Foundations of Classical Electrodynamics
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Lecture-69
Maxwell's Equation in Matter

Hello student to the foundation of classical electrodynamics course. Under module 4 today we have lecture number 69, where we are going to discuss the Maxwell's equations in matter.

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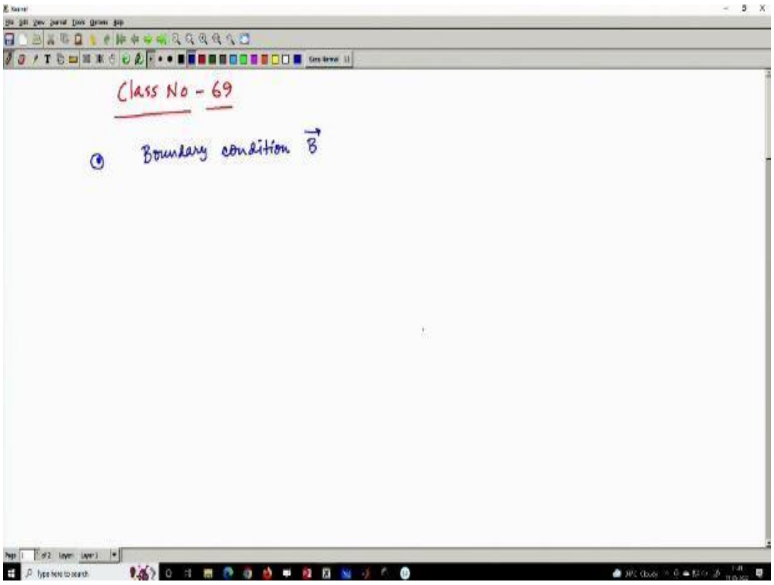

Today we have class number 69 and mainly today we are going to discuss the Maxwell's equation in the matter, but before that let me cover one important thing that we missed actually. So, that is the boundary condition of the magnetic field $\vec{B}$. There is interface how the magnetic for boundary condition of the magnetic field appears. So, that we are going to discuss quickly because, that we will use this in understanding the Maxwell's wave equation in matter later.

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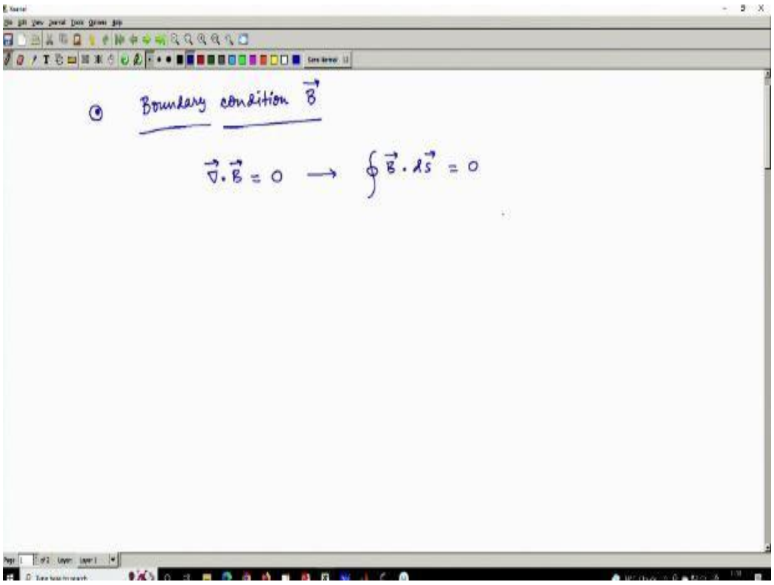

So, we know the $\vec{\nabla} \cdot \vec{B}$ is 0, which gives us that the closed surface integral of $\vec{B}$ that is the flux should be 0 that is the second equation Maxwell's equation. So, exploiting that expression we can figure out, what is the boundary condition of $\vec{B}$?

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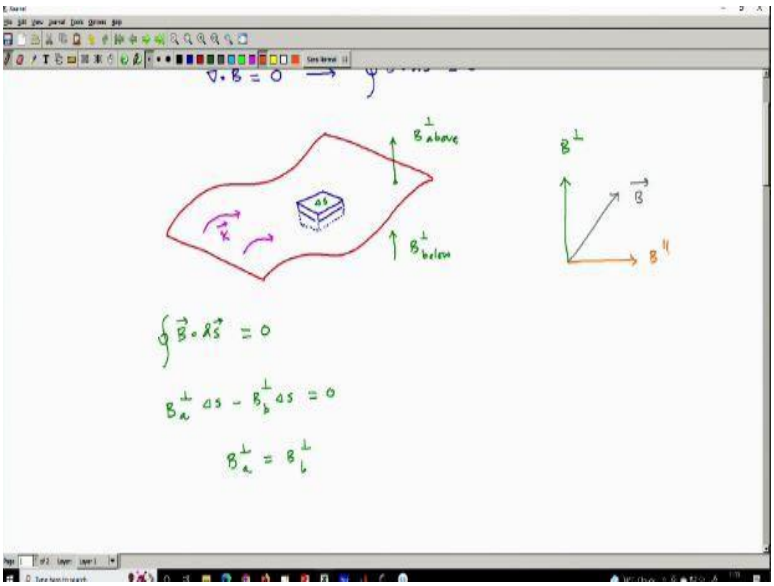

Suppose, we have a surface boundary like this, suppose this is a surface and below here we have $\vec{B}$ say perpendicular component and this is below. And over that also we have $\vec{B}$ perpendicular but this is above. So, 2 sections are there below and above. And the magnetic field the perpendicular component of the magnetic field I write $\vec{B}$ below and $\vec{B}$ above.

And the surface current that is flowing here corresponds to $\vec{k}$ and that is flowing over the surface. So, we have the surface current that is flowing. Now if I want to calculate these integral that is the closed surface integral like the Gaussian surface we need to draw a pill box. If I draw that here. And then this is the small area, we are drawing. And below that also so this is basically having a cross section.

So, this is the small surface we are having. And now the condition we are going to put here is this, closed surface integral = 0. And we know the perpendicular component of $\vec{B}_b$ and $\vec{B}_a$ and if I do that, so if I execute that considering the fact that the surface of these things is say Δs. Then this quantity has to be $\vec{B}_a^{\perp}$ I just write $\vec{B}_a$ here. Then Δs and then - $\vec{B}_b^{\perp}$ and Δs = 0.

Because other component will not be there in the surface because, when I take the $\vec{B}^{\perp}$ component then I am just taking this one along over this and the surface is the same direction. So, that is why $\vec{B}_a^{\perp} \cdot d\vec{s}$ and the negative sign is because the surface and $\vec{B}$ is opposite direction. However we have also the surface here in this side and this side.

And in this side surfaces I can make this length tends to 0, so, that this component will no longer alive. So, only the meaningful component we have is this perpendicular component and they will follow this equation. And that means we have a condition that $\vec{B}_a^{\perp}$ is equal to $\vec{B}_b^{\perp}$. That means the perpendicular component of the $\vec{B}$ is continuous.

That is the outcome that we had here. Also if I want to calculate the parallel component because, this is the condition of the perpendicular component but for any arbitrary direction $\vec{B}$, if I have an arbitrary direction here. Say this is the direction of $\vec{B}$ suppose it should have a perpendicular component that I already mentioned along this direction it should be the perpendicular component.

And also one can expect or one can have a parallel component here which we write $\vec{B}^{\parallel}$. So, this is the way the components are distributed. So, the parallel component is still there. And I can use another equation, which is the equation for the amperes law essentially.

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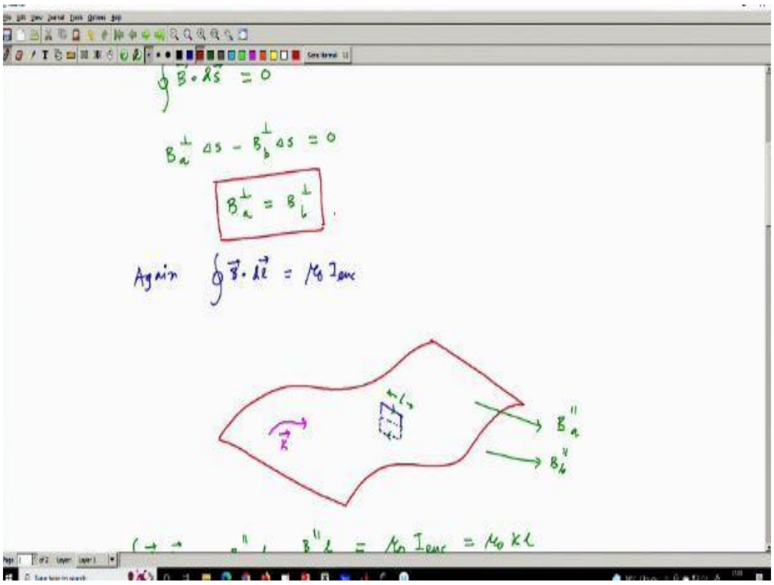

And we had this again we have $\oint \vec{B} \bullet d\vec{l}$ that quantity is μ_0 and then I_{enc}. This is the amount of current I_{enc} I do not put the c. So, this is the amount of current that is enclosed by this closed loop so that is. So, let us now consider what is I mean how to make this closed loop. Again consider the same surface, the way we consider. So, suppose this is the surface same surface I am considering this is the surface and over this surface we have now my loop.

And the loop is from here to here and then I have here and have here. And then I go the downwards the downwards, which is below the loop below the surface. So, that is given by this dotted line**.** So, this is the way the loop is defined. Now the current flow the length of this loop is suppose from here to here is say l and this is the way the loop is there. And the surface current is flowing, so, surface current should play a role here the way we draw the surface current.

So, I just draw it like this. Now the $\vec{B}^{||}$ component like the way we draw here the $\vec{B}^{||}$ component should be like this. So, this is the $\vec{B}^{||}$ component and $\vec{B}^{||}$ this is above and below also we have a $\vec{B}^{||}$ component like $\vec{B}$.

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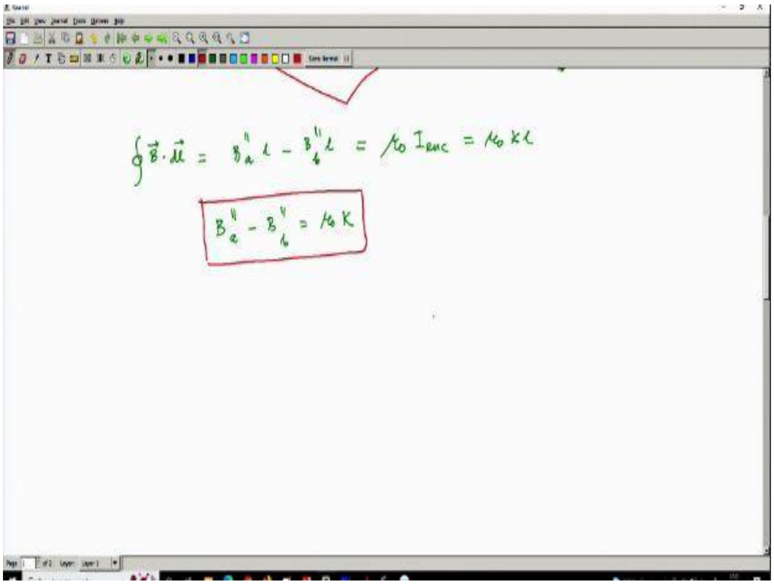

Now I integrate this closed line integration and that integration means I need to integrate over this closed loop whatever the loop I draw here, amperian loop. So, if I draw you can see that I have $\vec{B}_a^{\parallel}$ l - $\vec{B}_b^{\parallel}$ l and that should be equal to other component will no longer be there because the direction of the loop and the direction of the $\vec{B}$ is perpendicular to each other.

So, they will not going to interfere. Here also the similar thing happened actually. So, here $\vec{S}$, the direction $\vec{S}$ and the direction $\vec{B}$ they are perpendicular. So, these components are not there. Here we have the opposite case; the perpendicular component is no longer there because this is perpendicular to the length itself. But, here only the parallel component will survive and this will be the condition the negative sign again because of the opposite direction.

The direction is opposite for the loop in this region. So, now this quantity is equal to μ_0 I_{enc}. And this μ_0 I_{enc} I should write simply μ_0 the surface current density multiplied by the l that is all. Then we simply have $\vec{B}_a^{\parallel}$ l - $\vec{B}_b^{\parallel}$ = μ_0 and then k. So, here we can see that in the previous case, we had the perpendicular component they are conserved for magnetic field.

But, here we see the parallel component is not conserved. It depends on thus if there is a surface current there, then it depends on the surface current k. If the surface current is 0 however, you can have the parallel component to be continuous again. So, what is the fate of the vector potential that we also calculate here?

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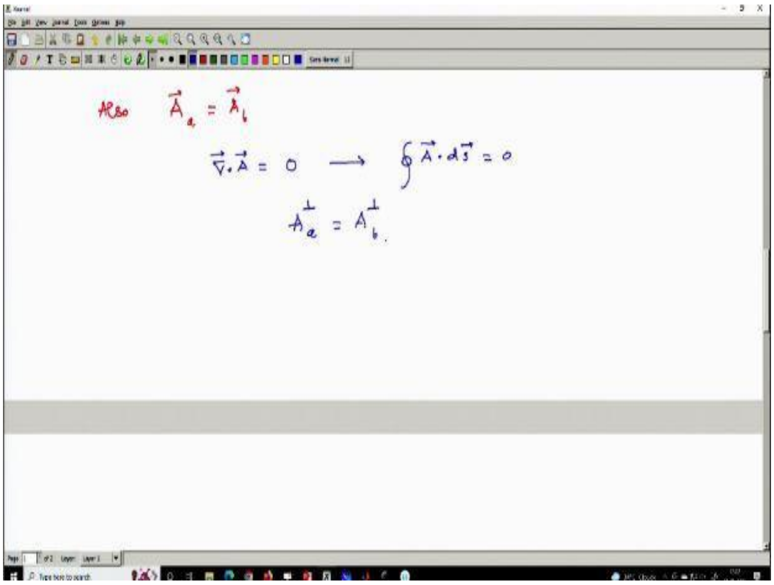

So, also what we have here is $\vec{A}_a$ and $\vec{A}_b$ these are continuous the vector potential seems to be continuous. A 0, we can do that, this vector potential is continuous whatever the vector potential we have in the below that is the same vector potential we have in the above part of this interface. And that we can show by that means, the parallel component and perpendicular component both are continuous.

So, we can use this expression the $\vec{\nabla} \bullet \vec{A} = 0$ that is the condition we have for magnetostatic this is the condition, coulomb gauge condition is there, that is the condition that need to be satisfied by $\vec{A}$. And then that leads to the equation $\oint \vec{A} \bullet d\vec{s} = 0$. And similarly the way we calculate for $\vec{B}$, we can see that the $\vec{A}_a^{\perp}$ is equal to $\vec{A}_b^{\perp}$ same way.

So, like the magnetic field the vector potential is also having the same relationship with the perpendicular component of above and below. They are continuous. Well, what about the other component that you can also figure out that. Let us do in the next page.

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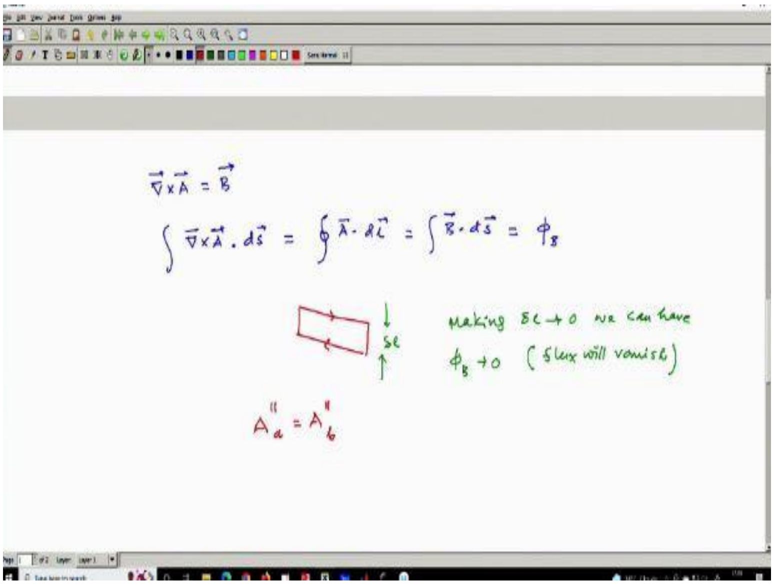

So, we know that $\vec{\nabla} \times \vec{A} = \vec{B}$ that is the relation we are having. Now if I try to find out that $\vec{\nabla} \times \vec{A} \bullet d\vec{s}$ suppose I am making a surface integral. So, that quantity is simply $\oint \vec{A} \bullet d\vec{l}$ and that is nothing but $\int \vec{B} \bullet d\vec{s}$ and that is the magnetic flux. We had this equation several time we know that. So, this magnetic flux calculation, so we can do so. So, here if I make a loop like this.

So, this is the loop and in this loop if I calculate. So, and this length suppose δl. So, by making delta tends to 0 we can make this ϕ be vanished. And in that case, what we get is simply so, what happened? That if I make making δl tends to 0 because this is my choice I can make δl according to my choice. So, we can have ϕ_B tends to 0 that means, the flux is almost 0 or the flux will simply the flux will vanish because the surface area I can make 0.

So, that basically gives me an expression like $\vec{A}_a^{\parallel} = \vec{A}_b^{\parallel}$. Exploiting the same technique that we used here. Only difference is that I can make this δl tends to 0 to make this surface area 0. So, flux can be 0 and then we had simply this equation. So, **so** this is the 2 conditions we get and perpendicular and parallel component. If both are conserved then we can say that the vector potential $\vec{A}$ is conserved in this interface, in this boundary.

If I have an interface between two, this is a boundary between the two region. Then the magnetic field in the lower and magnetic field in the upper should have the relationship like this. But the vector potential that is producing the magnetic field is continuous. So, that information may be we can use in our future calculations**.**

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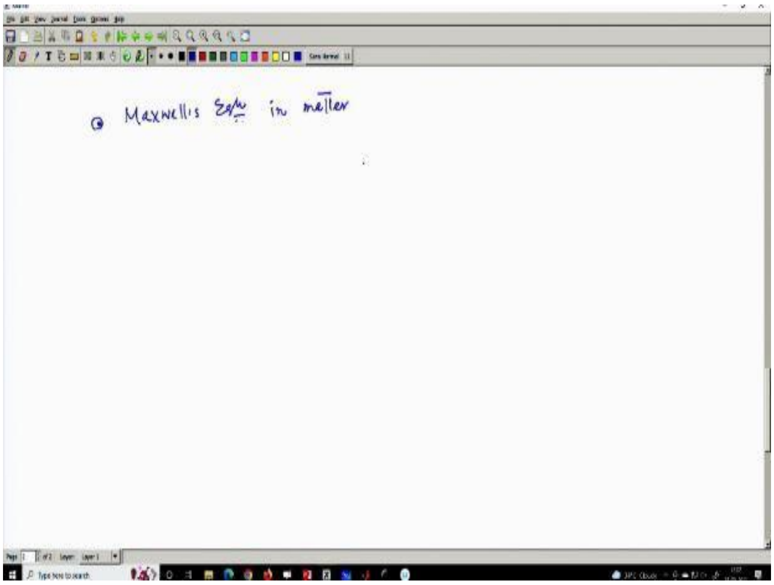

Now let us after having that let us now go to our main topic Maxwell's equation in matter. So, in Maxwell's equation in matter so that means, previously whatever the Maxwell's equation we discuss is mainly in the free space and then calculate the wave equation in this free space. But now we are going to deal with the Maxwell's equation in matter.

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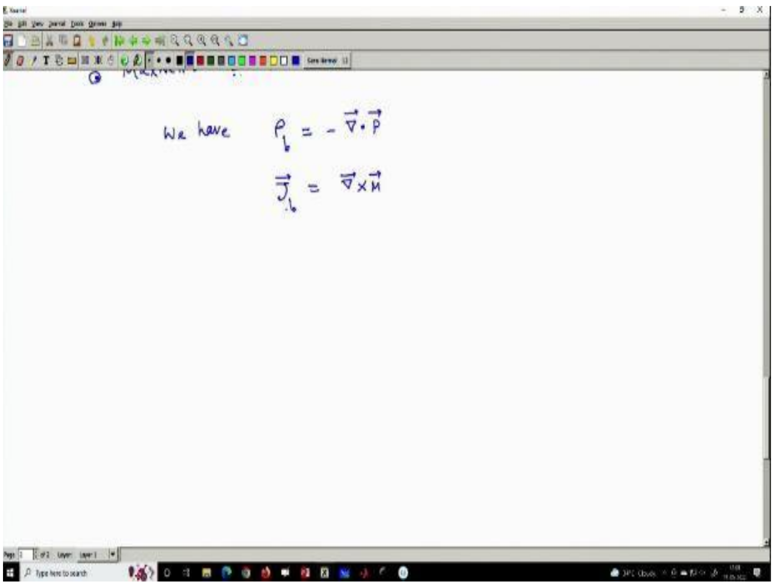

In matter we have 2 special things. So, we have whenever we consider the matter we have the bound current, bound surface charge density, which is associated with the polarization of the material. If it is a dipole, if it is dielectric then it is related to this polarization $\vec{P}$, which is a dipole moment per unit volume and that is the relation. Also we have a bound current here. If the system is magnetic and it is related to the magnetization and we have $\vec{\nabla} \times \vec{M} = \vec{J}_b$. So, these two additional components should be there in the Maxwell's equation.

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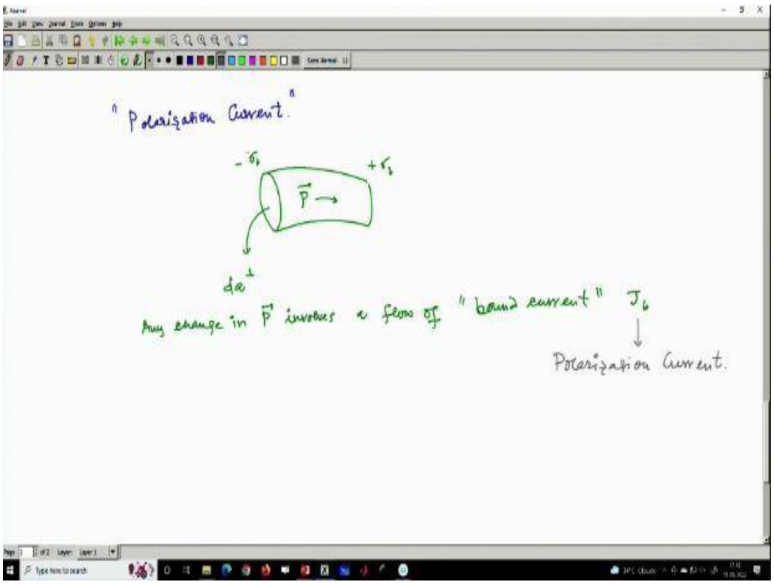

So, polarization current let me try to understand make you understand that polarization current. So, polarizations current suppose we have a rod like this. And this is a polarization and this is the area, small area say perpendicular direction to the polarization. And there is a difference between the two ends there is a difference between the bound surface charge density. So, one is negative and another is a positive.

So, one is this side and another is that side. So, there is a difference. So, the current flow ideally so let me write it first. So, any change in the electric polarization that means any change in vector $\vec{P}$ involves a flow of bound current $\vec{J}_b$, which we called this $\vec{J}_b$, actually we called the polarization current. So, here we are having a change over $\vec{P}$ over the distance and that is if I have a surface bound charge density minus and plus there is a difference. Then it leads to the flow of current.

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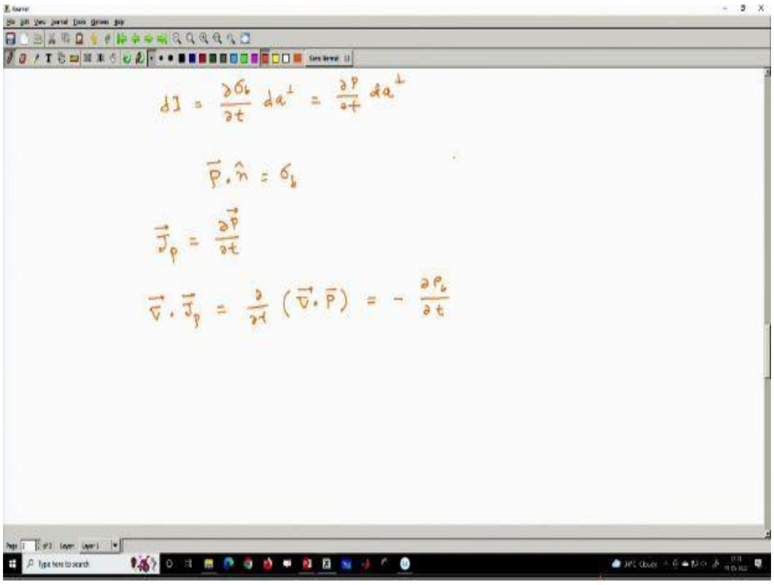

So, I can calculate this flow of current and that is dI is the amount of current that is flowing that is the change rate change of this bound surface charge density multiplied by the area, which is perpendicular. And that eventually leads to $\frac{\partial P}{\partial t}$ and then da perpendicular because, P the polarization P is σ. So, that we know that polarization $\vec{P}$ is polarization dot $\hat{n}$ is σ.

So, that is the relation we had, I do not need to write it several time. So, that expression we use here to find out that thing. So, let me write it. So, $\vec{P}$ what I am trying to say is, this $\vec{P} \bullet \hat{n}$ is our σ_b that we know. So, the bound current density that we are having here, which is the polarization current that we are having $\vec{J}_P$ that is equal to the rate of change of the polarization.

This is the time rate of change of the polarization. And if I make a divergence both the side, $\vec{\nabla} \bullet \vec{J}_P$ then I can have $\frac{\partial}{\partial t}$ and the $\vec{\nabla} \bullet \vec{P}$. And this $\vec{\nabla} \bullet \vec{P}$ is the ρ_b that is the bound charge density. So, this rate of change of bound charge density can leads to the $\vec{\nabla} \bullet \vec{J}_P$ that is the divergence of the polarization current. That is generated due to the change of time change of P. And that is due to the flow of bound current that leads to basically the flow of bound current.

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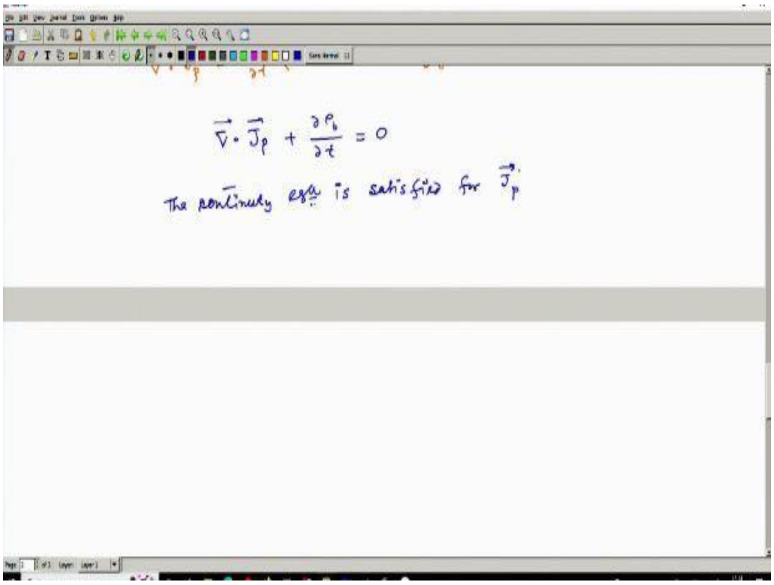

Now it follows that this equation is simply follows the continuity equation. That if I write it in a standard way. So, $\vec{\nabla} \cdot \vec{J}_P + \frac{\partial \rho_b}{\partial t} = 0$. So, the continuity equation is satisfied for the polarization current $\vec{J}_P$, the continuity equation is satisfied for that.

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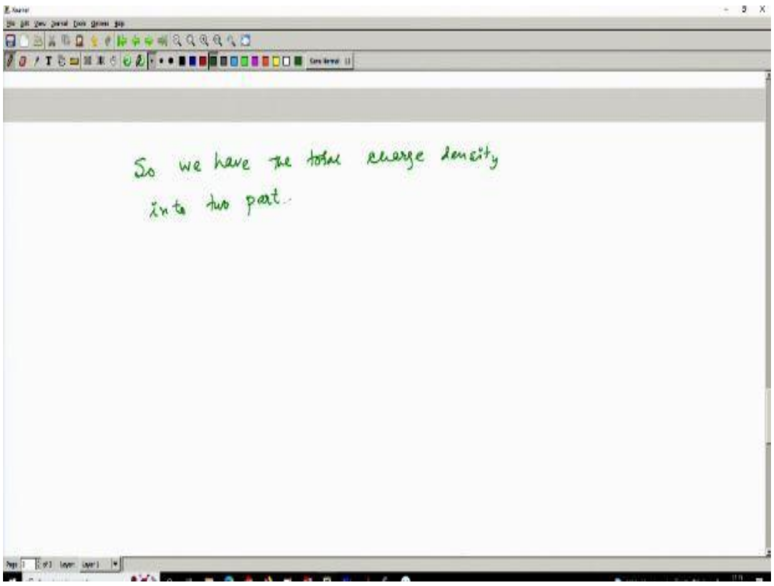

So, we finally what we have after doing that, so, we have the total charge density into two parts.

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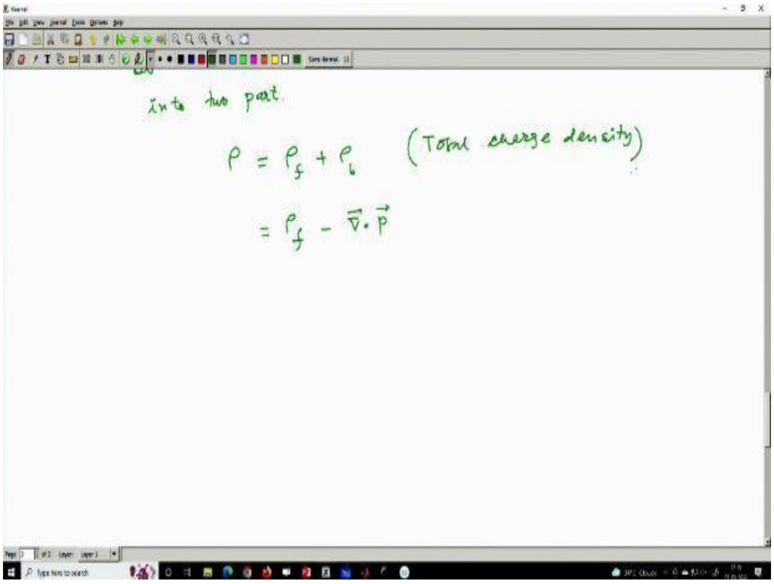

So, total charge density ρ can be divided into two parts, one is the free charge density that is the thing we always consider. But, for polarized system we also have the bound charge density ρ_b. So, this condition is simply $\rho_f - \vec{\nabla} \cdot \vec{P}$ that is my ρ. Also the total current this is for total charge**.** This is the total charge density.

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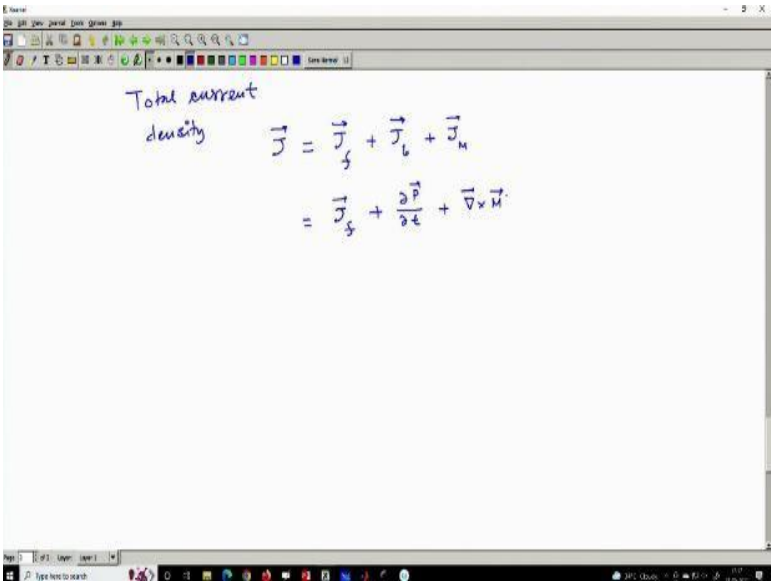

What about the total current density? The total current density $\vec{J}$ is now having 3 components; one is the free current density that we have. Apart from that we have a bound current density that we derived here that is due to the rate change of the polarization. And another we already had in magnetic material and that is for this magnetization we called it $\vec{J}_M$. So, it is simply $\vec{J}_f$ plus the rate of change of polarization.

If there is a time dependent variation of the polarization then we have $\vec{J}_b$ and if we have a space variation of the magnetization. Then we have $\vec{J}_M$ and if I make a curl of that quantity. Then you are going to have this current density $\vec{J}_M$.

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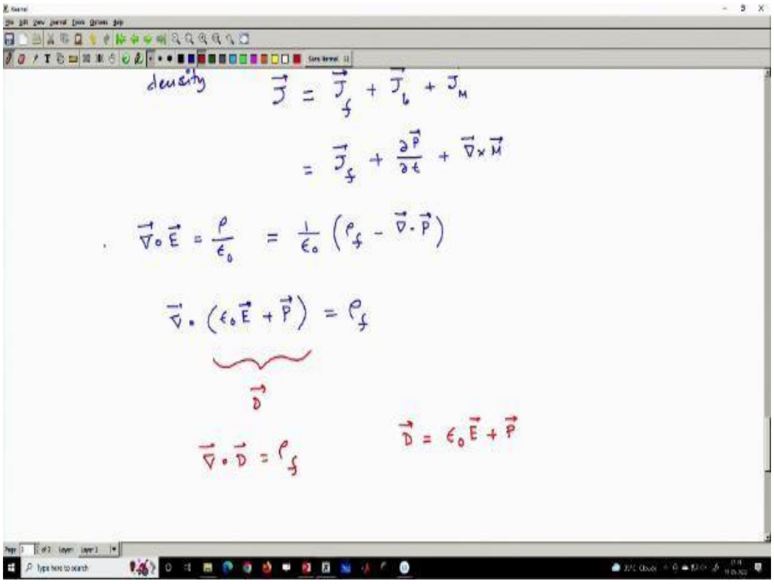

So, I have the equation now $\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\epsilon_0}$ that was our equation. But, this equation I am going to modify here because, ρ I need to write in terms of ρ_f and ρ_b, which I write minus of divergence of the polarization. And this equation now become ϵ_0 $\vec{E}$ considering this is and plus $\vec{P}$. I just put this side, under this divergence operator and we have ρ_f.

This quantity we know that we define this quantity as our $\vec{D}$ displacement vectors. Eventually we have $\vec{D} = \rho_f$, where my $\vec{D}$ is considered to be ϵ_0 $\vec{E} + \vec{P}$. This is the way we define the ρ_f. So, also the other equation because, these two equations we need to consider where we have the source term. So, this is the equation 1 where, we had the source term ρ. And equation 4 also is equation where we had the source term.

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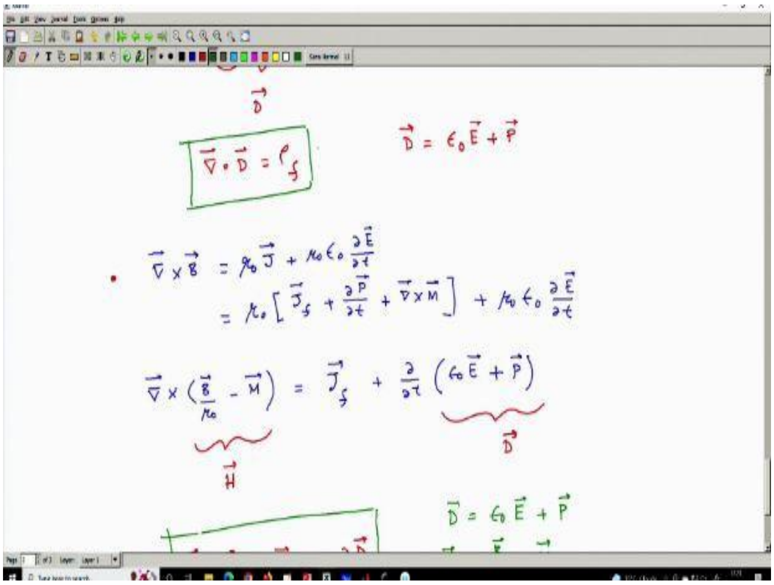

So, in principle we need to modify that well and that equation is $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, that is the Amperes law. And now we will modify that $\mu_0 \vec{J}$ can be divided into 3 parts $\vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M}$ that is the 3 part we had. And also I had let me add this because, this additional term is there in Maxwell's equation and that is the modification by Maxwell's himself and that is this term $\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$.

Let us add this term as well for the completeness. Now I rearrange, so, I have $\vec{\nabla} \times \vec{B}$ I just divide everything with μ_0. So, μ_0 and then I put this curl this side. So, I am having $-\vec{M}$ that quantity is equal to $\vec{J}_f$ this is simply the free current density plus I have $\frac{\partial}{\partial t}$ of $\epsilon_0 \vec{E}$ that is already there plus $\vec{P}$.

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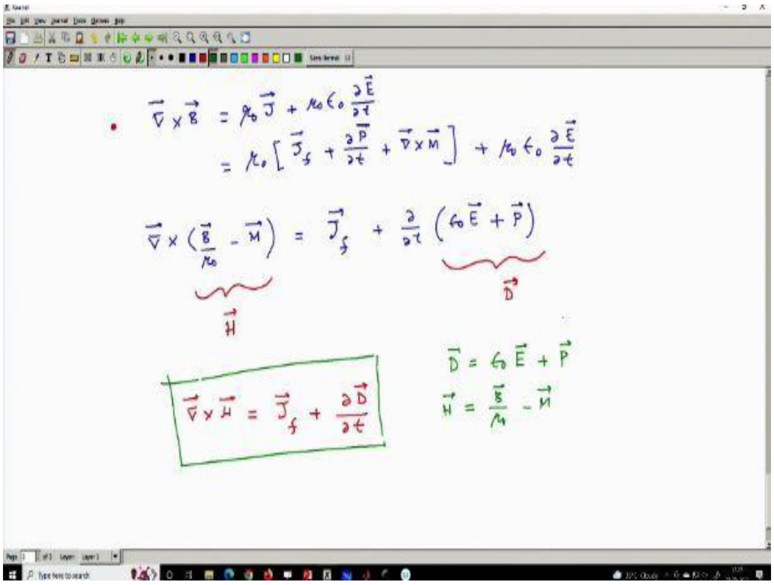

Again I have here this quantity, which is nothing but $\vec{D}$. So, eventually we can write this as $\vec{H}$ and we have our modified equation and that equation I am writing here is the $\vec{\nabla} \times \vec{H}$ is equal to $\vec{J}_f + \frac{\partial \vec{D}}{\partial t}$. So, this is the equation that one should follow. So, this is a modified version of the Maxwell's equation in material where the characteristics of the material can be entered through this value of $\vec{D}$, which is $\epsilon_0 \vec{E}$ plus the polarization $\vec{P}$.

This is the characteristics of the material. And $\vec{H}$ is $\frac{\vec{B}}{\mu_0}$ minus the magnetization. So, the polarization and magnetization are now involved, which is strictly the property of the material. And that property when you include in the Maxwell's equation, then the equation is modified. And these 2 equations, which is having the source term is only going to be modified because the contribution will be in the source term.

And we will have 2 new equations, other 2 equations will remain same that is the $\vec{\nabla} \cdot \vec{B} = 0$. And the $\vec{\nabla} \times \vec{E} = \frac{\partial \vec{D}}{\partial t}$. So, in today's class so what we did is we discussed the Maxwell's equation in a material. Before that, we also calculate the boundary condition of the magnetic field $\vec{B}$, which we did in the earlier classes. But anyway, so, that information will be going to use in the next part of the topic, when we discuss about the Maxwell's wave equation in matter. So, with that note I like to conclude today's class. See you in the next class and thank you for your attention.