Foundations of Classical Electrodynamics
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Lecture – 08
Differential Calculus, Gradient

Hello students to the foundation of classical electrodynamics course. So, today we will have lecture 8 where we are going to learn the differential calculus and start with gradient how what is the concept of gradient. Today we have class number 8.

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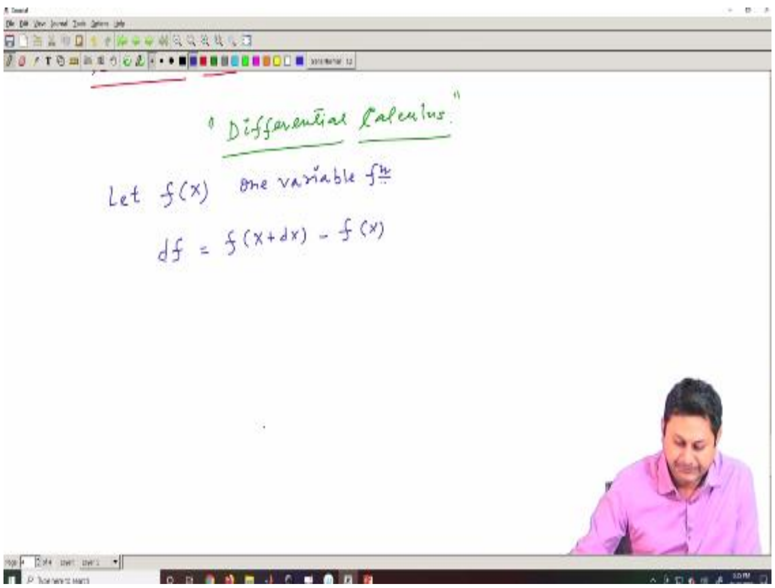

So, we will start a very new topic today very important and that is called the differential calculus. So, we know that if so, let f(x) is a function and this is a one variable function. Now, if I want to find out what is the deviation of the function when we have a slight change in x the dependent variable then this is the way we write small change dx is there over the function and then what is the change that is my df.

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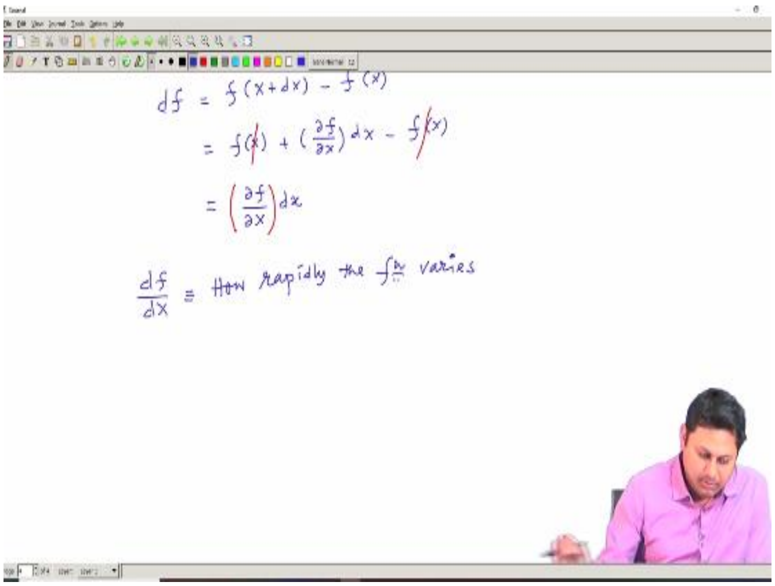

This is essentially f(x) if I make a Taylor series expansion of this first order. Then I have and eventually these things will cancel out I have this. So, what is $\frac{df}{dx}$ then ∂f this quantity inside the bracket is basically the rate at which the function changes. So, that means this if it is a single variable, I can remove this ∂ and I can simply write df. So, this eventually tells us how rapidly the function varies.

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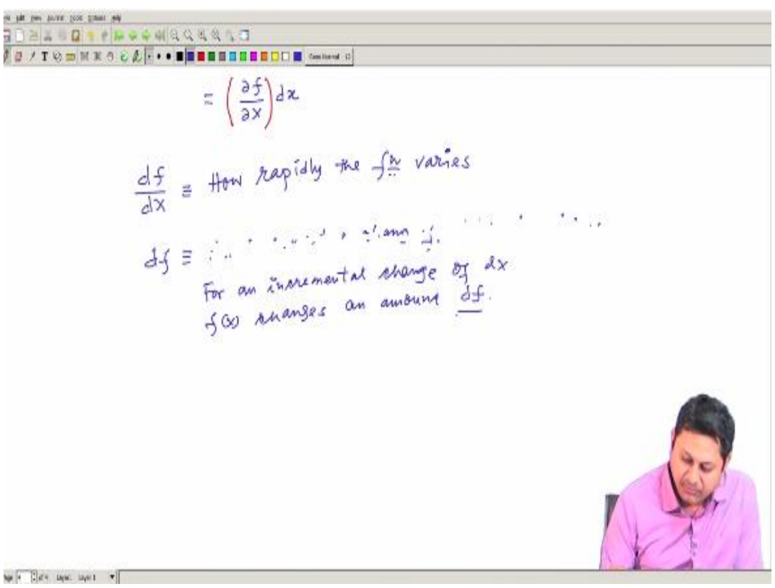

So, df is the, what is df as I mentioned an incremental change of f(x) and incremental change of f(x) due to the incremental change rather, I should say that simply that because I do not know whether it is incremental change or not. So, rather I should let me erase this and I rewrite this because the incremental I do not know whether the function will change incrementally because that is that the rate will be going to tell me whether this function will change incrementally or not.

So, I need to be careful enough to write this so rather I should say that, I am writing here I am not erasing this part properly. So, this is for an incremental change of dx because I am changing this dependent variable slightly. f(x) changes in an amount df, df is a change of the function due to the inclement change of dx that is all.

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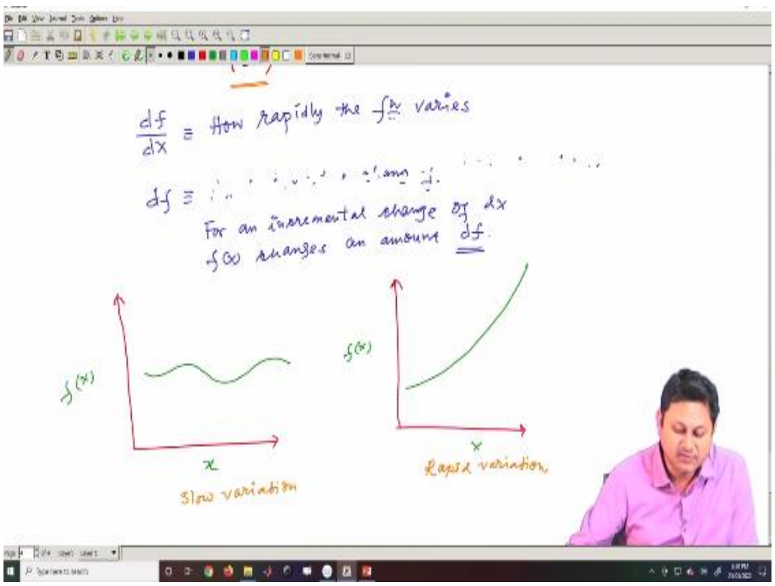

Well, different kinds of function we can have let us so, let us consider 2 different 2-dimensional functions, in one case, you know this say this is my f(x) and x so, suppose the function is changing like this and in other case, the function is changing like this. So, this is x so, in this case, this is a slowly varying, this function varies slowly so, these were I can say this function varies slowly so, the slow variation, but in this case, this function varies rapidly, so this is a rapid variation.

So, these are rapid variation. This is a slow variation, this is a rapid variation and that whether this is a slowly varying or rapid varying function that basically one can understand by calculating this term here, this rate of the change of the function can give us the idea whether the function is varying rapidly or slowly.

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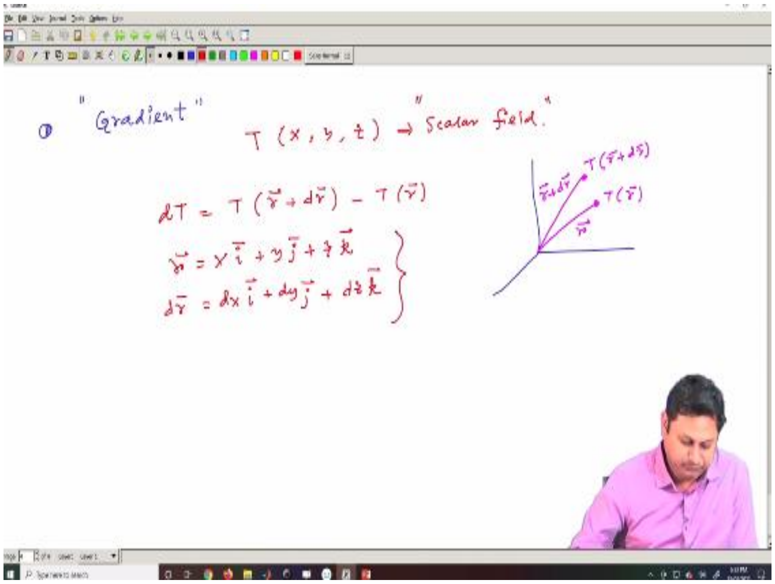

So, with this note we can now extend the concept and now, the concept of gradient appear here. So, now, we will understand what is the meaning of gradient. So, suppose this is a 1-dimensional, 2-dimensional function or 1 variable function we are discussing, but let us consider a function T(x) varying with x y z so, eventually this is a scalar function or a scalar field. This is a scalar field. Now, this scalar field is a function of x y z. So, if I change x y z it will be going to change.

Now, x y z, is can be defined in a coordinate system, can be defined by a position vector and if the position vector changes, then how this T changes that is the thing I want to write here. So, dT is a change of the scalar field when we have say slight change of r + dr and previously where it is r. So, in the coordinate system if I try to understand, so, in this coordinate system I am having here this point is r and the function here I defined as function of r.

This is the location I am talking about not but this is overall T is a scalar field mind it, it is a location and here this is r + dr and T here the value is this that is the thing just I wanted to mention. So, 2 different points 2 different scalar values, but the change here is dT. Now, what is r here? It is a position vector so; we know how to define a position vector very first class we defined that and what is dr? The increment or the length small length element that we also defined in the previous class so, r and dr are now define.

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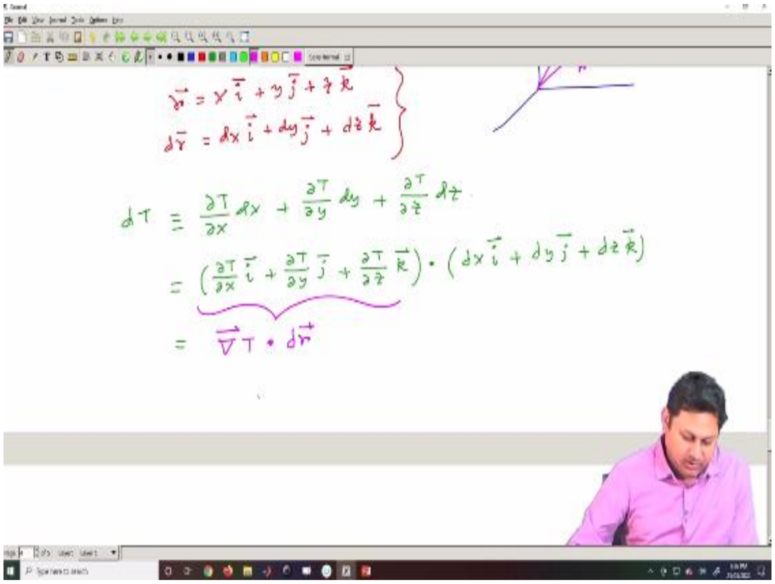

Now, if I expand this dT for this 3-dimensional case, so, how we define in 1-dimension we know that I can define this df is $\frac{\partial f}{\partial x}$dx. Now, f is a function of x y z so, what should be the value of ∂f? It should add in this way with the other 2 variables exactly that thing I will do here. So, dT, T is a function of x y z. So, I should write it as $\frac{\partial T}{\partial x}$ dx + $\frac{\partial T}{\partial y}$ dy + $\frac{\partial T}{\partial z}$ dz. Now, this quantity I write in a nicer way a very compact way and that is the beauty here.

So, please check it. I can write this quantity as in dot product of 2 vectors dx i dot because dx dy dz I know this is dr so, I can write it as this way so, if you make a dot product you are going to get this. Now, I define this quantity very important thing whatever I get here this quantity with a notation and that is this dot whatever we have here is simply dr. So, what is then this notation this deal this is an operator and operated over T.

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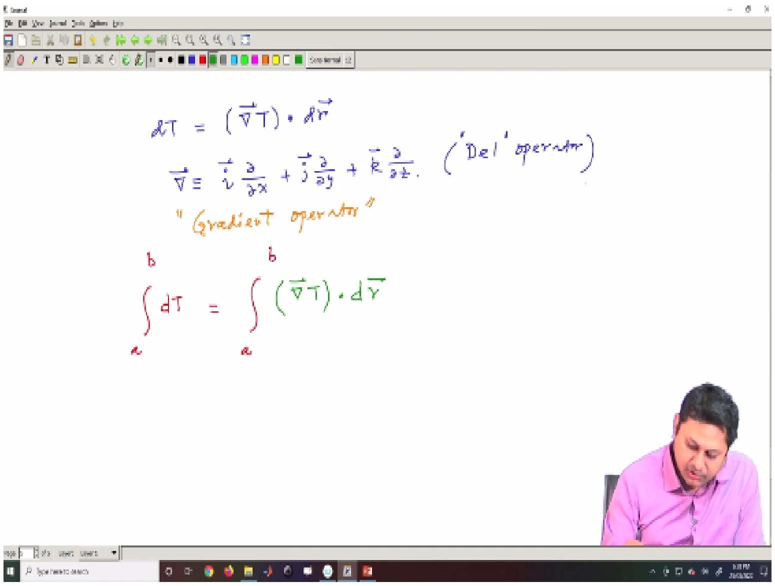

So, now, we write once again so, I am writing dT the variation of T as an operator operating over T I am getting something dot dr vector here. So, this operator if I now compare with this full expression the operator should be this way one can write this operator in this way. This is a very unique kind of operator. This is called the vector operator associated with the partial derivative but with a vector sign. This operator is called the del operator ($\vec{\nabla}$).

The del operator or the gradient operator also if I want to find out the gradient what is gradient I will discuss. So, you should remember that this is a partial derivative $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial z}$ and this $\vec{\nabla}$ is associated with this expression and this is an expression. This is eventually at the end of the day is operator and it will be going to operate over a scalar field. Here for example, like T. Now, if I want to integrate both the sides, you can see that one interesting thing appears here.

So, if I integrate here both the side say dt with some initial point a to b and in the right-hand side, if I look carefully a to b I am having 2 terms here, one is this associated with operator and dot dr. So, eventually, I am doing the line integral here.

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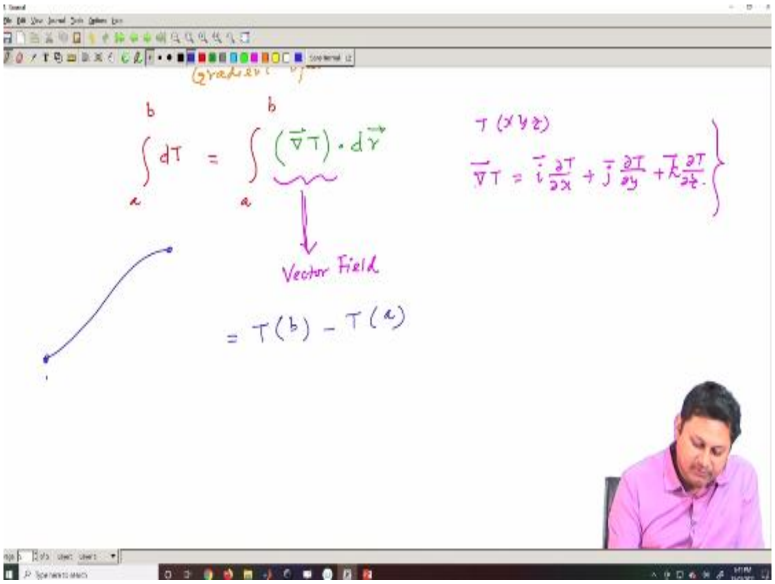

If you look carefully this quantity, this is a quantity, which leads to a vector field this quantity at the end of the day leads to because this is operating over T, T is a scalar, but this operator itself is a vector. So, this leads to a vector quantity or a vector field and we know that when we have integration of vector field over dr and integrate with certain lines, it gives you simply line integral. So, you should be very careful that whenever we have so what is this operator then T, T is a function of x y z and then I have j y partial derivative and k z.

So, this is the entire picture T is scalar field and I am operating this $\vec{\nabla}$ over scalar field what the outcome what is I am getting is a vector field. So, T is a scalar field, but when I operate all these things in this way with the $\vec{\nabla}$, I am getting a vector field. Now, what is this thing, this is a complete integral. So, close this. So, I can have these as because this is a dT. So, I can have this as the result is T(b) – T(a). So, only the final points matter I can have T is a scalar field then I will operate over this $\vec{\nabla}$, I have vector field.

Now, I am asking to find I am asked to calculate what is the line integral and I calculate the line integral I find that this line integral say point a to b if I calculate this quantity only depends on the initial value and final value.

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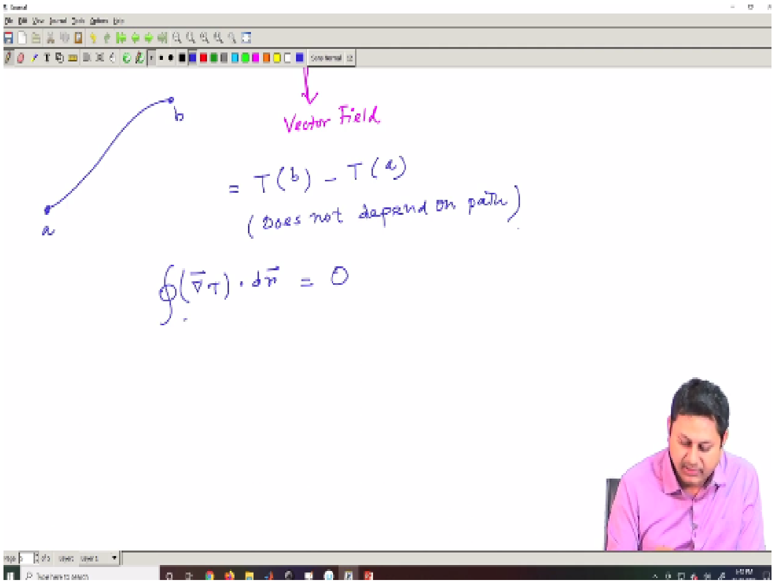

So, from that I can also find so, it does not so, what is the meaning of that it does not depend on the path, does not depend so, it is a path independent integral you can have if you have something like this. Now, one very important conclusion we can make is that if I am having a closed line integral for this quantity, which is that this over T and then dot product of say dr you will be going to get this value you will be going to get 0.

For close line integral if you find in right-hand side this quantity so that has to be 0. Well, what is the general geometrical interpretation so that we quickly understand and then maybe I will discuss more about and do some problem so geometrical.

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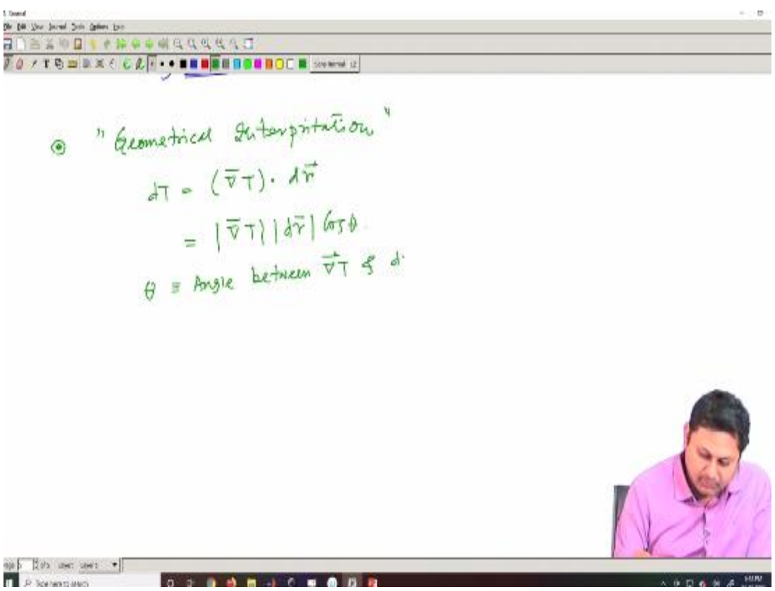

So, dT is equal to I am having this quantity, which is it nothing but a dot product so I can write it as mod of this quantity and then mod of this quantity and cos θ of that where the theta is the angle between dr and this quantity whatever I figured out. So, θ is the angle between this vector quantity and dr.

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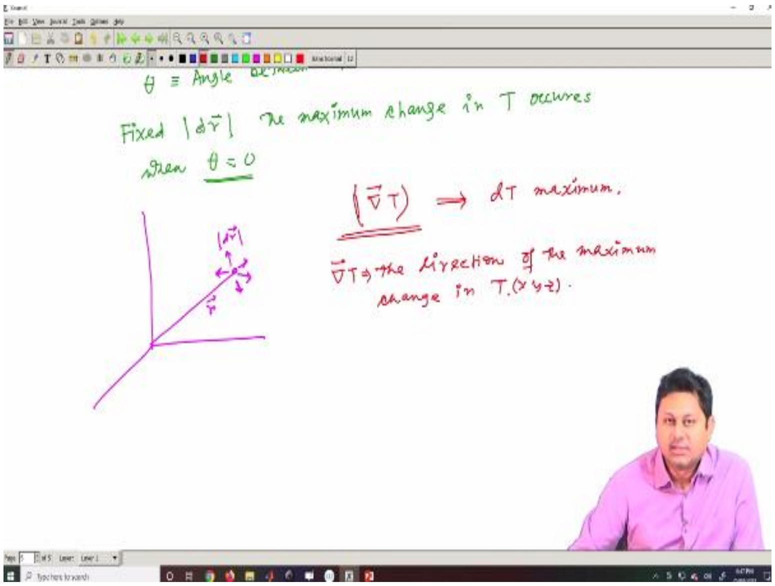

So, now for a fixed if for a fixed dr so, dr is fixed you are in a point and you can go any direction you want, but dr is a fixed amount. The maximum change in T occurs because dT is the maximum dT is a change. So, I want to find out what is how this thing become maximum occurs when θ = 0 that means, you can see that for θ = 0 what happened cos θ will be 1 and then I am going to get dT maximum.

So, for maximum, so, try to understand these, this is a coordinate system and I am having a point here this is r now, I can change my point in any direction I want everything so, any direction I want. So, this is dr vector, but I am saying that for a fixed dr that means all these directions are possible, but I want to change dr fixed direction either this direction, I can move either this direction, I can move either this direction, I can move either this direction, this is my choice that I can move whatever the direction but my dr mod dr is fixed.

Now, in that condition, you can see that if I move the dr in the direction along this quantity so, this is vector quantity along this quantity if I move my dr then in that case, it ensures that the change of dT is maximum. So, what is the meaning of that? So, the meaning is this quantity physically gives the direction of the maximum change in the scalar field T. So, I am having a scalar field T, which is a function of x y z it is varying over this space.

Now, I want to find out that at which distance this T is changing in the maximum way. So, if I calculate that quantity, this gradient of T if I calculate, so, that quantity eventually gives me that this is the direction along which the change is maximum. I can give a quick example here example means how to evaluate these things.

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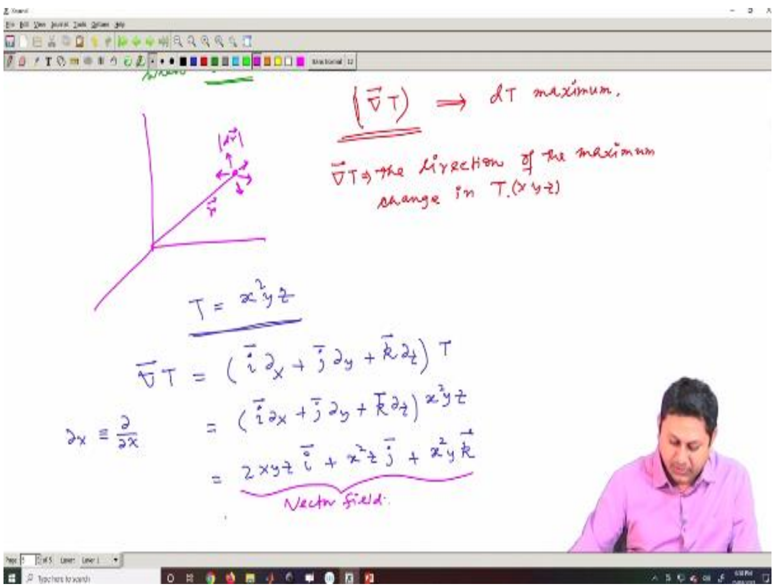

And suppose T is given like a function, a scalar field is given like x^2yz, this is a scalar field that is given. If this scalar field is given, I can find out. So, how to find out this quantity? This is i ∂x, this is a short form, so, ∂x I am using as a short form of d, dx + j dy + k dz that will operate on T and here let me write it T and this T is x^2yz. So, I am operating eventually i ∂x + j ∂y + k ∂z and this is x^2yz.

Now, if I figure out what is the result, so, first is the partial derivative with respect to x. So, the first term will be simply $2xyz\,\hat{\imath} + x^2z\,\hat{\jmath} + x^2y\,\hat{k}$ so, I am having a vector field so, this is nothing but a vector field so, whatever I am getting is a vector field here so, at a given point this vector field should have a value.

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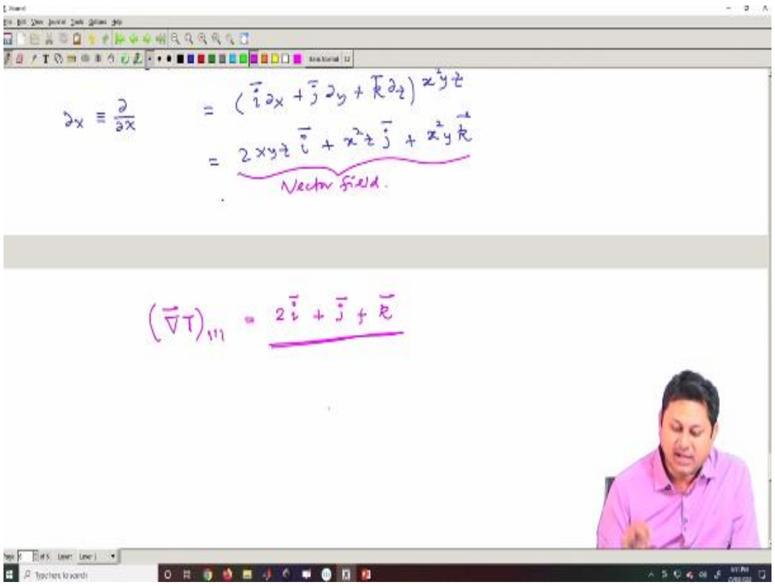

So, if I want to find out what is the value of this quantity at 111 So, I am now in 111 point and try to find out that in 111 point what is the direction for which that the scalar field T is the change of scalar field T is maximum. So, I already figured out this vector field. And now at 111 point, I will just put the value. It will be simply 2 i + j + k. So, I am having a vector here and if I find out the unit vector, so, this is the direction this is a vector if I only find the direction of this vector.

That direction eventually gives me that at this point 111 point that is the direction along which the value of this T that is the scalar field is having a maximum change. Now, if I change my point to other say 123 or 211 or 012 accordingly this field this direction will change and this direction basically gives me that at that particular point, if I move to along this direction, then the change of this scalar field will be maximum.

Let us do more because I have already and very useful expression here that if I find the closed line integral of any given scalar field and then I make a gradient and then if I want to find out the close line integral, I will be going to get a 0 that is the path independent integral I will get. So, if I want to find out the close integral it should be path independent. So, let us check whether we can get something like that or not.

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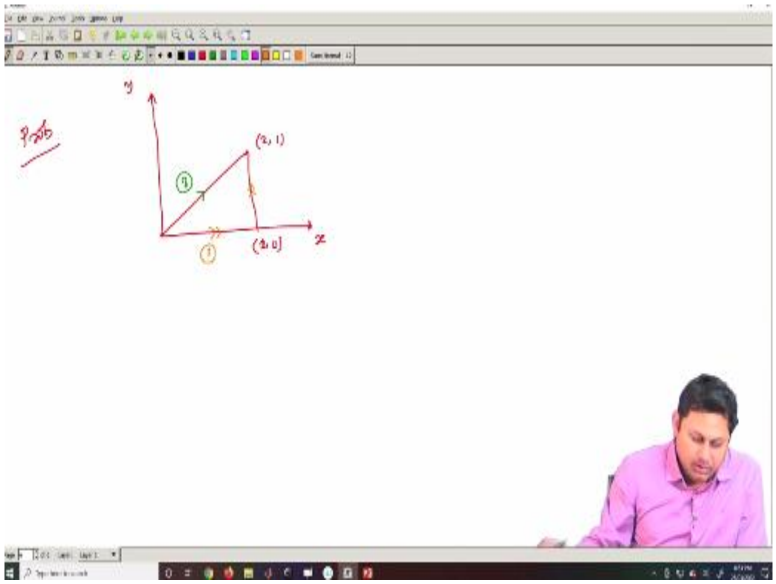

So, the problem so, now, let me consider this as a problem. So, what is the problem? The problem says that let us make it 2-dimensional, I have a x coordinate, x y coordinate as Cartesian coordinate and I am having a path like this last day we were doing a similar kind of problem. So, this coordinate point is say 1 0. No, this is say 2 0 and this point is 2 1. So now, I am going a path here from this path to this path because I am going to calculate the path integral here and another path is this one. So, this is my path 2 and this is my path 1.

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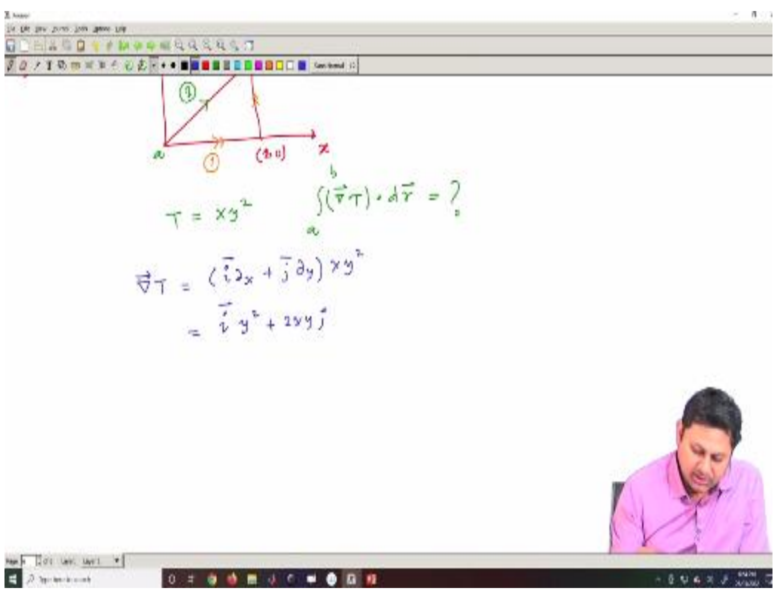

The problem is the scalar field is now given; the scalar field T is given, which is xy^2. Now, the problem is find a to b this point say a and this point b, a to b what is the value of this quantity. What is the value of this quantity? So, this is how much that I need to figure out. So, T is given so, I can find out what is the gradient of T. So, first thing first so, this is i ∂x + j ∂y I do not need to include k here because it is 2-dimensional, z is not there and it is operating over this. So, I can have i y^2 + 2xy j, this is my vector field.

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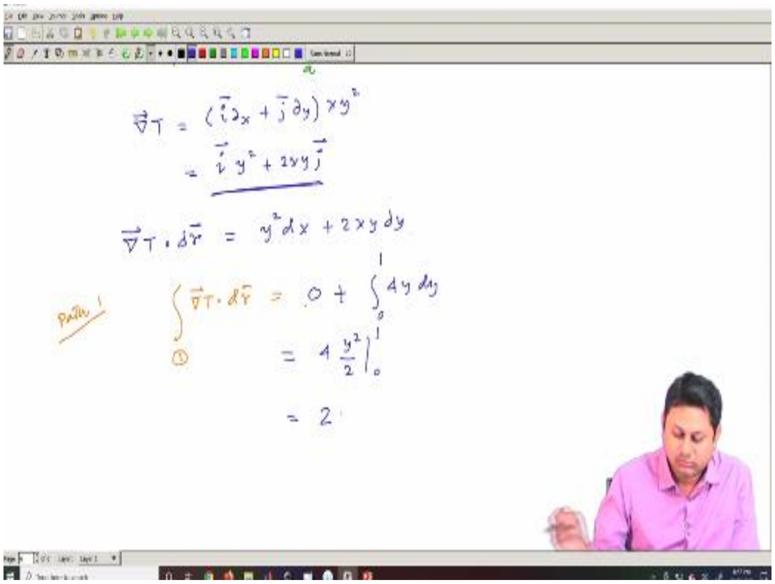

And next thing is that I need to calculate this quantity T dot dr. This is y^2 dx + 2xy dy, that is all. Now I need to calculate this integral for 2 paths so, for path 1 integration so, in path 1, so what we are getting, so, first we need to check. So, this integration I need to calculate for, say path 1 I just write it here, path 1, so path 1 is having two you know two paths, one is 0, it is from here to here. And when I am going from here to here, my dy is completely 0 only dx is changing and when dx is changing y value is 1.

So, for the first path of this path, it is simply 0 without doing anything I can simply say this value is 0. What about the next path? In the next part, when I go from this 2 0 point to 2 1 point, you can see that my x value is fixed it is 2 but y is changing from 0 to 1. So that means for this path, my dx is not changing, so dx value is 0. So, y is changing. So, dy is non zero and y is 2. So, I should have integration of 4y dy, that is all because x is 2, it is not changing.

So, what value I am getting here 4 $\frac{y^2}{2}$ so, integration 0 to 1 this is for the second part of this path 0 to 1. So, eventually I am going to get 2. This is the result for path 1.

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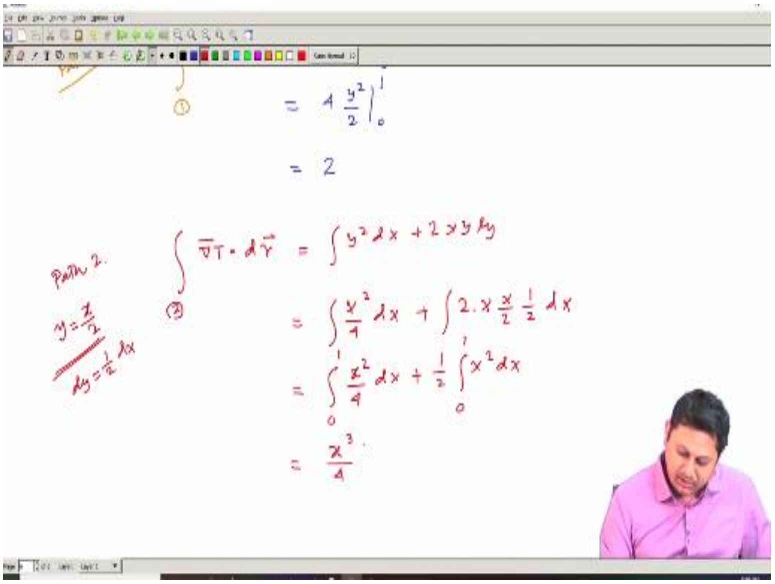

What about path 2, I need to calculate the same thing, but for a different path. And this same quantity I am going to calculate, but in path 2 what we are getting? In path 2, you can see that I am directly going from this point to this point. So, there is a relationship with x and y and the relationship is simply y is equal to so you can see the coordinate here. So, this coordinate is 2 1 so simply y = $\frac{x}{2}$ that is the relationship between x and y.

So, I just used this relation, so here dy, I can replace as half of dx. And then this quantity, which I am having, I can already figure out this here. So, this is integration of y^2 dx + 2xy dy, everything is changing. So, now I will be going to exploit this expression y, so I can put everything in terms of either y or either x, so let us put everything in x, if I do, y^2 is simply becoming $\frac{x^2}{4}$ dx. And then this quantity 2x is there, y I replace like $\frac{x}{2}$ and dy, I replace $\frac{1}{2}$ dx.

So eventually, I am having integration of $\frac{x^2}{4}$ dx + one 2 will be going to cancel out so $\frac{x^2}{2}$ dx. So, 0 to 1, 0 to 1. So, eventually, I will have $\frac{x^3}{4}$ seems to be sorry, limit is not 0 to 1, because x is changing from 0 to 2, we have to be careful here.

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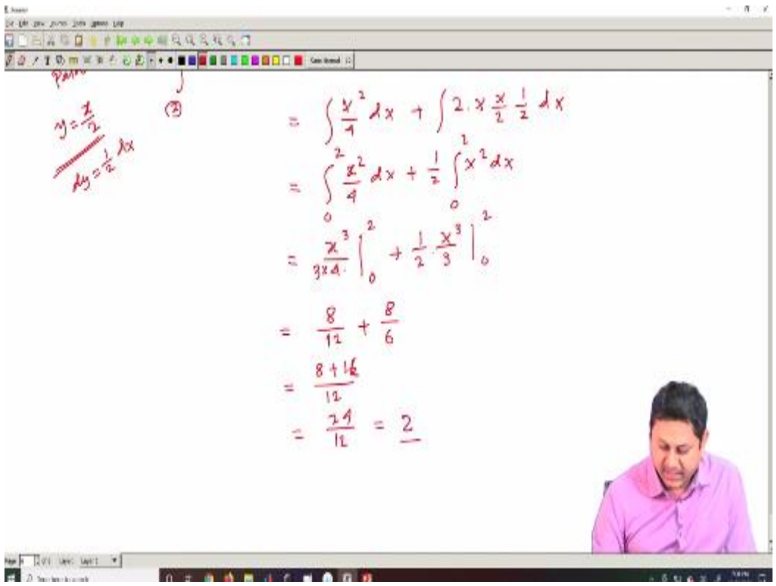

So, this limit is not 1, but 2, 2 and 2, so it is 0 to 2 and then $\frac{1}{2}\frac{x^3}{3}$. So, here I am having so $\frac{x^3}{3}$ also. So, 3 multiplied by 4 so 0 to 2 so, I should have $\frac{8}{12}+\frac{8}{6}$. So, it should be $\frac{8+16}{12}$. So, it should be $\frac{24}{12}$. And I will be going to get the value 2. The interesting thing that is these 2 values and here also I am getting 2 that means what I am saying here is these quantities is a path independent integral.

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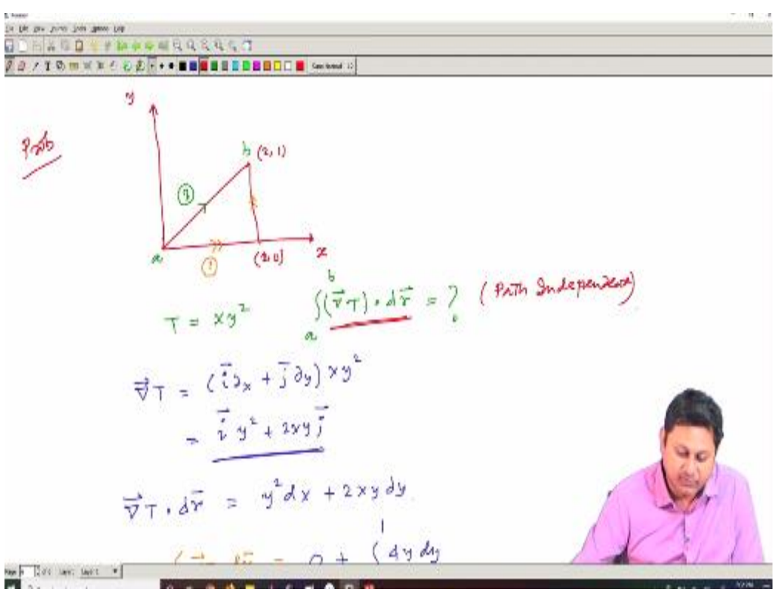

This is a path independent integral. This is path independent that means if I am having an integral in this particular form, then if you start from point a to this and then you for example, you take you can go a to b in infinite amount of infinite way, you can choose whatever the path you like there are infinite amount of way you can reach from a to b. We have taken 2 very simple paths, one is path 1 and one is path 2.

And you can see that for path 1 and path 2, both the cases you are getting the same result and which is expected because this quantity has to be path independent. So, you will get the value 2 both the cases if you now have a close line integral for example, if you start from a then go to this point and then go to b and then come back a, you will get a simply you will get to 0 because in that case you there you have a minus 2 sign here and if you integrate the total thing you should get a 0. So, with this note, I like to conclude I do not have much time to discuss.

So, today we learned a very important thing that the gradient of function if a scalar field is there. If you calculate the gradient if you operate the gradient operator over the scalar field, this operator physically gives the direction along which the scalar field is increasing in a maximum way. So, the calculation is straightforward. And I believe you can also calculate even if a scalar field is given if you can calculate the gradient. And you can find out what is the line integral also if the line is defined properly.

So, thank you very much for your attention. In the next class, we will discuss more about these problems and we will see that how other things like divergence and curl emerges 2 very important operator. So, thank you and see you in the next class.