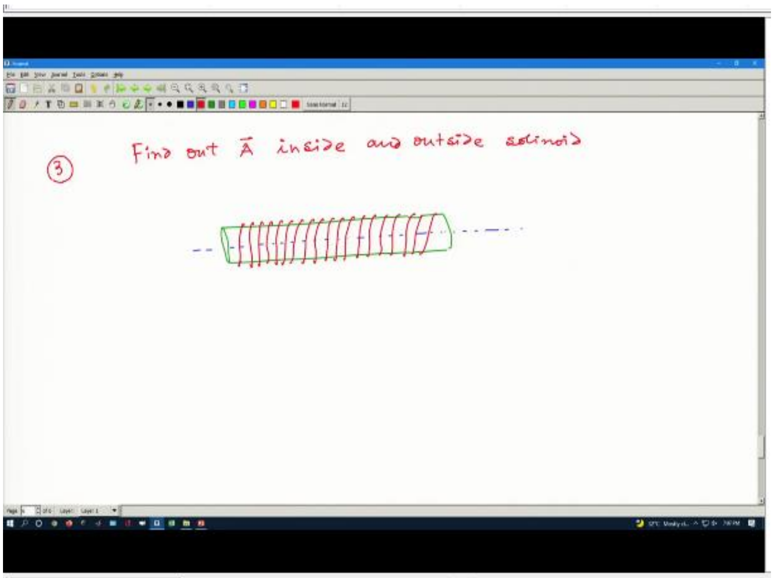

Lastly we did it for two cases. So, find out $\vec{A}$ inside and outside solenoid I want to find out $\vec{A}$. So, what happened in the solenoid because this problem we already so this is the solenoid. So, suppose I have a cylindrical system and over that we have the wires and this say let me draw axis first, maybe I can use different colours? So, inside the solenoid and outside the solenoid we need to calculate the magnetic vector potential.

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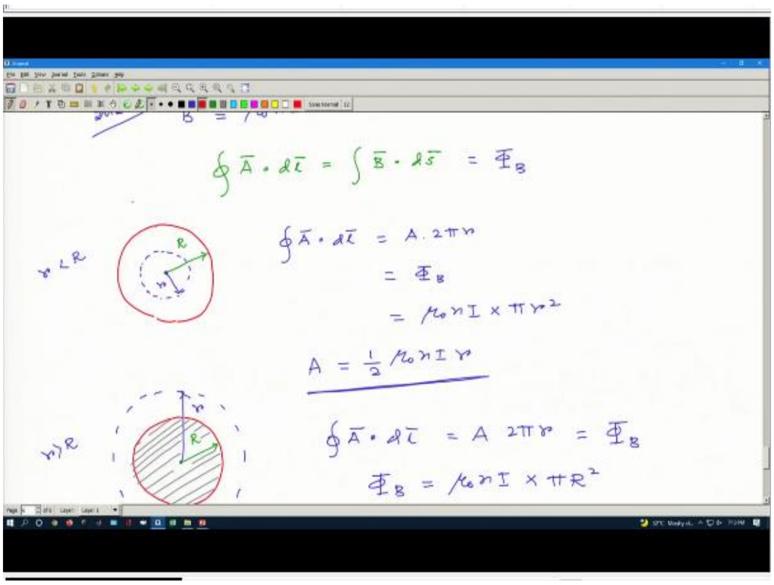

So $\vec{B}$ inside the solenoid value of $\vec{B}$ is simply μ_0 n I, we calculated this earlier and also we have so we just learned one thing that integral $\vec{A} \cdot d\vec{l}$ is equal to integration of $\vec{B} \cdot d\vec{s}$ the flux. So, this is the cross section, let us make a cross section of the solenoid and so this is the radius of the solenoid R and we have a section inside the solenoid with r, so r is less than R. So, $\vec{A} \cdot d\vec{l}$ if I calculate because $\vec{B}$ is perpendicular to this plane.

So, $\vec{A}$ is circulating this, this, this, line, whatever the line we are having, so I can calculate this quantity $\vec{A} \cdot d\vec{l}$. So, $\vec{A} \cdot d\vec{l}$ for this case is simply $\vec{A}$ and it is 2πr that quantity again should be equal to the flux ϕ_B. So, this thing is nothing but the magnetic flux and magnetic flux here $\vec{B} \cdot d\vec{s}$, $\vec{B}$ is uniform here. So, that quantity I should write $\vec{B}$, magnetic flux here is simply $\vec{B}$, which is μ_0nI, which is uniform inside this multiplied by the area, which is πr^2. So, from here I can find out the value of $\vec{A}$ very easily.

The value of $\vec{A}$ is simply $\frac{1}{2}$ μ_0 n I and then r, that is the value of the magnetic vector potential. What about outside case because this is inside and again this is the cross section we are having. So, let us draw the cross section of the solenoid. This is the cross section of the solenoid and from here to here we have R and again we have a circular path where we have the value of $\vec{A}$.

And that now is my r. In this case, r is greater than R. Now again we go to calculate this quantity $\vec{A} \cdot d\vec{l}$. That value is simply A 2πr and ϕ_B here this quantity is equal to ϕ_B magnetic flux. ϕ_B here the amount of magnetic fields μ_0nI, which is only in this region not outside multiplied by the area but the area is πR^2, because outside that the magnetic field is not there. So, only the area here we need to consider is the region, which is said A, this is the region where we have the magnetic field. This is the region where we should have only the magnetic field because outside there is no magnetic field. So, the area will be this one.

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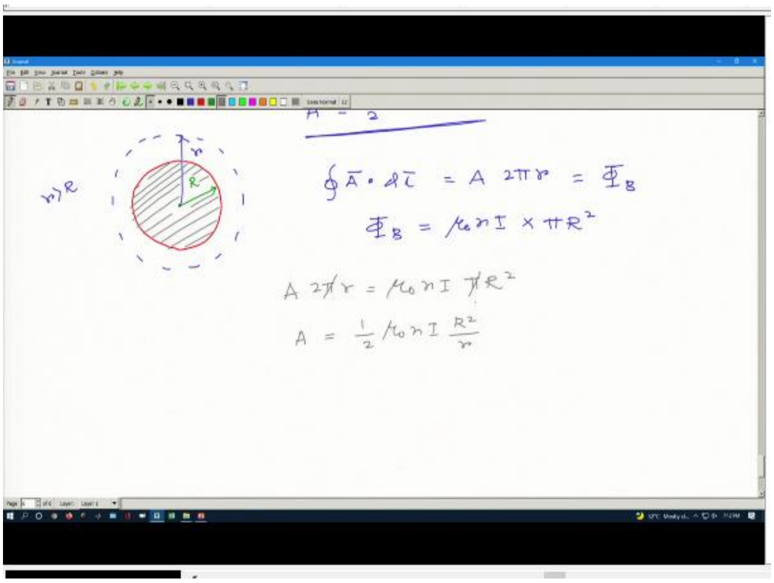

And now if I calculate A out of that, so my A then $2\pi r$ is μ_0 n I then πR^2. So, A from here I can calculate so π will cancel out and it should be simply $\frac{1}{2}$ μ_0 n I then $\frac{R^2}{r}$. So you can see that outside the magnetic field is 0 but the value of magnetic vector potential is non zero.

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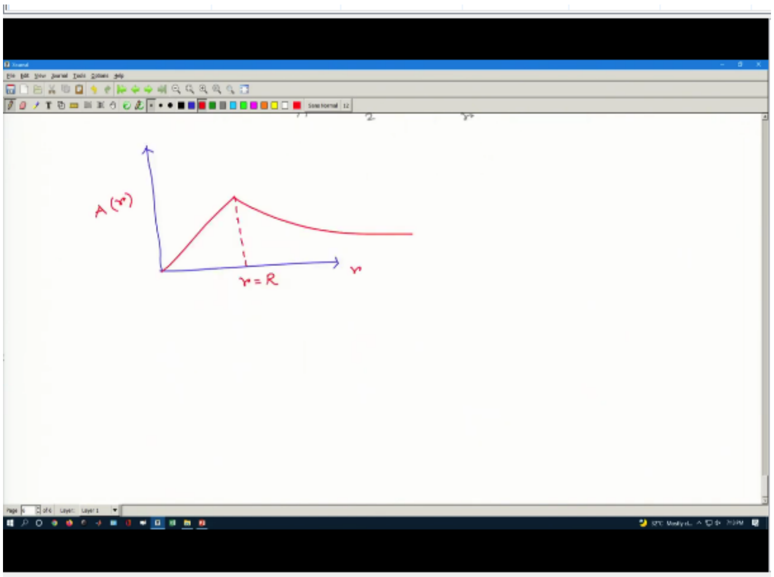

So if I plot these things how the magnetic field magnetic vector potential is changing with respect to r, then we have something like this, which is, so it is linearly changing and then changing at 1 by r. So, this is when r = R and r is this side and I am plotting magnetic vector potential as a function r. This is the way the magnetic vector potential will going to change inside the solenoid.

So, now, we will do another important thing and that is the monopole expansion of vector potential. So for a charge distribution for static electricity for the charge distribution we calculated potential and then we find that the potential can be written in terms of so let me quickly remind what we did in electrostatics.

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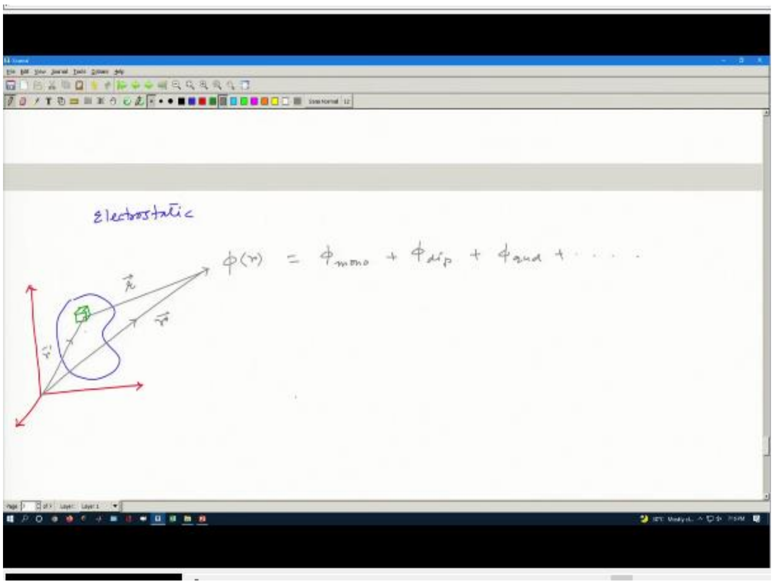

So, in electrostatics, what we did, we had a charge distribution and if I want to find out what is this charge. So, this is placed in a coordinate system like this, for a small section here I want to find out what was the value of the potential here what is ϕ, which is a function of $\vec{r}$ and this is from here to here we had $\vec{r}$ ' and from here to here, we had $\vec{Л}$ this was Л.

But the point is when we calculate this we find that this is the contribution of the monopole, it is the contribution of the electric dipole, the contribution and so on, the contribution of the monopole, dipole then quadrupole and so on. So, the vector potential this is the scalar potential for electrostatic field can be expanded in this way. So, that was the multipole expansion we had. So, that is the outcome that we figured out. The similar thing we had here.

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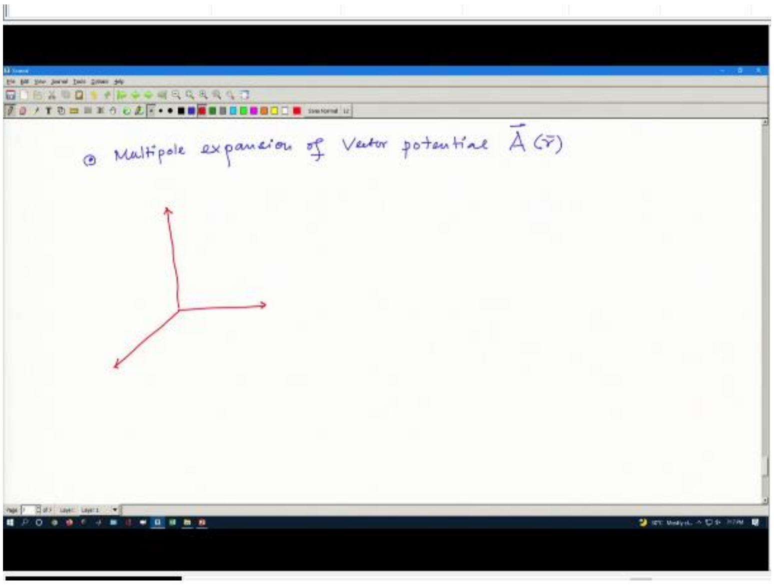

So, here what we do is multipole expansion of vector potential $\vec{A}(\vec{r})$. So, previously we did it for electrostatic and that is the multipole expansion of scalar electrostatic potential and we find that the contribution is summation of monopole, dipole and quadrupole. Here we will do the similar thing and see that what happens. So, let me again draw. So, in this case there was a charge distribution let here we will have a current loop. So, let me draw the coordinate system first. So, this is my coordinate system.

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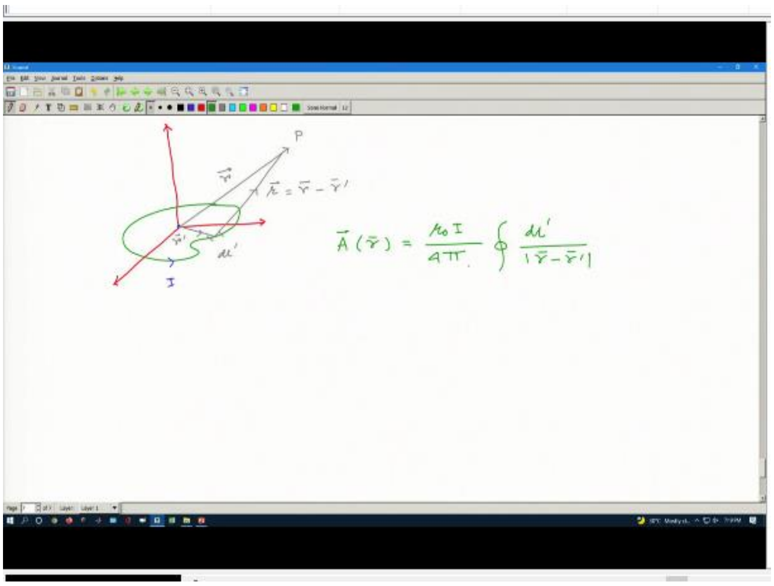

And I have a current loop like this, this is my current loop and the current is flowing here is I suppose the current that is flowing through this is I and this is the origin and I want to find out what is the magnetic field produced at some point here, which is say let me draw this here at point P what is the amount of magnetic field produces. So, I can have a section here say dl'.

And from here to here this is say $\vec{r}\,'$, this is the vector $\vec{r}\,'$ and this vector from here to here is my $\vec{\mathcal{R}}$, this is $\vec{\mathcal{R}}$, which is $\vec{r}$ - $\vec{r}\,'$. So, this is the geometry we are having. So, my $\vec{A}$ is simply at $\vec{r}$ we know the expression and that is $\frac{\mu_0 I}{4\pi}$ and then it is a closed line, so I should have a $\oint \frac{dl'}{|\vec{r}-\vec{r}'|}$. This expression we already defined that when the line current is there how the magnetic vector potential can be represented.

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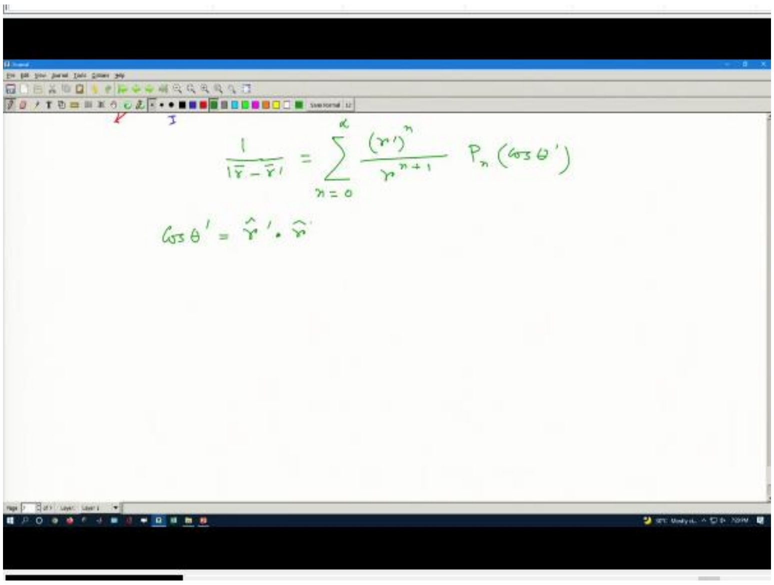

So, again like in the previous case I need to expand $\frac{1}{|\vec{r}-\vec{r}'|}$ and we did it earlier. So, if this angle is θ then this quantity we did it in the earlier case we are not going to do it here is $\sum_{n=0}^{\infty} \frac{(r')^n}{r^{n+1}}$ and then P_n (cos θ). That was the expression, maybe I can put θ' because everything is prime frame θ'. And cos θ' is simply $\hat{r}\,' \bullet \hat{r}$ and A I can now write as so what is my A finally?

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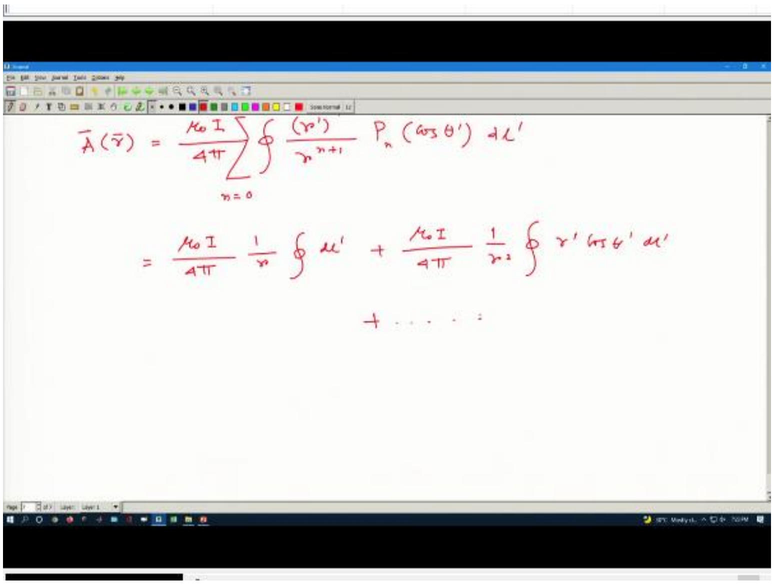

So, $\vec{A}$ vector potential is equal to $\frac{\mu_0 I}{4\pi} \oint \frac{(r')^n}{r^{n+1}}$ and then P_n (cos θ ’) and dl’. That is the expression we have. Now what we do? We just expand one summation I am missing, so it should be sum over n from 0 to infinity. So, now what we do that we will just expand for different n values. And if you do then I get the first term like $\frac{\mu_0 I}{4\pi}$ for n = 0 my first term will be $\frac{1}{r}$.

Because I can take it outside the integral and integral 0 $\oint dl'$ plus second term $\frac{\mu_0 I}{4\pi}$ and then $\frac{1}{r^2}$ $\oint r' \cos\theta' \, dl'$ and so on. Now I can write this term in this way.

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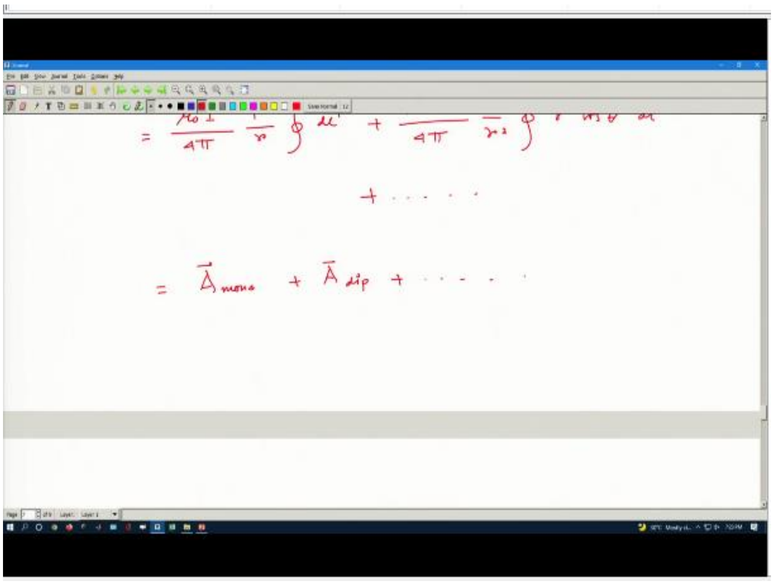

I can write the first term as $\vec{A}_{monopole}$ + $\vec{A}_{dipole}$ + so on. Now let us find one by one what is the meaning of monopole.

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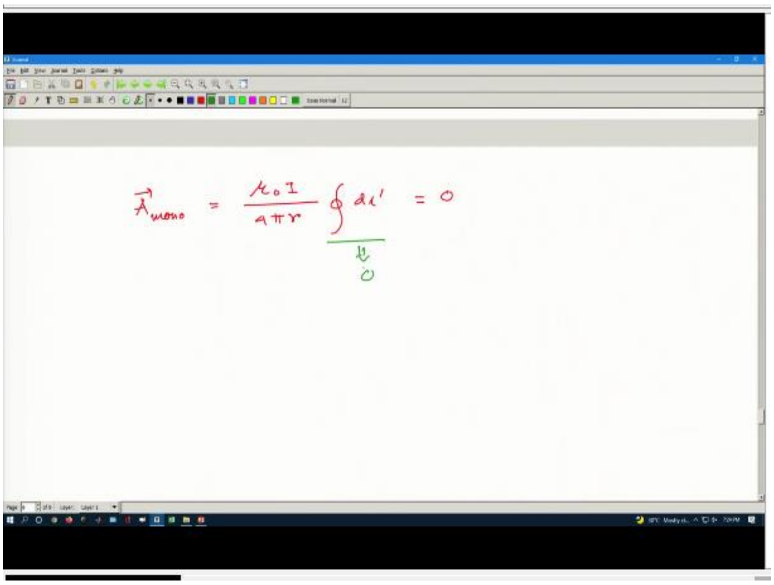

So, $\vec{A}_{\text{monopole}}$, this is the monopole contribution of the potential that is contributed due to the monopole here but we know that for magnetic field there is no magnetic monopole and also the expression is suggesting that we have $\frac{\mu_0 I}{4\pi r}$ and $\oint dl'$ and this is a closed line integral of dl and we know that this value has to be 0, this is zero because this quantity should be 0.

So, magnetic monopoly is not there at all. So, that means that contribution should not be reflected in calculating the vector potential of $\vec{A}$. What about the dipole the next important term is dipole.

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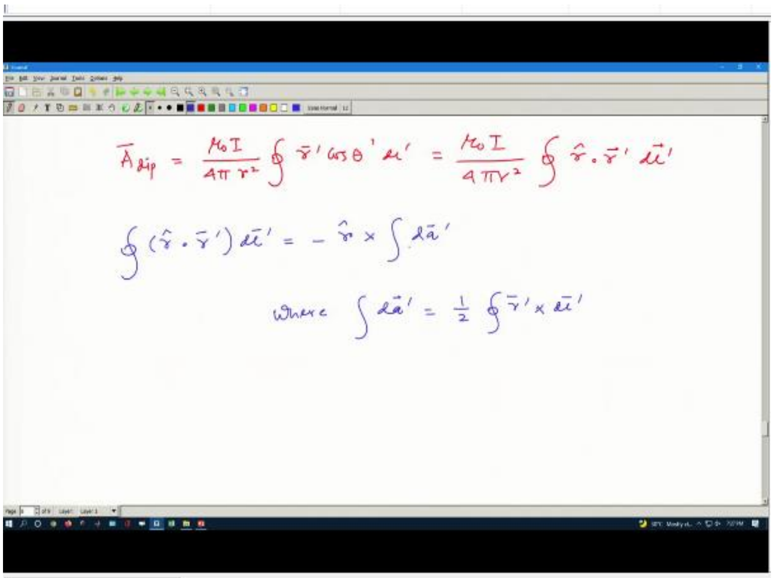

So, a dipole term is simply $\frac{\mu_0 I}{4\pi r^2}$ and then we have $\vec{r}'$ cos θ' and dl'. This I can simplify $\frac{\mu_0 I}{4\pi r^2}$ and then $\hat{r} \bullet \vec{r}'$ dl', $\hat{r} \bullet \vec{r}'$ take care of this r' cos θ'. So, I just replace this in this way. Now I will be going to use one identity and that is integration $\oint \hat{r} \bullet \vec{r}' \, dl'$ is equal to -$\hat{r}$ × $\int d\vec{a}'$, where $d\vec{a}'$ is half.

So, this is area. So, I mean I just write it but I did not prove, so I will be going to prove that maybe in the next class, this is a lengthy proof to be very honest and it is not very trivial. So, we will be going to do in the next class because today I do not have that much of time. But if I consider for the time being that okay this is the way I can represent $-\vec{r}'$ × $d\vec{l}'$ and this is the area.

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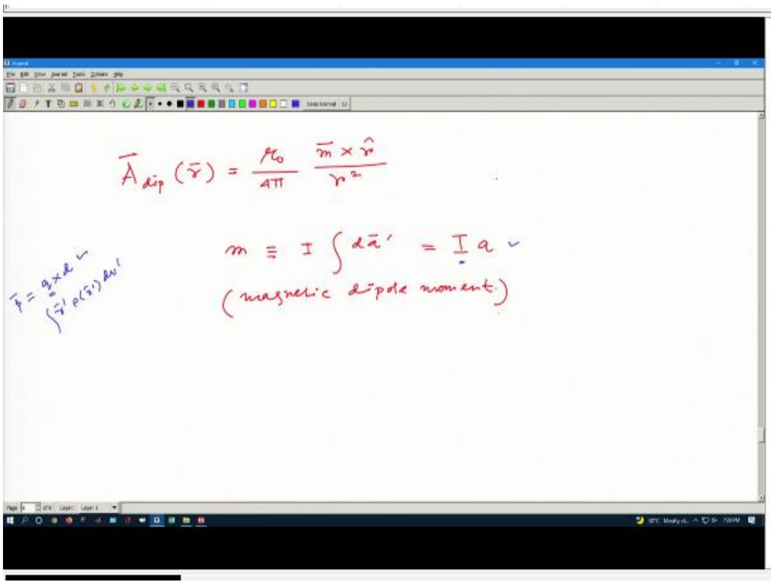

Then my magnetic dipole potential term contribution of that thing becomes $\frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$ where my m, which we call the magnetic dipole moment very, very important term is simply current into area or this into area. So, this is called the magnetic dipole moment. So, this is the way magnetic dipole moment is defined. So, electrostatic dipole moment if you remember that was defined like P is equal I am just writing in this side.

That was charge multiplied by the distance, that was the way and it is also defined in the integral from like r' and then r' ρ(r') dv that is the way also it was defined, but it was charge multiplied by the distance. But here it is current multiplied by the area. In electrostatic charge is there and in magnetostatic current is there because the static charge give rise to electrostatic and the steady current gives rise to the magnetic field.

So, that is why q is here, we have I their distance is there, so q into distance here we have area. So, these you need to at least appreciate that how the magnetic dipole and electric dipoles are defined. So, in the next class we will be going to discuss more about this proof because I am exploiting one identity here, this one but without any proof. So, in the next class we are going to prove that which will be a little bit lengthy proof.

So, we will be going to do this proof in the next class to understand how these areas are there in the picture. So, with that note I like to conclude here, thank you very much for your attention and see you in the next class.