B.Sc. DEGREE EXAMINATION,NOVEMBER 2018
III Year V Semester
Core Major - Paper IX
MODERN ALGEBRA

Time : 3 Hours **Max.marks :75**

Section A $(10 \times 2 = 20)$ Marks

Answer any **TEN** questions

1. If $\mathbf{G}$ is a finite group and $\mathbf{a} \in \mathbf{G}$, then prove that $\mathbf{o}\,(\mathbf{a}) \mid \mathbf{o}\,(\mathbf{G})$.
2. Is the converse of Lagrange's theorem true? Justify.
3. Find the orbits and cycles of the permutation $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \mathbf{2} & \mathbf{3} & \mathbf{8} & \mathbf{1} & \mathbf{6} & \mathbf{4} & \mathbf{7} & \mathbf{5} & \mathbf{9} \end{pmatrix}$.
4. When do you say that an integral domain is of finite characteristic?
5. If $\mathbf{U}$ is an ideal of a ring $\mathbf{R}$ and if $\mathbf{1} \in \mathbf{U}$, then show that $\mathbf{U}{=}\mathbf{R}$.
6. Define a ring homomorphism.
7. Define a Euclidean ring.
8. Let $\mathbf{R}$ be a Euclidean ring. Suppose that for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}$, $\mathbf{a}|\mathbf{b}c$ but $(\mathbf{a}, \mathbf{b})\,{=}\mathbf{1}$. Show that $\mathbf{a}|\mathbf{c}$.
9. State the division algorithm for polynomials.
10. Define unique factorization domain.
11. Show that every subgroup of an abelian group is normal.
12. If $\mathbf{R}$ is a ring, then show that $\mathbf{a0}{=}\mathbf{0a}{=}\mathbf{0}$ for any $\mathbf{a} \in \mathbf{R}$.

Section B $(5 \times 5 = 25)$ Marks

Answer any **FIVE** questions

13. Show that the subgroup $\mathbf{N}$ of $\mathbf{G}$ is a normal subgroup of $\mathbf{G}$ if and only if every left coset of $\mathbf{N}$ is a right coset of $\mathbf{N}$ in $\mathbf{G}$.
14. Let $\mathbf{G}$ be a group and let $\mathcal{A}\,(\mathbf{G})$ be the set all automorphisms of $\mathbf{G}$. Show that $\mathcal{A}(\mathbf{G})$ is also a group.
15. Let $\mathbf{U}$, $\mathbf{V}$ are ideals of a ring $\mathbf{R}$ and let $\mathbf{U}{+}\mathbf{V}{=}\{\mathbf{u}{+}\mathbf{v} \ \mid \mathbf{u} \in \mathbf{U},\, \mathbf{v} \in \mathbf{V}\}$. Show that $\mathbf{U}{+}\mathbf{V}$ is an ideal of $\mathbf{R}$.
16. State and prove that unique factorization theorem.
17. State and prove Gauss' lemma.
18. Let $\mathbf{H}$ and $\mathbf{K}$ are subgroups of a group $\mathbf{G}$. Show that $\mathbf{H}K$ is a subgroup of $\mathbf{G}$ if and only if $\mathbf{H}K{=}\mathbf{K}H$.

19. Show that every finite integral domain is a field.

Section C $(3 \times 10 = 30)$ Marks

Answer any **THREE** questions

20 Define the kernel of a homomorphism. If ϕ is a homomorphism of $\mathbf{G}$ into $\overline{\mathbf{G}}$ with kernel $\mathbf{K}$, then show that $\mathbf{K}$ is a normal subgroup of $\mathbf{G}$.

21. State and prove Cayley's theorem.

22. Let $\mathbf{R}$ be a commutative ring with unit element and $\mathbf{M}$ is an ideal of $\mathbf{R}$. Show that $\mathbf{M}$ is maximal if and only if $\mathbf{R}/\mathbf{M}$ is a field.

23. (a) Define Gaussian integers.

    (b) State and prove Fermat's Theorem.

24. (a) Show that F[x] is an integral domain.

    (b) State and prove the Eisenstein criterion.

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