Foundations of Classical Electrodynamics
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Lecture-57
Magnetic Dipole Moment (Contd.)

Hello student to the foundation of classical electrodynamics course. Under module 3, today we have lecture 57 and we will continue our discussion on magnetic dipole moment in this class also.

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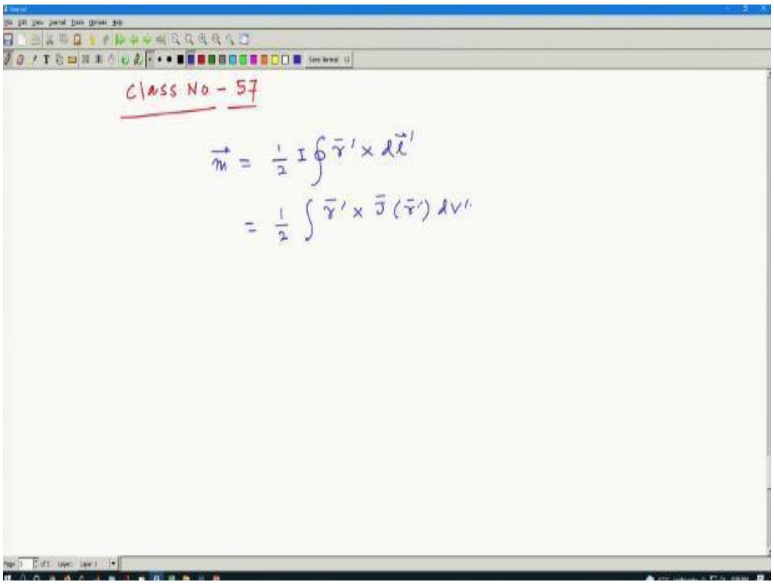

So, we have class number 57. So, we already mentioned that the magnetic dipole moment $\vec{m}$ is defined like $\frac{1}{2}$ I this is one way $\int \vec{r}' \times d\vec{l}'$ and also it can be defined that is $\frac{1}{2}$ $\int \vec{r}' \times \vec{J}(\vec{r}')$ dv' because, I am using the volume current. So, it should not be l, it should be v.

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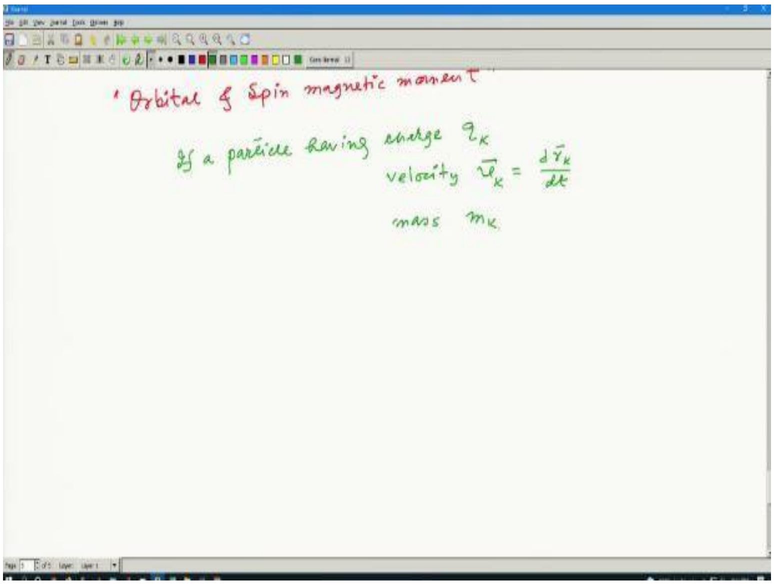

So, now today we will so with that note today what we do is try to understand the orbital and spin magnetic moment. Orbital and spin magnetic moment is important in atomic and molecular physics. So, this is a very important physical concept to try to understand that these here in the light of electrodynamical theories. So, what we have here is, suppose, if a particle having charge say q_k discrete particle.

So, charge q_k and velocity say $\vec{v}_k$ vector, which is $\frac{d\vec{r}_k}{dt}$, charge q and velocity $\vec{v}_k$ and also mass say m_k**.** The associated current density if that is the case. So, if a charged particle q is there and then the velocity is this and mass is m_k. Now if there are many particles not one particle, but there are many particles that is flowing discrete charge particle with mass m is flowing.

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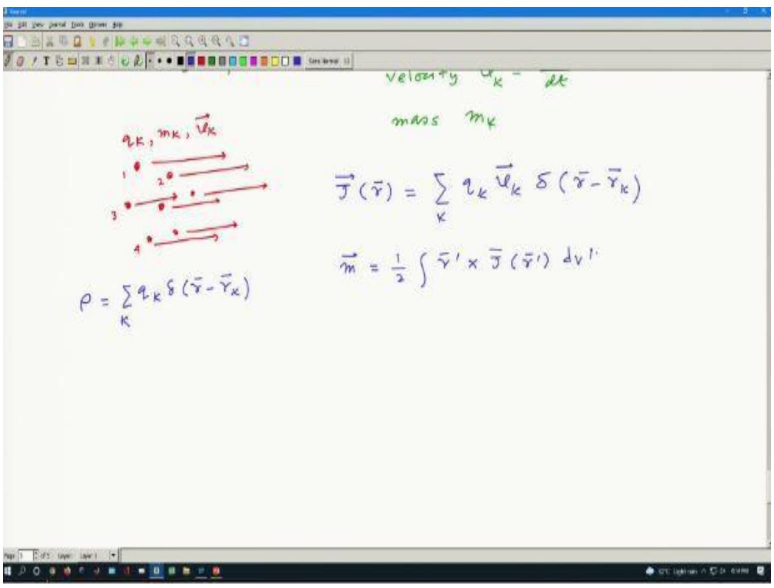

So, suppose I have this one charge particle, this is another charge particle, this is another. So, these are the charge particles and they are in motion. All are moving. And this is q_k the charge, k^{th} particle, mass is m_k and velocity is $\vec{v}_k$. These are given, for all the particles 1, 2, 3, 4 etcetera. And for k^{th} particle, these are the thing. Now the current density if they are flowing then the corresponding current density if I want to write $\vec{J}$. And we know that for discrete charge that is moving the current density can be written in terms of delta functions.

Like the way we write the volume charge density for a discrete charge. Here they are moving, I am writing the current density for these discrete charges. So, it should be q_k and then $\vec{v}_k$ and then $\delta(\vec{r} - \vec{r}_k)$. Because, for if they are in stationary state then the ρ is that quantity q_k and $\delta(\vec{r} - \vec{r}_k)$ that was the value of the volume charge density for discrete charge. And now I write the current density for this discrete charge, which is moving with the velocity $\vec{v}_k$ different.

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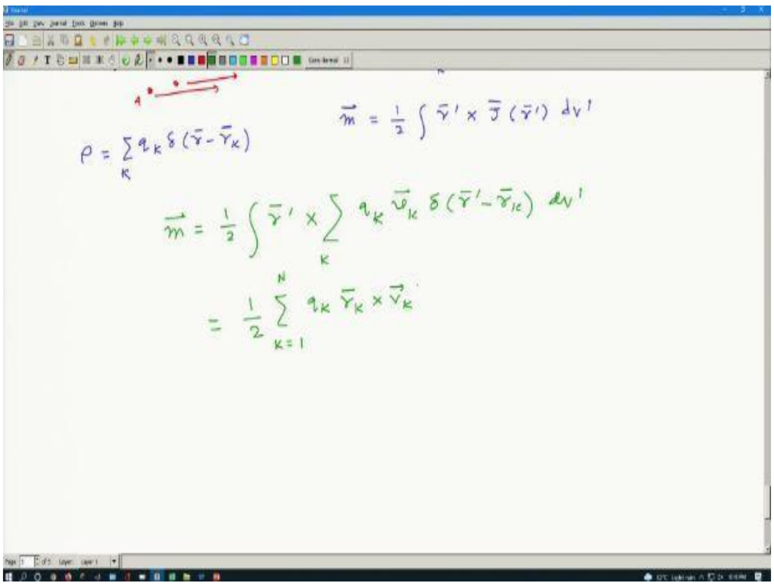

So, the magnetic moment already I mentioned, that the magnetic moment is $\frac{1}{2}\int \vec{r}' \times \vec{J}(\vec{r}')\, dv'$ this is a general form, in terms of current density. So, exploiting the value of the $\vec{J}$ here what we had I can write my $\vec{m}$. Let me write it here, $\vec{m}$ now is $\frac{1}{2}\int \vec{r}' \times$ and $\vec{J}$ is the summation of all this charge particle over k.

And then we have q_k and then we have $\vec{v}_k$ and $\delta(\vec{r}' - \vec{r}_k)$ and over dv'. So, this integral is over dv' and delta function is associated with that. So, that simply reduces the equation and I can simply have half of the summation I put this integral inside and it should be k say from 1 to n number of particles, say N. Then q_k and then this will be $\vec{r}_k \times \vec{v}_k$. So, that should be the magnetic moment.

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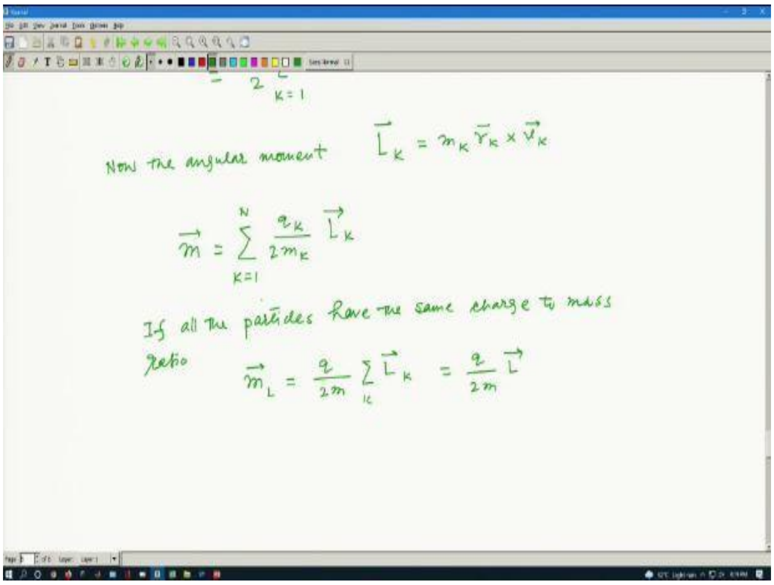

Now the angular momentum can be represented like $\vec{L}_k$, the k[th] particle angular momentum is $m_k\, \vec{r}_k \times \vec{v}_k$. So, that means I can write my magnetic moment in terms of this angular moment. So, magnetic moment now if $\vec{m}$ is simply $\sum_{k=1}^{N} \frac{q_k}{m_k}$ and the rest of the term 2 m_k because, half is there and the rest of the term is $\vec{L}_k$ and now if all the particles have the same charge to mass ratio.

So, if all the particles have the same charge to mass ratio, then we simply have $\vec{m}_L$ as $\frac{q}{2m} \sum_k \vec{L}_k$ I can take this $\frac{q}{m}$ outside, this integral because and then it should be simply $\frac{q}{2m}$ angular momentum $\vec{L}$. So, this is the relationship with the magnetic moment with angular moment.

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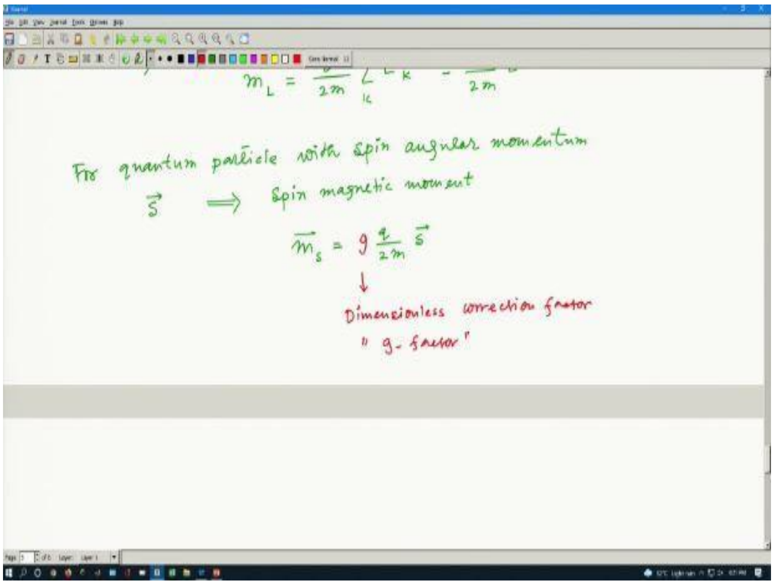

And for quantum particle with spin angular momentum. So, this is the angular momentum with the spin angular momentum, say the vector $\vec{S}$ the intrinsic. So, that spin angular momentum can produce an intrinsic spin magnetic moment. So, that intrinsic spin magnetic moment can be defined as the spin magnetic moment is equal to $\frac{q}{2m}\vec{S}$ like before. But we have a factor here, very important factor g and this is a dimensionless correction factor or g factor.

So, this is I mean just to you know in this statement just to show that how one can correlate with the magnetic moment with the spin angular momentum or linear angular momentum or spin angular momentum. So, this is the way one can correlate that. Now after that we will proceed further. So, one thing I missed in the last class. So, let me write here. The magnetic field due to dipole so, we know that whenever we have a tiny so current loop is essentially.

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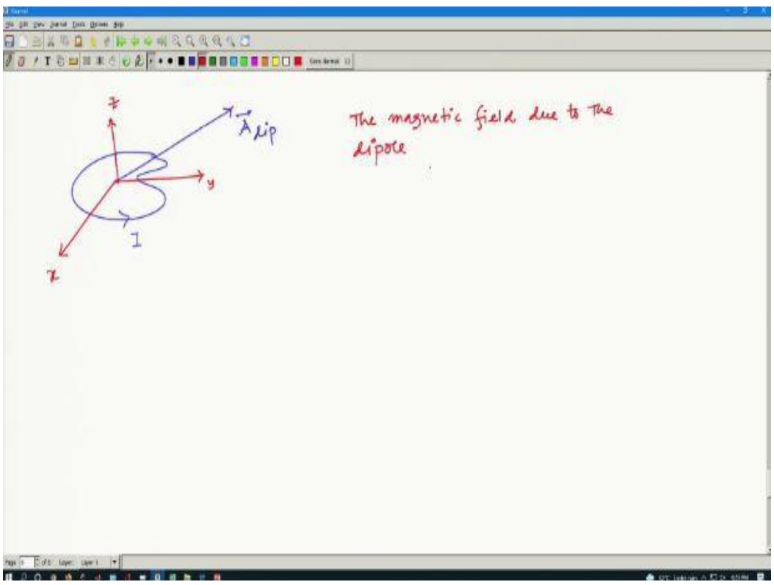

So, this is the current loop. And this current loop can produce a vector field here. And the dipole contribution, if I write that is the first important contribution because, monopole is not there in case of magnetic field. So, this current loop, which behaves like a dipole can produce a vector potential, which we called the magnetic vector potential with the dipole contribution. So, now this dipole potential can give rise to some magnetic field. So, here we try to understand the magnetic field gives rise to the magnetic field due to the dipole. Dipole here means magnetic dipole.

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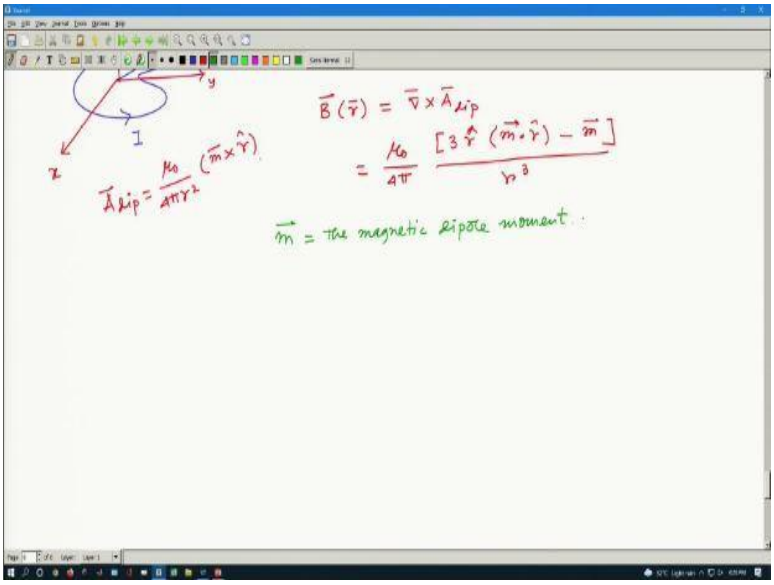

So, how I get that? So $\vec{B}(\vec{r})$ this is the magnetic field that is supposed to produce. Due to the magnetic dipole, so I should write as $\vec{A}_{\text{dip}}$. So, that is $\frac{\mu_0}{4\pi}$ and I now have here so $3\hat{r}$. I am just writing the result here. This is a homework for you. I like to give this as a homework because, the same problem that was okay**.** This is the vector sign.

A similar problem for electrostatic, we had if you remember. Exactly the same expression one can get, so divided by r^3**.** So, the $\vec{A}_{\text{dip}}$ we know what is $\vec{A}_{\text{dip}}$? $\vec{A}_{\text{dip}}$ is say this is the dipole contribution one can have for this magnetic. So, it is μ_0 what we extracted last day $4\pi r^2$ and then we have $\vec{m} \times \hat{r}$. Now if I want to find out $\vec{B}$, then you just need to make a curl of both sides and check that whether you are getting this result or not.

You just need to make curl of both side and try to figure out whether you are getting this result or not. This is a straight forward exercise and I hope you can do that. But the thing that I like to show here $\vec{m}$ is the magnetic dipole moment. Now for electric field what we what we got let me remind.

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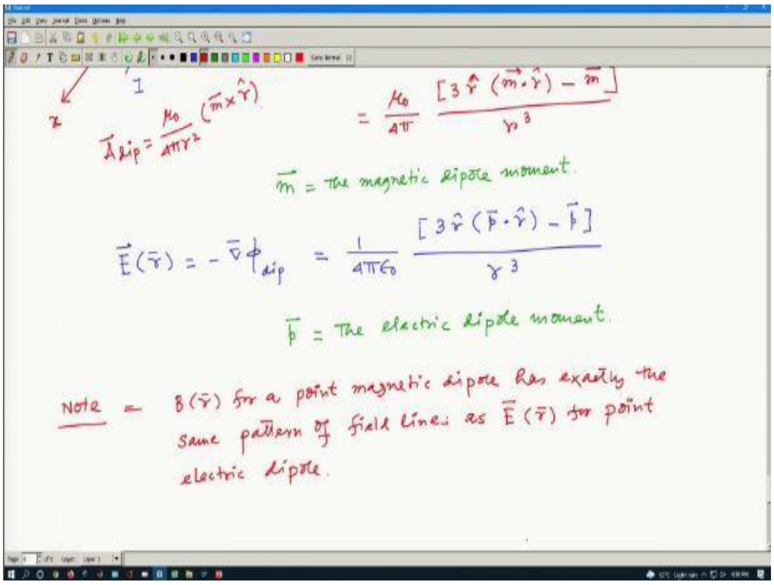

For electric field in electrostatic $\vec{E}$ was $-\vec{\nabla}\phi_{dip}$ and for dipole. We had the dipole potential and the electric field due to dipole, the electric field due to electric dipole that was $\frac{1}{4\pi\epsilon_0}$. And the term here was exactly same except the magnetic dipole here we have the electric dipole. So, $(\vec{p} \cdot \hat{r}) - \vec{p}$ I should have a vector sign here. Let me erase this. Whole divided by r^3. Because I am calculating electric field r^3.

So, here you can see only the $\vec{m}$ is replaced by $\vec{p}$ here for in. So, here $\vec{p}$ is my electric dipole moment. Since, the expressions are very similar. So, you should note that expressions are almost identical. So, what happened $\vec{B}(\vec{r})$ for a point magnetic dipole has exactly the same pattern of field lines as $\vec{E}(\vec{r})$ for point electric dipole? So, what does it means that since, the expression of $\vec{B}$ and $\vec{E}$ are very much identical.

So, the field line that due to the magnetic field that is produced due to the magnetic dipole should be identical to the electric field that is produced for the point electric dipole because, the expressions are very much same that you need to remember.

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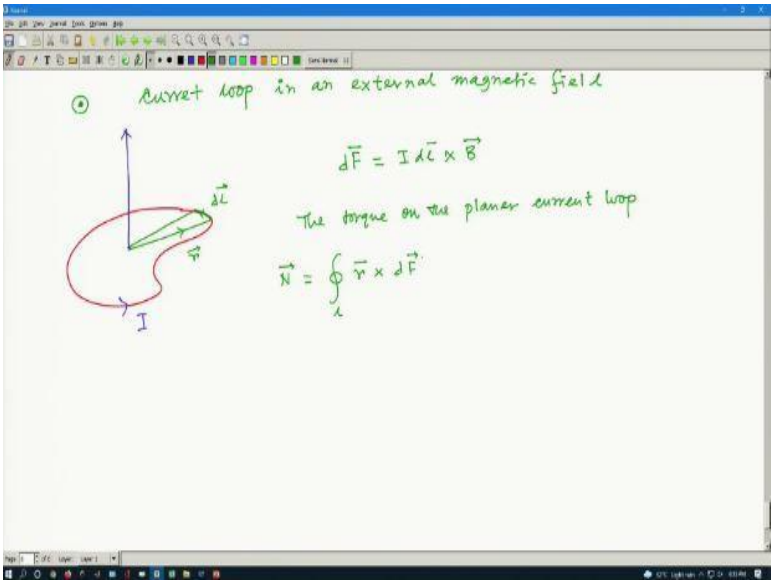

After that the thing that we do today is this. Like in electrostatic case, we placed the electric dipole in an external electric field. Here we will do characteristically a similar thing that we place the magnetic dipole that is a current loop in an external magnetic field. So, current loop, which gives rise to a magnetic dipole moment is placed in an external magnetic field. And so I placed a current loop in external magnetic field.

Now first we need to find out the force. The force on a current element, so if I have a current element I $d\vec{l}$. So, let us have a loop here. Say this is my current loop. Some current is flowing I and we have so this is say $\vec{r}$ and here it is $d\vec{l}$. So, the force on this current element is $d\vec{F}$ is equal to the current element that we are having cross external magnetic field, that is the force that one can have.

So, I have an external magnetic field here in some direction. And that will produce some kind of force here in this current element $d\vec{l}$. Now the torque on the planar current loop, this is the force. The torque on the planar current loop is $\vec{N} = \vec{r} \times d\vec{F}$ this is the torque one can have.

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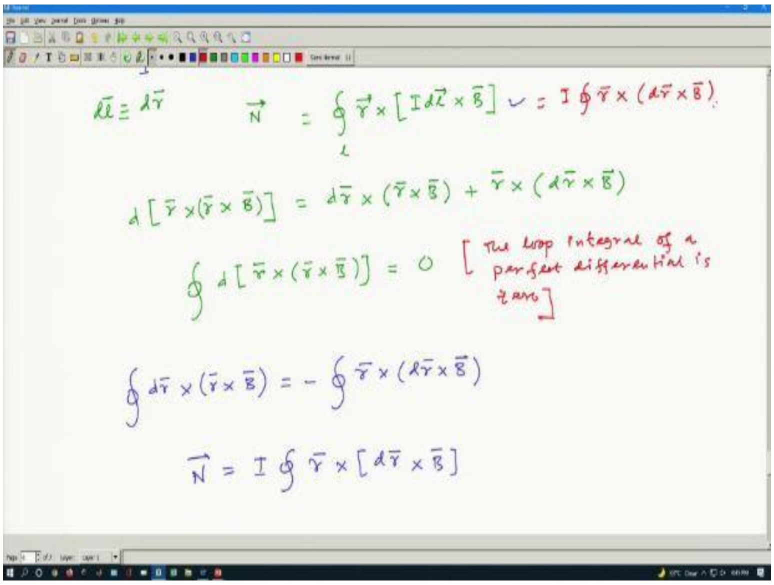

So, that eventually gives me $\vec{r} \times d\vec{F}$ I already defined here. So, it is I $d\vec{l}$ then $\times \vec{B}$**.** From here you can note one thing. So, $d\vec{l}$ is nothing but $d\vec{r}$ because, I am writing. So, this is equivalent. So, it will be easier for me because, this is $\vec{r}$ and this is $\vec{r} + d\vec{r}$. So, $d\vec{l}$ should be equivalent to $d\vec{r}$. So, now what we do is this. So, this is my torque. So, d $[\vec{r} \times (\vec{r} \times \vec{B})]$ if I want to calculate this quantity.

It should be $d\vec{r} \times (\vec{r} \times \vec{B}) + \vec{r} \times (d\vec{r} \times \vec{B})$. $\vec{B}$ is a constant. So, and also this is a closed integral, if I make a close integral of this quantity, which is a complete integral. So, $\vec{r} \times (\vec{r} \times \vec{B})$, it will simply give me 0 because we are calculating the loop integral of a perfect differential. So, this is the loop integral and we have integrate perfect differential. So, this has to be 0. So, let me comment here, the loop integral or the close line integral of a perfect differential is zero**.**

So, that quantity should be zero. Now if that is the case then we can exploit this to find the relation that $d\vec{r} \times (\vec{r} \times \vec{B}) = -\vec{B}$. Now once we have this let us go back to the expression of the $\vec{N}$ that we figure out. So, $\vec{N}$ = I close line integral, so what was that? $\vec{N}$ was I close line integral here**.** Then I can take I outside and $d\vec{l}$ is $d\vec{r}$. So, from here I can write is $\vec{N} = \vec{r} \times (d\vec{r} \times \vec{B})$. So, $\vec{r} \times (d\vec{r} \times \vec{B})$**.** Now let us manipulate this.

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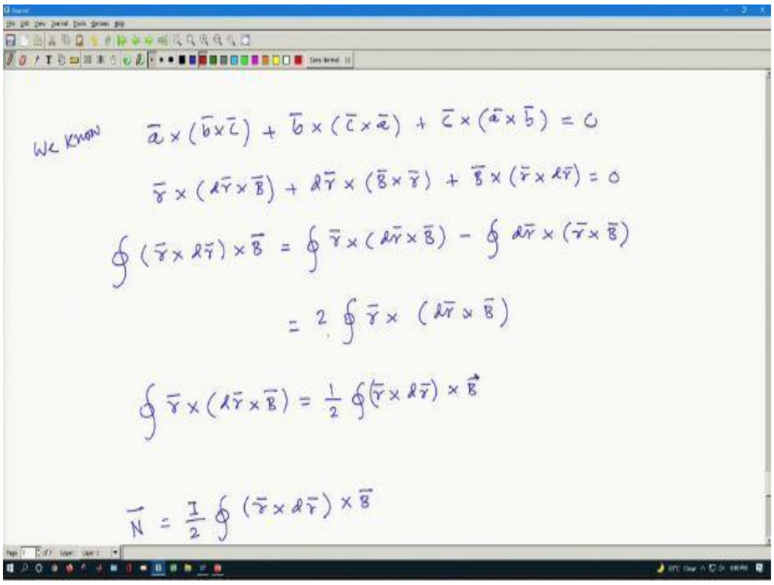

So, we know that. That $\vec{a}$ cross this is a cyclic rule $(\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$ is 0. So, I can write as $\vec{r} \times (d\vec{r} \times \vec{B}) +$ d$\vec{r} \times (\vec{B} \times \vec{r}) + \vec{B} \times (\vec{r} \times d\vec{r}) = 0$. So I can have $\oint(\vec{r} \times d\vec{r}) \times \vec{B}$ $= \oint \vec{r} \times (d\vec{r} \times \vec{B}) - \oint d\vec{r} \times (\vec{r} \times \vec{B})$. So, that basically gives me this quantity as $2 \oint \vec{r} \times (d\vec{r} \times \vec{B})$.

Because from here I just write it as $\vec{r} \times (d\vec{r} \times \vec{B})$ and that should be a negative sign. So, that makes it plus and then we have this. So, just we exploit the expression that is here. And I can have $d\vec{r} \times (\vec{B} \times \vec{r}) = \vec{r} \times (d\vec{r} \times \vec{B})$. If I go this side it should be negative sign and I will be going to get this. So, what I finally have is this one, that $\vec{r} \times (d\vec{r} \times \vec{B}) = \frac{1}{2} (\vec{r} \times d\vec{r}) \times \vec{B}$.

So, $\vec{N}$ is eventually, so what was the value of the $\vec{N}$ that is $\vec{r}$ cross, so I can put I outside. So, here $\vec{N}$ is simply torque is $\frac{I}{2}$ and then $\oint(\vec{r} \times d\vec{r}) \times \vec{B}$. So, $\vec{B}$ is constant here you should note. So, the point here is to write in then these things in $\vec{r} \times d\vec{r}$ to get the feeling of the magnetic potential, the magnetic dipole moment.

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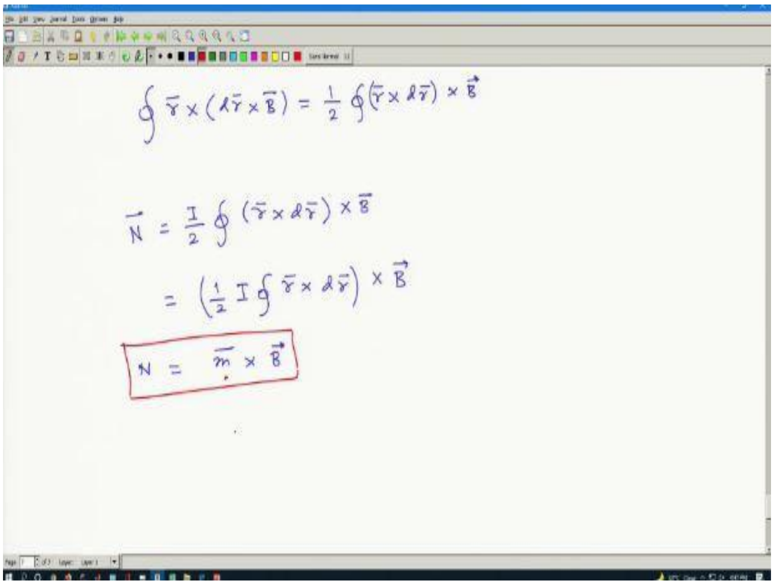

So, I can write this $\frac{1}{2}$ I $\oint(\vec{r} \times \mathrm{d}\vec{r})$ bracket it and then we have cross $\vec{B}$. So, this quantity is $\vec{m}$. So, I simply have $\vec{m} \times \vec{B}$. So, the point of this entire calculation is to find out the amount of torque that a current loop will be going to experience in terms of the magnetic dipole moment $\vec{m}$ and the external magnetic field $\vec{B}$. And simply so this calculation looks little bit clumsy.

But if you go to step by step I think you will understand. So, let me go back and check once again. So, $\vec{N}$ is simply the $\vec{r}$ cross the force that we are going to experience by this loop. The current element is I $\mathrm{d}\vec{l} \times \vec{B}$. The I $\mathrm{d}\vec{l}$ and the force that we are going to experience by the magnetic field is this cross $\vec{B}$. So, this is $q\vec{v} \times \vec{B}$. So, I can write $q\vec{v}$ is in terms of current and $\mathrm{d}\vec{l}$. So, it is the current element cross $\vec{B}$.

So, $\vec{N}$ the torque should be $\vec{r} \times \mathrm{d}\vec{F}$, $\mathrm{d}\vec{F}$ I just replace here I $\mathrm{d}\vec{l}$. And then I can take this I outside because, I is a steady current that is flowing. So, eventually I have I so this thing is simply I and then I have close line integral of $\vec{r}$ cross and then I have $\mathrm{d}\vec{l}$ I replace $\mathrm{d}\vec{r}$ for my convenience because, these are two same things cross $\vec{B}$. The next part was to just write $\vec{r} \times (\mathrm{d}\vec{r} \times \vec{B})$ in terms of $\vec{r}$ cross the last line what I get $\vec{r} \times (\mathrm{d}\vec{r} \times \vec{B})$.

So, we just use few vector identities like d of these things is this quantity. So, if I make a close line integrals I have so, $\vec{r} \times \vec{r} \times \vec{B}$ should be $-\vec{r} \times (d\vec{r} \times \vec{B})$, so that we use here. And I put this inside. So, whatever the value we had here $\vec{r} \times (d\vec{r} \times \vec{B})$, I just then use the $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times$

$(\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$ and then just replace $\vec{a}$, $\vec{b}$, $\vec{c}$ in terms of $\vec{r}$, $d\vec{r}$ and $\vec{B}$, and carry forward the calculation.

When you carry forward the calculation, we find that $(\vec{r} \times d\vec{r}) \times \vec{B}$ can be represented in terms of $\vec{r} \times (d\vec{r} \times \vec{B})$ with twice of that. So, $\vec{r} \times (d\vec{r} \times \vec{B})$ is now half of this quantity. And then I replace this $\vec{r} \times (d\vec{r} \times \vec{B})$ from here. So, this $\vec{r} \times (d\vec{r} \times \vec{B})$ is finally replaced here, with the half of $(\vec{r} \times d\vec{r}) \times \vec{B}$ and eventually we find our magnetic dipole moment.

So, my torque becomes the magnetic dipole moment cross the amount of magnetic field that is there. So, when the loop is there. So, I know what is the magnetic moment and if the $\vec{B}$ is known and the magnetic moment is known, which is current into area. So, that is the amount of the torque it will be going to experience under the external magnetic field. So, please check this calculation and try to do that.

So, and I believe you can understand this. We are just using some vector identities, some mathematical jugglers are there. But at the end of the day, you will be going to get the result quite straight forward results. So, with that note I think I should conclude here in today's class. So, I think in the next class I will like to discuss about the magnetostatic in magnetization problem.

That means, inside a magnetic system if I apply the magnetic field and if I put a magnetic system the magnetization will be there. And how this magnetization can be calculated, what should be the physical significance, how physically we can calculate all these aspects? We will be going to discuss. It is exactly same like putting a dielectric material in an external electric field and how the polarization appears.

So, here in terms of magnetic field we have magnetization. And how the magnetization can be represented in terms of this magnetic dipole moment etcetera we are going to discuss. So, thank you for your attention and see you in the next class.