Foundations of Classical Electrodynamics
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Lecture-68
Maxwell's Wave Equation (Contd.)

Hello student to the foundation of classical electrodynamics course. So, under module 4 today we have lecture 68. And today we will be going to continue our discussion on Maxwell's wave equation, class number 68.

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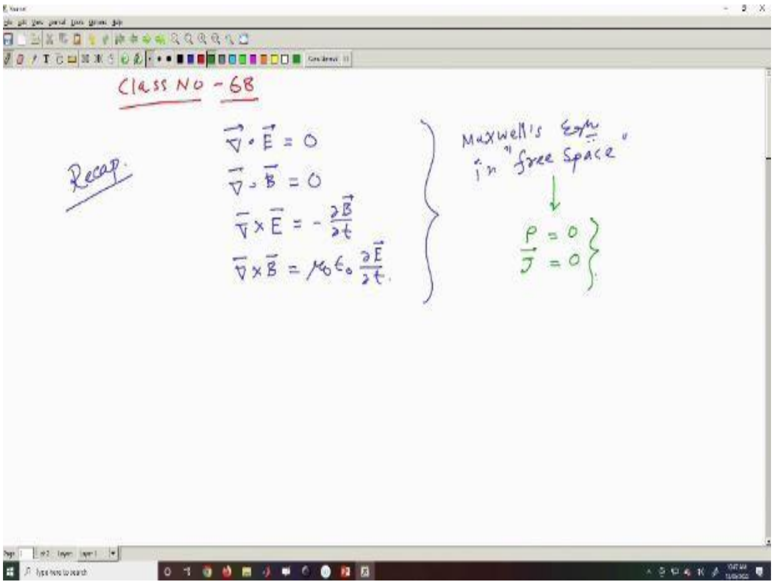

So, in the last class let me recap quickly. So, we had the Maxwell's 4 equations in free space and that 4 equations were this $\frac{\partial \vec{B}}{\partial t}$ and then $\vec{\nabla} \times \vec{B}$ was $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. So, this was Maxwell's equation in free space. So, in free space means, there is no source term. So, $\rho = 0$ and $\vec{J} = 0$. These 2 are the condition.

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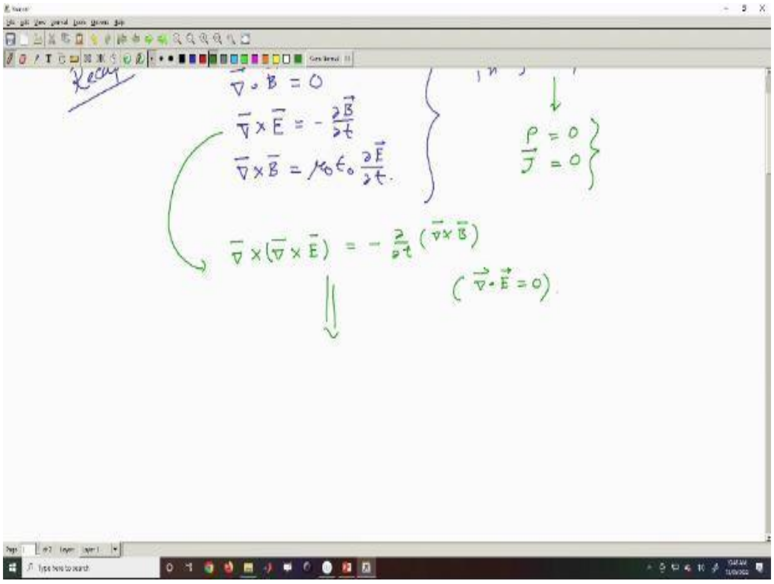

Now from that exploiting this curl equation this equation taking the curl both sides, I am not going to do the entire process 2, 3 steps and there we discussed last day. So, taking and using the condition that $\vec{\nabla} \cdot \vec{E} = 0$, we had this wave equation.

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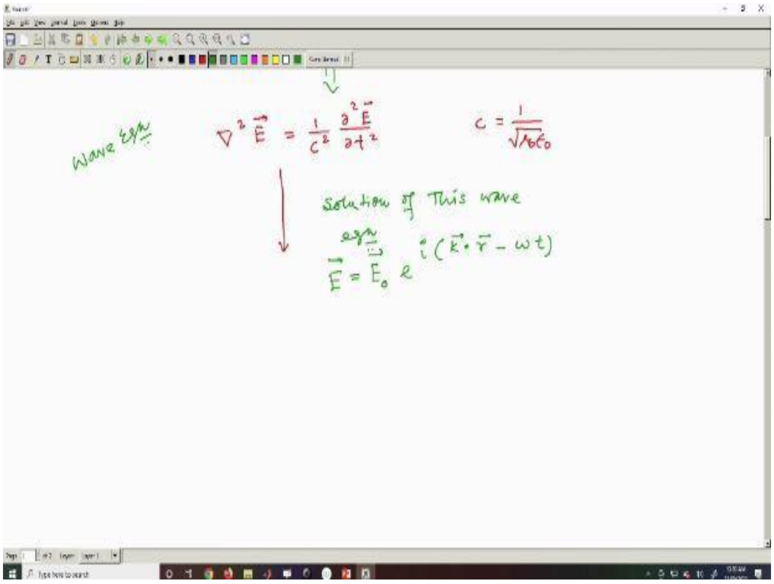

The wave equation was in the form like this, $\frac{1}{c^2}$ where c was $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ that is the expression we had. And then after that what we get is the solution. We find a solution for that, solution of this wave equation was something like this in this form. This is called the plane wave solution. Why it is called plane wave solution? We will discuss. So, also we can discuss.

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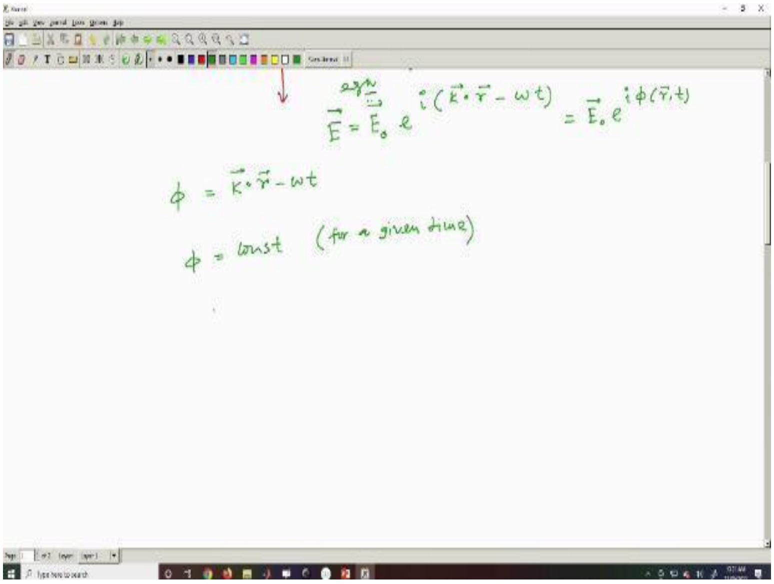

So, here whatever the phase we are having, so, it is eventually an equation like $\vec{E}_0$ e^{i}, a phase is there, which is a function of $\vec{r}$ and t. So, the phase here is like $\vec{k} \cdot \vec{r}$ - ωt. Now for a fixed time or if I take a particular instant of time. Then the equation for the constant phase, if this is constant for a given time.

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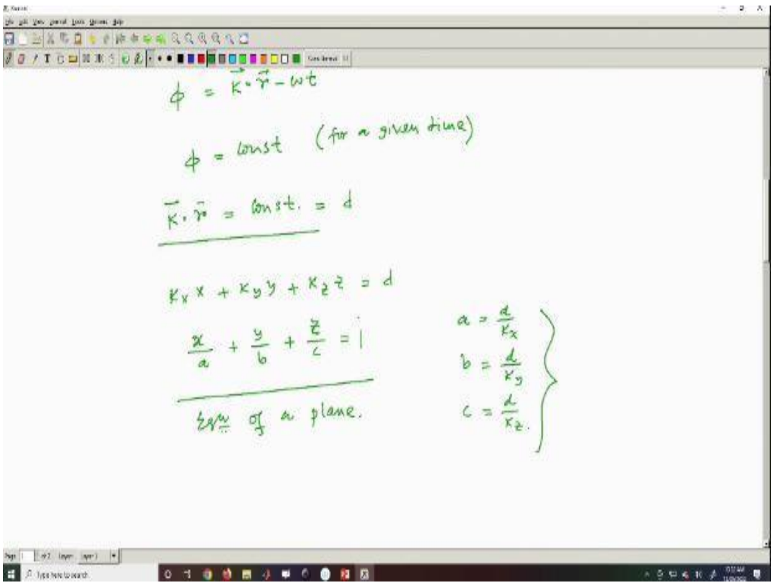

The locus of this constant phase the equation of this constant phase simply gives us $\vec{k} \cdot \vec{r}$ equal to some constant. Because, when ωt is given, that means, ωt is a constant. And I am saying that ϕ is constant, that means, $\vec{k} \cdot \vec{r}$ this expression should be constant. So, I am trying to find out the point where, all the phase value is constant. So, in that case I should have this is the equation to get all the phase constant.

Now if I consider this equation, say this constant is some value d. So, this equation simply k_x x + k_y y + k_z z is equal to this constant d. And I can write this in this particular form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ = 1 where, a is equal to say $\frac{d}{k_x}$, b = $\frac{d}{k_y}$ and c is $\frac{d}{k_z}$. So, this is the way a, b, c are there. So, this is the equation of a plane. So, which suggests that when this wave is propagating for then the phase font is making a plane and that is why it is called a plane wave.

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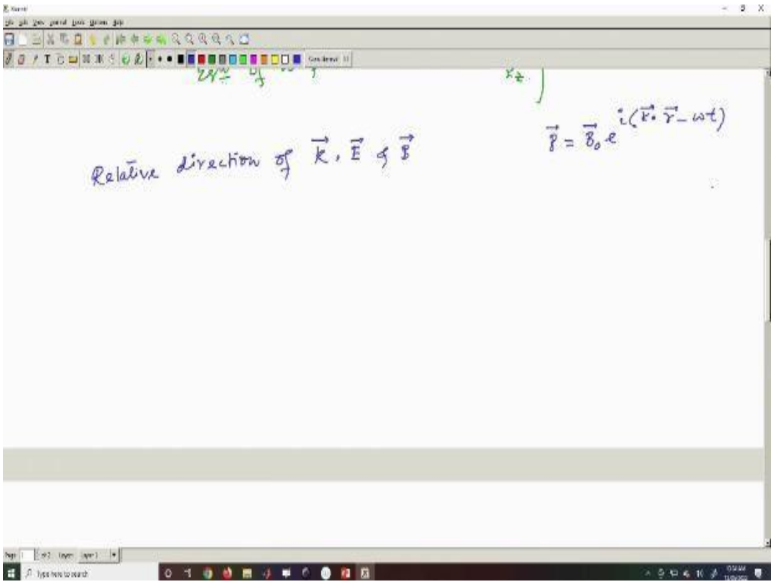

Now we will try to understand one important thing and that is the relative direction of $\vec{k}$, $\vec{E}$ and $\vec{B}$. So, there are 3 vectors are associated if you look carefully. There is a $\vec{k}$ propagation constant and inside the $\vec{B}$ also we have because, if you solve the wave equation for $\vec{B}$ you will be going to get the similar solution. And also $\vec{E}_0$ and one should have a solution for $\vec{B}$.

Let me write it is like $\vec{B} = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. It will have also the same velocity that of $\vec{E}$. But the direction of the $\vec{B}_0$ and the direction of the $\vec{E}_0$ they are not same. So, that is why we need to find it out. So, how you get that? So, we will be going to exploit. So, let me write down once again.

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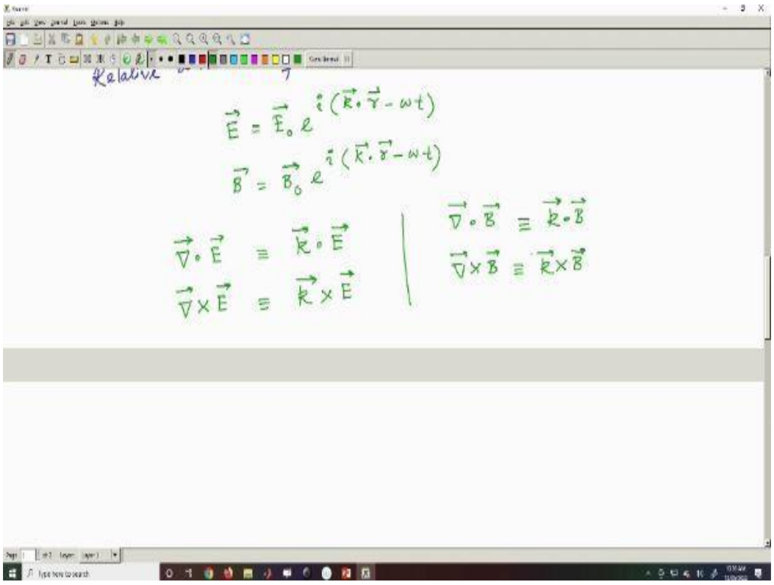

So, my $\vec{E}$ the solution is $\vec{E}_0\ e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and my $\vec{B}$ is $\vec{B}_0\ e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Now if you have this solution one very important thing, you can find out by yourself. That if I do this operation over this plane wave this operation divergence, then it should be equivalent to $\vec{k}\bullet\vec{E}$, if the solution in this plane waveform. So, this is an exercise if you can check it last day also I mentioned that and I derive it but, you can check it what happened here.

In the similar way if I do the $\vec{\nabla}\times\vec{E}$ then that should be equal to $\vec{k}\times\vec{E}$ that is that means, it is also true for $\vec{B}$ as well. So, because $\vec{B}$ is in the same way, so, that means if I do this operation over $\vec{B}$, so this operation is eventually tells me that it will be like $\vec{k}\bullet\vec{B}$. And if I do the $\vec{\nabla}\times\vec{B}$ if the $\vec{B}$ is in this plane wave form, then it should be $\vec{k}\times\vec{B}$. You can prove that, by simple using the vector algebra, which we learn.

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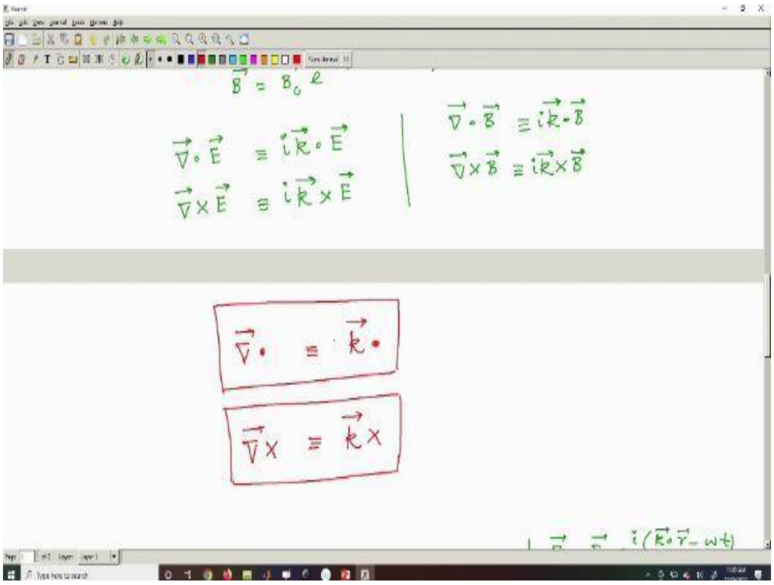

So, what we get is this operator is equivalent to $\vec{k}$ •. So, that is one and another thing is this curl operator if it is operating over some function having the plane waveform like here it is valid only for this kind of solution. So, then this can be replaced by $\vec{k}$ × these things**.** So, this operator can be replaced in this way some sort of Fourier transform thing is involved.

So, this is a space operator and we can simply make it an inferior plane we are just simply making a dot and cross by just changing this operator. But, you should note this is very handy thing. Now let us go back.

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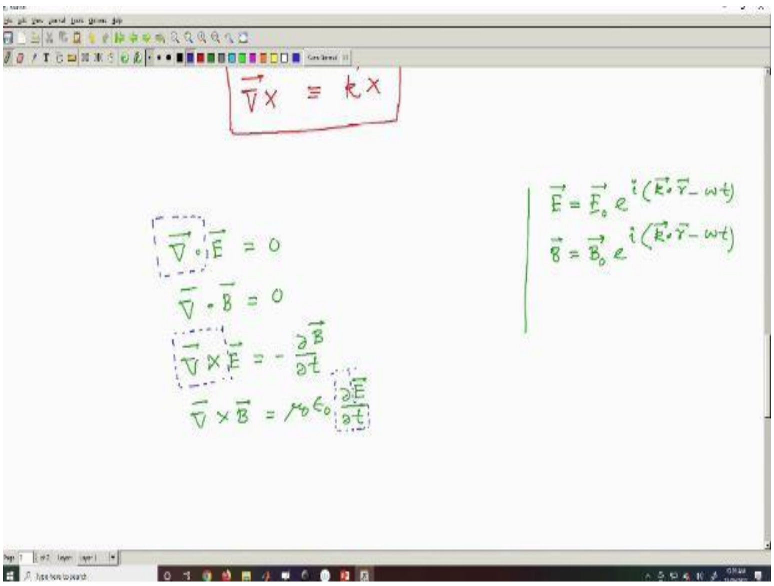

What is the equation we had? We had $\vec{\nabla} \cdot \vec{E} = 0$. You should note that my solution is already known. And this solution is like $e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ that is there and $\vec{B}$ is $\vec{B}_0\, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$. So, if this is 0, so I can write. Let me write down all the 4 equations first. $\vec{\nabla} \bullet \vec{B}$ is 0, $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$ and $\vec{\nabla} \times \vec{B}$ is $\mu_0\ \epsilon_0\ \frac{\partial \vec{E}}{\partial t}$. Now we had this, parallely I like to also mention here, that we have another, so, another operator.

So, one operator here is this. If I note so one operator is this, which we know how to handle with curl and divergence this is the operator, which is operating over that. And another is called operator that is operating. So, these 2 kinds of operator are there and we know what is the recipe. But also there is operator sitting here, which is $\frac{\partial}{\partial t}$. So, what should be this form of this operator because, I have the full form explicit form.

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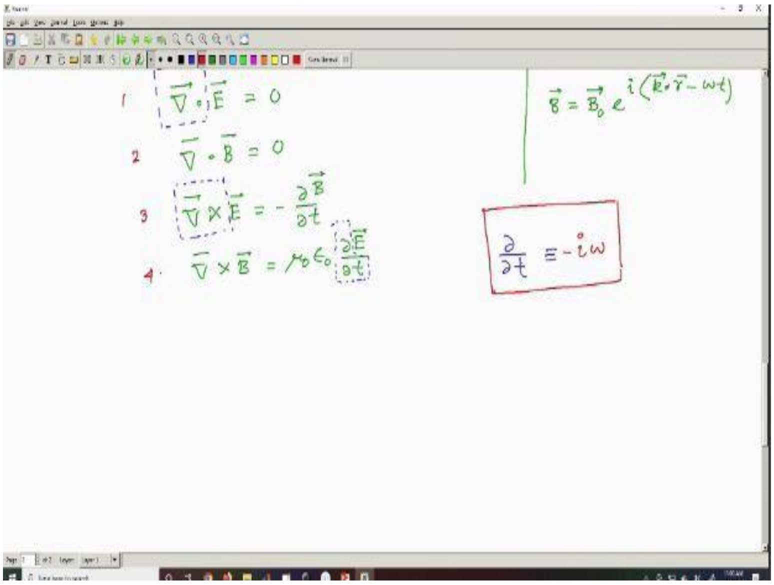

So, here you should also note that. I am writing here $\frac{\partial}{\partial t}$ this operator should be equivalent to. I missed i here because, whenever you have this operator. So, it should be equivalent to there is a mistake. Let me so it should be because, there is i involved here so it should be associated with one i. So, it should be i of these things. So, now what is $\frac{\partial}{\partial t}$? $\frac{\partial}{\partial t}$ this operator, now will be replaced by simply - iω.

So, that is the replacement for $\frac{\partial}{\partial t}$ operator. So, these 3 things you should know and exploiting these 3 informations let us now find out what we get from 1, 2, 3 all these 4. So, I have 1 equation here, 2 equations, 3 and 4. So, let us check what we get.

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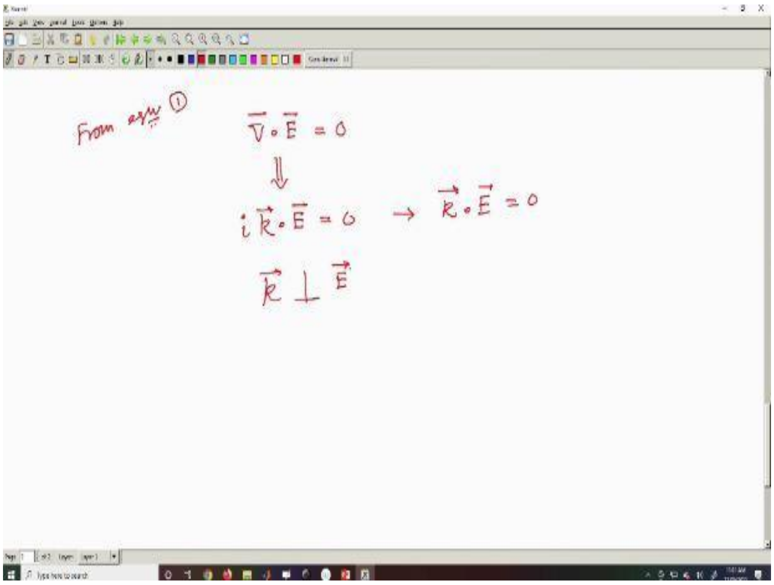

So, from equation 1, we have $\vec{\nabla} \cdot \vec{E} = 0$ this simply means, $i\,\vec{k} \cdot \vec{E} = 0$ or simply $\vec{k} \cdot \vec{E}$ is 0. It simply says that the $\vec{k}$ is perpendicular to $\vec{E}$. That means the propagation vector should perpendicular to the electric field. Now let us do let us exploit equation 3. So, that is one information I get. So, my goal is to find out the relative direction between the $\vec{E}$, $\vec{B}$ and $\vec{k}$. So, I find one information that $\vec{k}$ should be perpendicular to $\vec{E}$.

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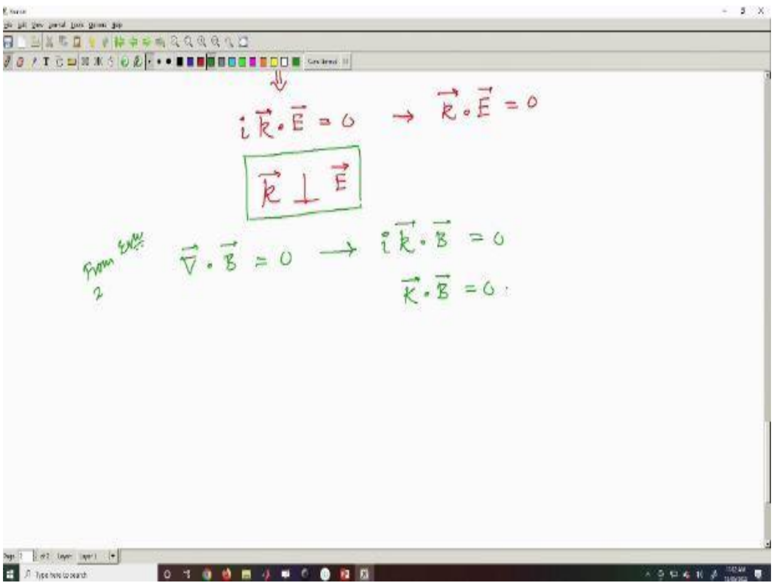

Now also we had picked from equation 2, we had $\vec{\nabla} \cdot \vec{B} = 0$, which leads to the expression $i\,\vec{k} \cdot \vec{B} = 0$ or simply I can have $\vec{k} \cdot \vec{B} = 0$.

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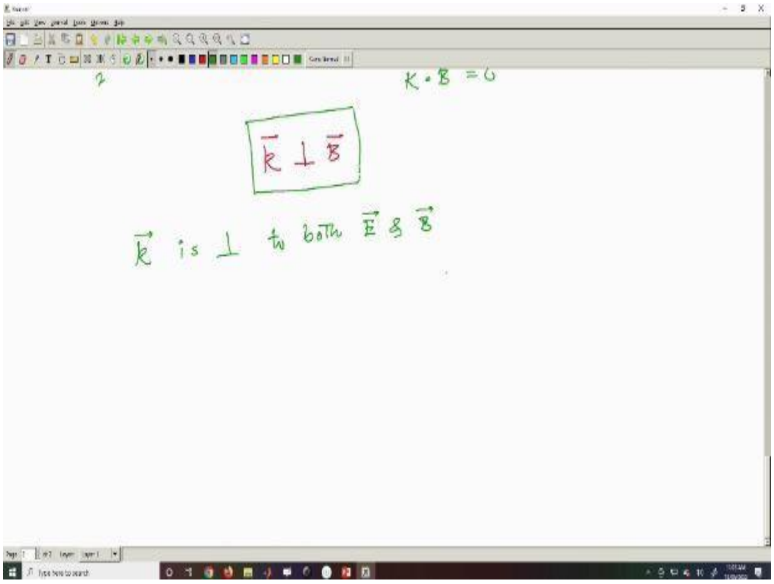

So, again I can write that $\vec{k}$ is perpendicular to $\vec{B}$ as well. So, that is the second information I get. So, eventually I find that $\vec{k}$ is mutually perpendicular to both $\vec{E}$ and $\vec{B}$. So, $\vec{k}$ is perpendicular to both $\vec{E}$ and $\vec{B}$**.** So, now what is the relative? So, I know $\vec{E}$ is perpendicular to both $\vec{B}$ and $\vec{E}$ but, what is the relative direction between $\vec{E}$ and $\vec{B}$ to find that, I need to exploit equation 3.

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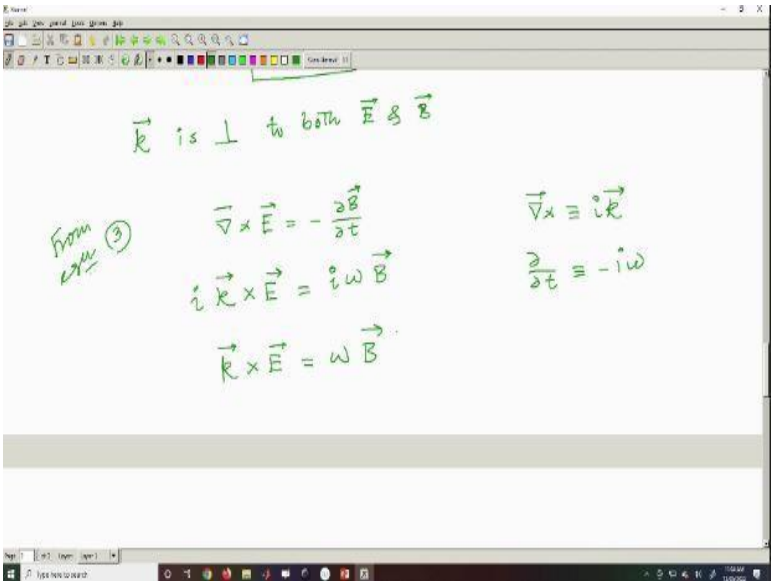

So, from equation 3 what we have. We have $\vec{\nabla} \times \vec{E}$ is $-\frac{\partial \vec{B}}{\partial t}$. So, $\vec{\nabla} \times \vec{E}$ I can write as I cross because, curl I can replace. So, these is equivalent to $i\vec{k}$ and $\frac{\partial}{\partial t}$ is equivalent to -iω. So, I can replace it and I can have that i $\vec{k} \times \vec{E}$ = i of 1 minus sign is already there $\omega\vec{B}$ or $\vec{k} \times \vec{E} = \omega\vec{B}$. Now $\vec{k}$ is perpendicular to $\vec{E}$ and $\vec{B}$ both. And now I can find that $\vec{k} \times \vec{E} = \vec{B}$.

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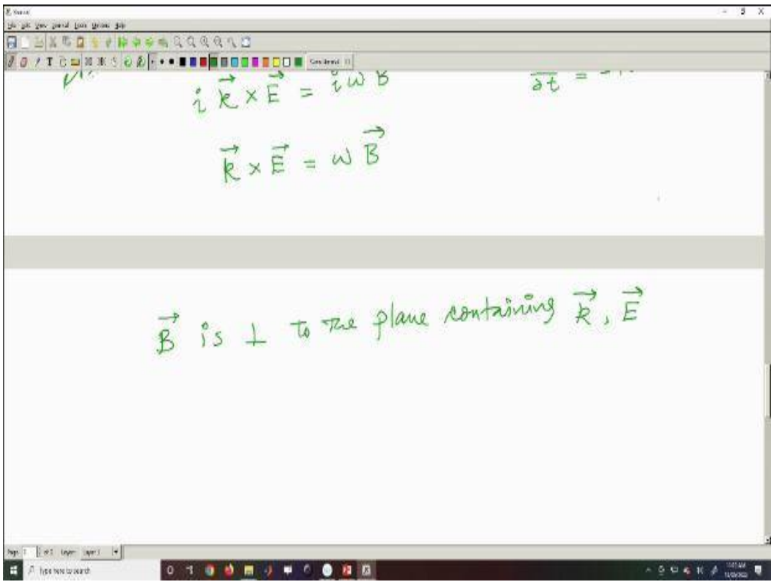

So, that means the $\vec{B}$ is perpendicular to the plane containing the vector $\vec{k}$ and $\vec{E}$. So, in other word $\vec{B}$ is perpendicular to both $\vec{k}$ and $\vec{E}$. So, that means $\vec{E}$, $\vec{B}$, $\vec{k}$ they are mutually perpendicular. $\vec{k}$ is perpendicular to $\vec{B}$, $\vec{k}$ is perpendicular to $\vec{E}$, $\vec{B}$ is perpendicular to $\vec{E}$. So, they are mutually perpendicular.

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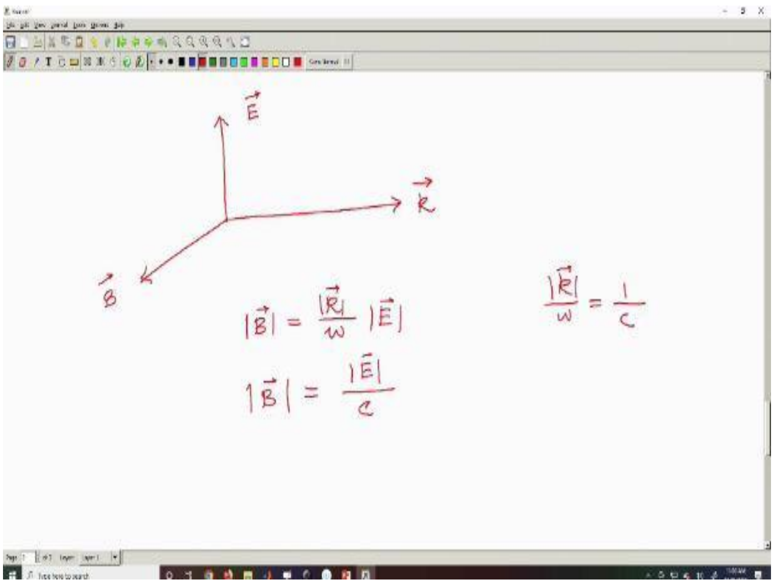

So, that means if I draw it should be like electric field if it is vibrating here like this magnetic field should be perpendicular vibrating here, $\vec{k}$ the propagation distance should be propagation vector should be this. Not only that, so I should have also the magnitude of $\vec{B}$, so if I find the $|\vec{B}|$ it should be simply $\frac{|\vec{k}|}{\omega} |\vec{E}|$.

Now $\frac{|\vec{k}|}{\omega}$ is simply $\frac{1}{c}$. So, the $|\vec{B}|$ is whatever the magnitude you have for electric field divided by c. So, you can see that in electromagnetic wave the $|\vec{B}|$ is very, very small compared to $|\vec{E}|$.

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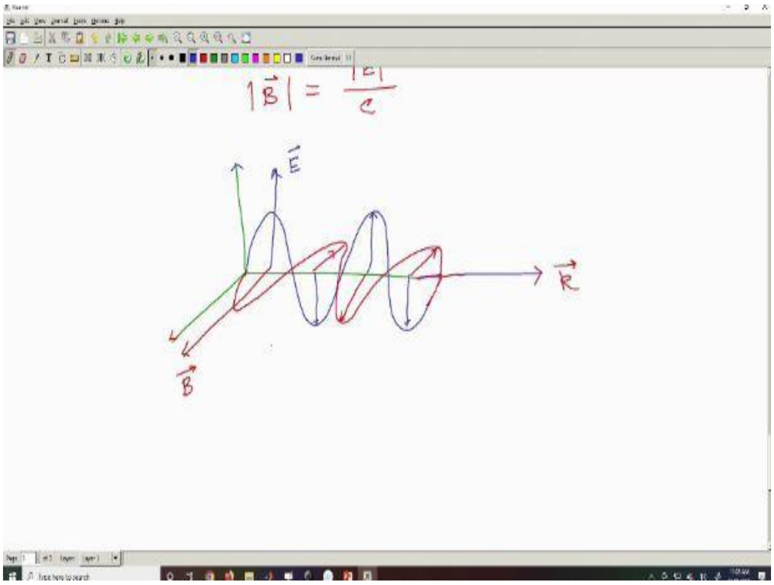

Now so if I plot quickly. So, how these things, so, in this 3 dimensional plot, this is a plot you always find that how the electric field and magnetic field propagates. So, there is a variation of the electric field like this. There should be a variation of the magnetic field like, this in perpendicular plane. So, this is the electric field, which is vibrating along this. And this is the magnetic field that is vibrating perpendicular to that.

And they are propagating along this direction. So, this is in opposite direction here. So, this is in this direction. And they are propagating along the direction of $\vec{k}$, which is $\vec{z}$ in this case. So, this is the way the electromagnetic wave is propagating. And this figure is let me erase this part because, let us confined up to this because, my drawing is not good here. So, $\vec{E}$ is in one plane, $\vec{B}$ is in the perpendicular plane and $\vec{k}$ is perpendicular to both $\vec{E}$ and $\vec{B}$.

And this is the way the electromagnetic wave should propagate. So, this is the electric field vector, this is the magnetic field vector, which is polarized along this perpendicular direction. This is the magnetic field vector in this plane, the electric field vector perpendicular. So, this is the way it should propagate roughly this is the figure for the electromagnetic wave that is propagating.

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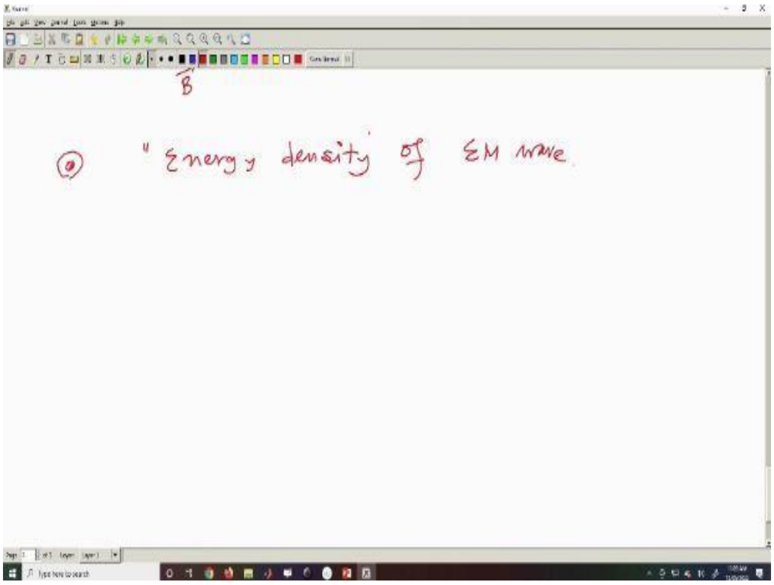

Now, next we discuss about the energy density. So, when the electromagnetic wave is propagating, we should have some energy density. So, energy density of EM wave, electromagnetic wave.

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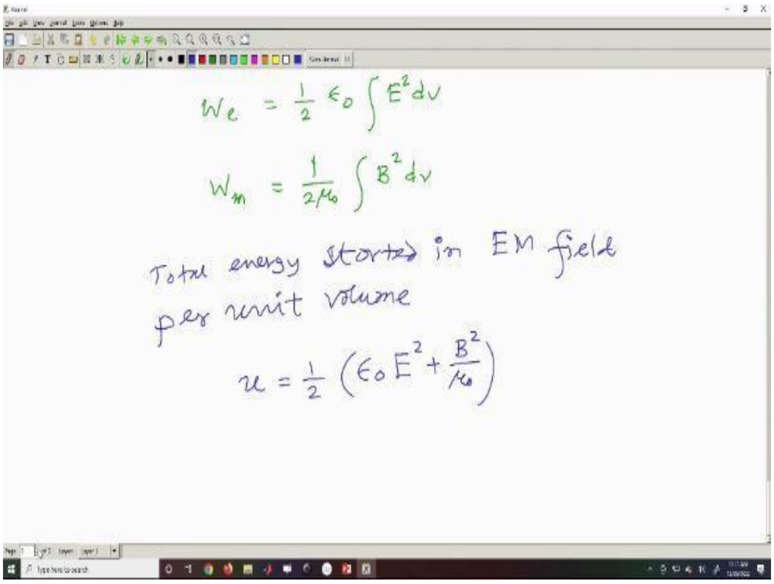

So, already we have the energy of and from electrostatic and magnetostatic. The energy of electrostatic field is simply $\frac{1}{2}$ ϵ_0 $\int E^2\,dv$ we calculated that. And also the energy for magnetic field is $\frac{1}{2}$ μ_0 $\int B^2\,dv$. So, the total energy stored. So, this is the total energy. So, total energy stored in EM electromagnetic field per unit volume, which is energy density, that is simply u = $\frac{1}{2}$ (ϵ_0 E^2 + $\frac{B^2}{\mu_0}$).

This is the amount of energy density that is stored there. Now let us quickly calculate a very important quantity in respect to that and that is the pointing vector.

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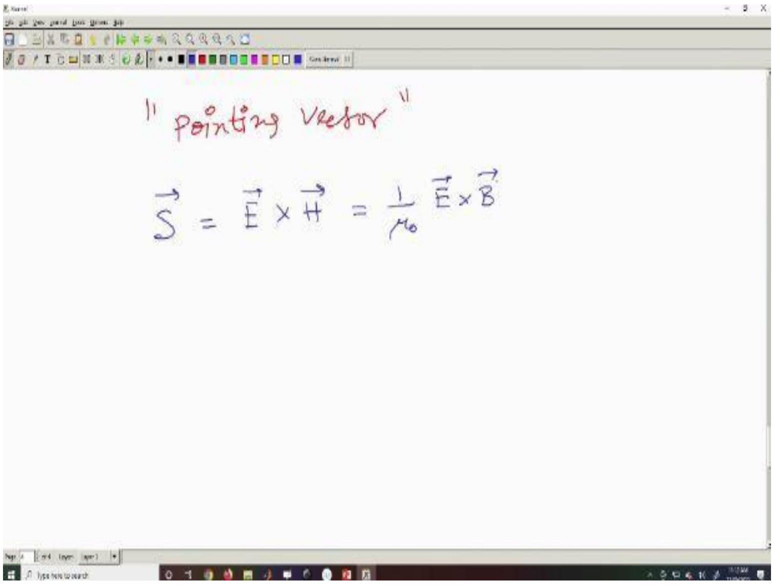

So, by definition the pointing vector. So, let me first define, then I am going to explain it is defined at $\vec{S}$ by definition it is $\vec{E} \times \vec{H}$. And if I write in terms of $\vec{E}$ and $\vec{B}$ it should be $\frac{1}{\mu_0} \vec{E} \times \vec{B}$. So, what is this $\vec{E} \times \vec{B}$?

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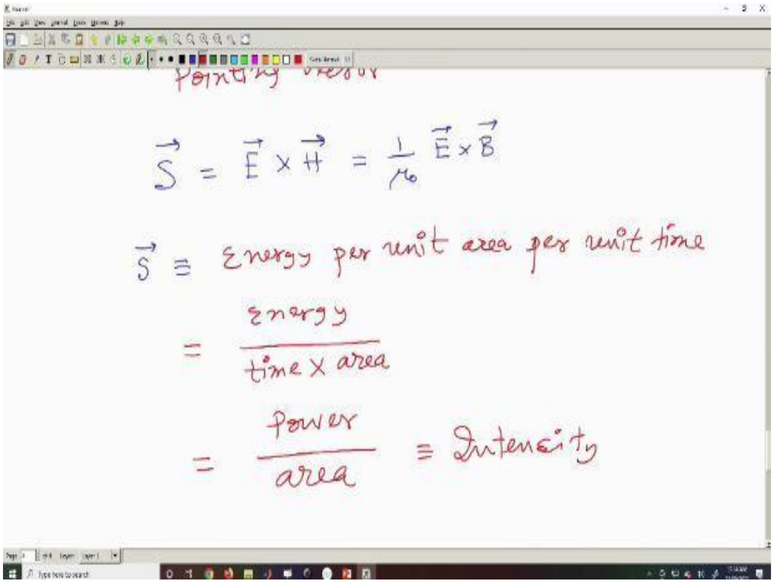

So, $\vec{S}$ is simply equivalent to energy per unit area per unit time. So, eventually it is $\frac{Energy}{time \times area}$. And that means $\frac{Power}{area}$ and that is eventually the intensity. So, it basically measures the intensity of the electromagnetic wave that is propagating. So, now we are going to calculate it is value, that if we know the electric field, then what should be the value. So, it is very straightforward calculation.

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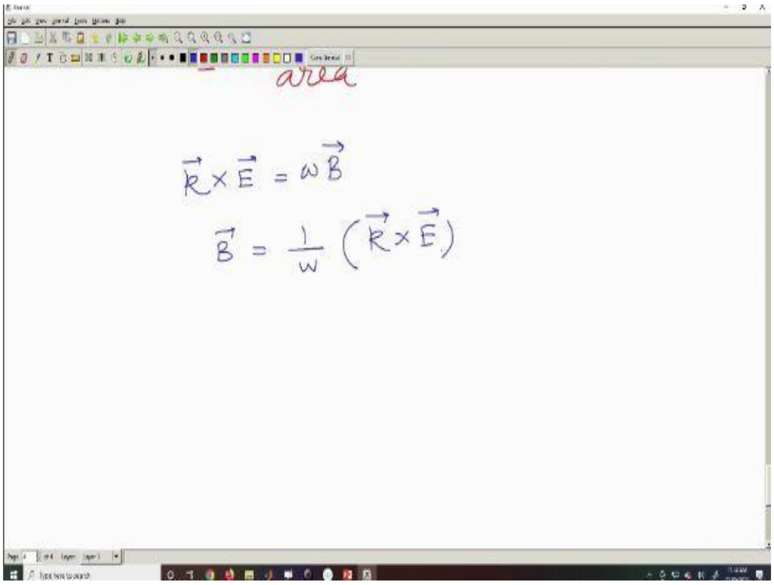

So, we already have this expression $\vec{k} \times \vec{E} = \omega\vec{B}$, this expression we just find. So, $\vec{B}$ is $\frac{1}{\omega}$ and then $\vec{k} \times \vec{E}$. Now if I put this value of $\vec{B}$ in the expression of the $\vec{S}$, then let us see what we get.

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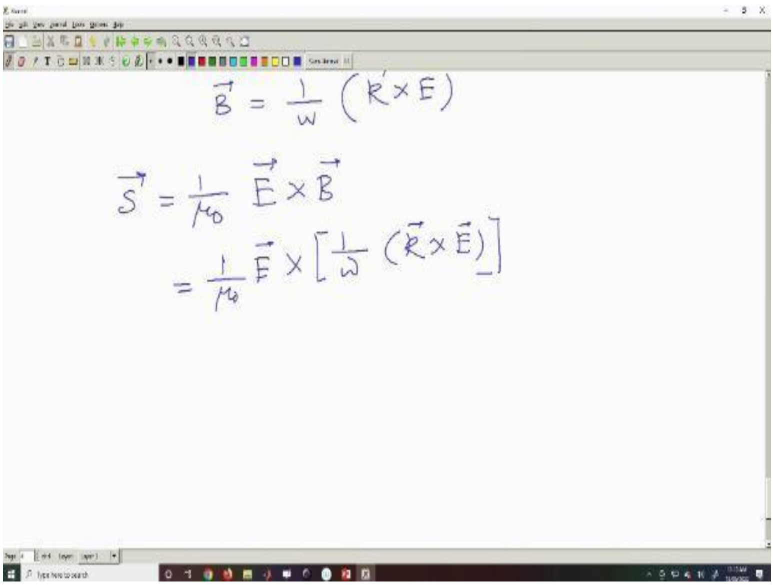

If $\vec{S}$ is $\frac{1}{\mu_0}$ $\vec{E} \times \vec{B}$ that we know. And in place of $\vec{B}$ now I am going to put this. So, $\frac{1}{\mu_0}$ $\vec{E}$ × in place of $\vec{B}$ I just put $\frac{1}{\omega}(\vec{k} \times \vec{E})$.

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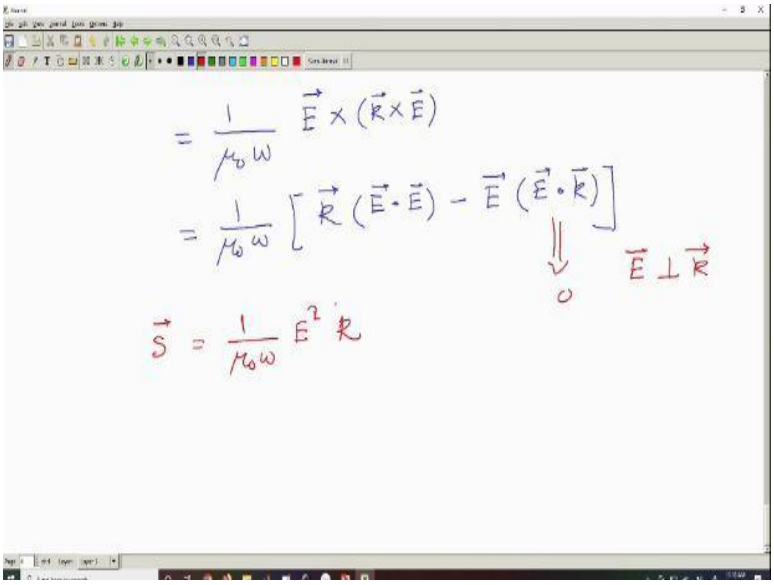

So, I should have $\frac{1}{\mu_0 \omega}$ and then we have $\vec{E} \times (\vec{k} \times \vec{E})$. And we know that what is $\vec{A} \times \vec{B} \times$ it is like a $\vec{A} \times \vec{B} \times \vec{C}$. So, I just expand this portion $\vec{E} \times (\vec{k} \times \vec{E})$ and it should give like $\vec{k}$ it is $\vec{B}\ \vec{C}\ \vec{A}$ – $\vec{C}\ \vec{A}\ \vec{B}$ this is the formula. So, this $\vec{B}\ \vec{C}\ \vec{A}$, which you should be $\vec{E} \bullet \vec{E}$ – $\vec{C}\ \vec{A}\ \vec{B}$ that means, $\vec{C}$ is $\vec{E}$ and $\vec{A}$, $\vec{B}$ is $\vec{E} \bullet \vec{k}$. Now we know that $\vec{E}$ and $\vec{k}$ are perpendicular.

So, this term should be 0 because, $\vec{E}$ is perpendicular to $\vec{k}$, that we already proved. So, my expression eventually is $\vec{S} = \frac{1}{\mu_0 \omega}$ and then E^2 and then $\vec{k}$. So, one thing that we find here that the pointing vector should be in the direction of the propagation that is in the $\vec{k}$ in free space.

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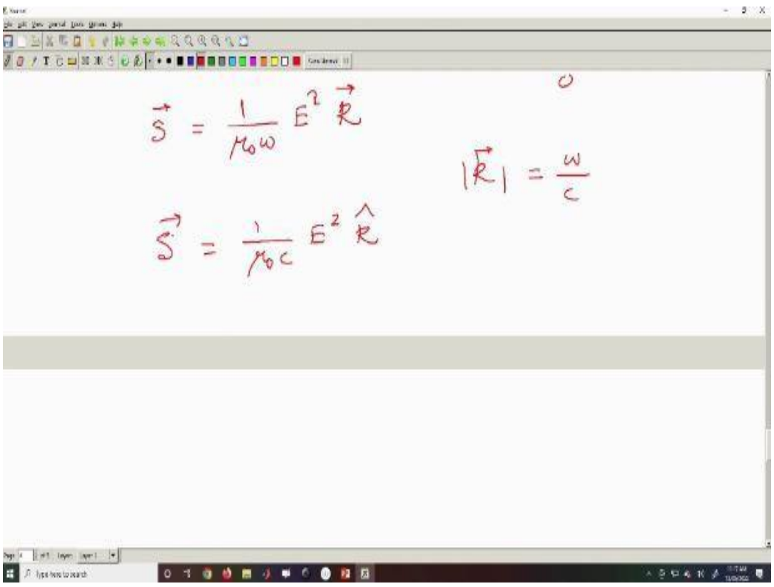

So, $\vec{k}$ if I want to find out the magnitude it should be $\frac{\omega}{c}$. So, I can simply write it at $\frac{1}{\omega}$ will be going to cancel out when I put the $|\vec{k}|$ μ_0 c E^2 and then $\hat{k}$.

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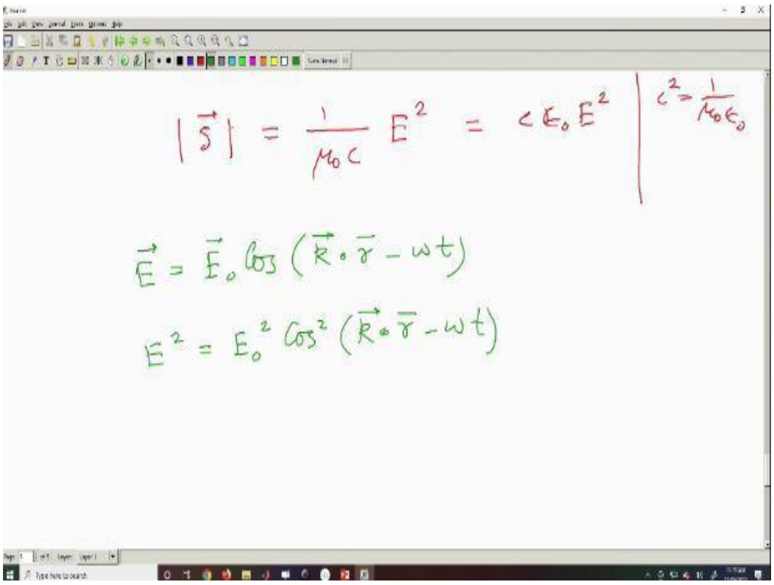

So, the $|\vec{S}|$ the pointing vector magnitude if I want to find out what is the intensity only the $|\vec{k}|$ shows that how the energy is flowing. What is the direction of the flow of the energy and that is in $\vec{k}$. That is why the $\vec{k}$ comes here but, when you calculate the intensity it should be the magnitude and you have $\frac{1}{\mu_0 c}$ and then simply E^2. Also it can be written in terms of say c ϵ_0 E^2 but, just replacing μ_0 in terms of ϵ_0 and c**.**

This is this is another way to now. The point is because, c^2 I should note it here, c^2 is $\frac{1}{\mu_0\epsilon_0}$. Now my $\vec{E}$ is $\vec{E}_0$ cos ($\vec{k} \cdot \vec{r} - \omega t$). So, E^2 is a $\vec{E}^2$ is simply E_0^2 and then $\cos^2$ ($\vec{k} \cdot \vec{r} - \omega t$). So, now it varies with respect to time in this cos square form. So, if I when I put this **so** then it will it will vary with respect to time in this form cos square form. So, I need to take the average, we know that when there is a time varying thing.

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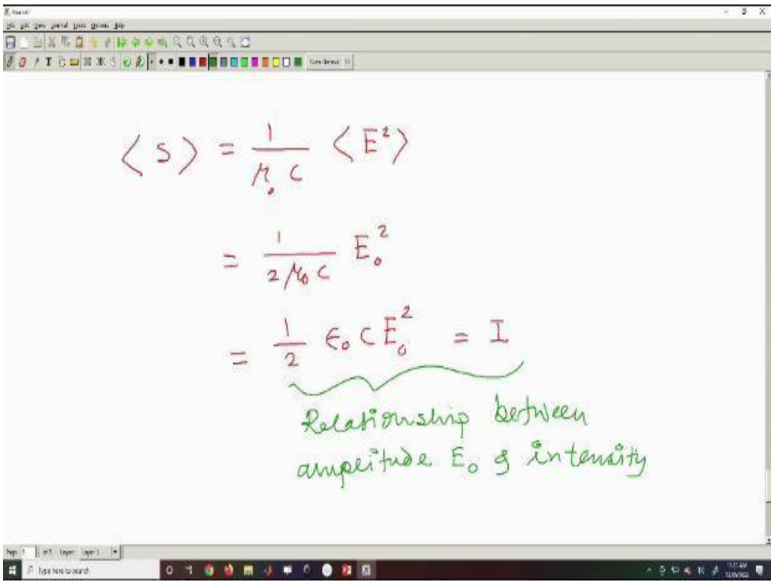

So, normally we take the average, so, this average if I take. Then it should be simply $\frac{1}{\mu_0 c}$ and then the $\langle E^2 \rangle$, which is a function of time. Now the time average if I take for this cos square function we know that we get half. So, the value should be $\frac{1}{2\mu_0 c}$ and it should be simply E^2 or in other way it is $\frac{1}{2} \epsilon_0 c$ and then E_0^2 that is basically the measurement of the intensity I.

So, this is the relationship between the intensity and the amplitude of the electric field. So, this is the relationship you should note that. This is the relationship between amplitude E_0 and intensity. And the direction of the $\vec{E}$ is along $\vec{k}$. So, that is the energy is propagating.

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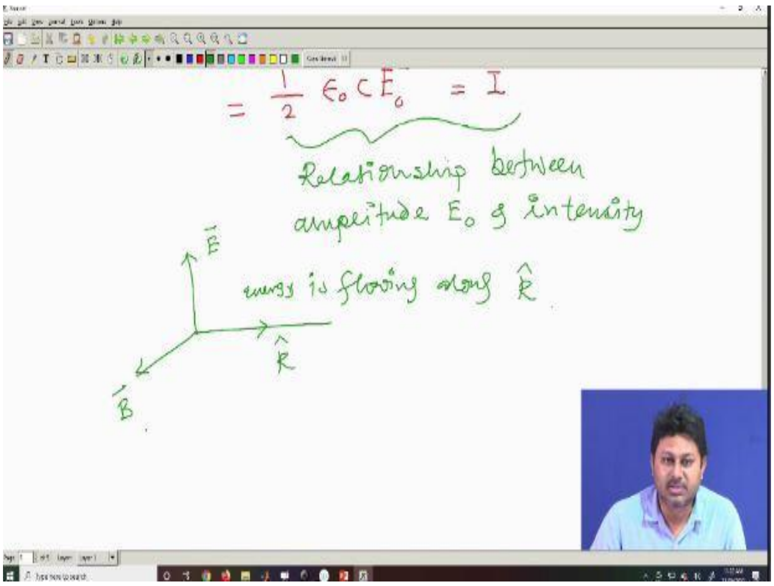

So, if this is the way we are having, so suppose the electric field is in this direction magnetic field is along this direction in electromagnetic wave but the energy that is flowing should be in the direction of $\hat{k}$. So, energy is flowing in this direction. So, that is why the $\hat{k}$ is there. So, energy is flowing along $\hat{k}$**.** So, with this note I would like to conclude, because I do not have much time to cover the next topic.

So, in the next class maybe I like to discuss something about the boundary condition of the magnetic field and then the boundary condition of the electromagnetic wave we will be going to discuss. If there is interface how the boundary condition will be there. And then after that maybe we will be going to discuss about the Maxwell's equations, wave equation in a material.

Now we are dealing with the free space but then we are going to discuss about what happened to a material. So, with that note let me conclude today then. Thank you very much for your attention and see you in the next class.